A STUDY OF SOME TOPICS IN q-SERIES

AND

CONTINUED FRACTIONS

A THESIS SUBMITTED TO THE PONDICHERRY UNIVERSITY

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE AWARD OF THE DEGREE OF

DOCTOR OF PHILOSOPHY

IN

MATHEMATICS

BY

YUDHISTHIRA JAMUDULIA

UNDER THE SUPERVISION

OF

DR. SYEDA NOOR FATHIMA

DEPARTMENT OF MATHEMATICS

RAMANUJAN SCHOOL OF MATHEMATICAL SCIENCES

PONDICHERRY UNIVERSITY, PUDUCHERRY- 605014

INDIA

DECEMBER, 2016

TO MY

PARENTS

DEPARTMENT OF MATHEMATICS

RAMANUJAN SCHOOL OF MATHEMATICAL SCIENCES

PONDICHERRY UNIVERSITY, KALAPET, PUDUCHERRY-605014.

DR. S. N. FATHIMA **E-mail: fathima.mat@pondiuni.edu.in**

Assistant Professor **Phone:**0413 – 2654704

CERTIFICATE

This is to certify that the thesis entitled "**A Study Of Some Topics In $q-$Series And Continued Fractions**" submitted by **Mr.Yudhisthira Jamudulia** to Pondicherry University, for the partial fulfilment of the requirements of **Doctor of Philosophy** degree in Mathematics, is a bonafide record of research work by him under my supervision at Department of Mathematics, Ramanujan School of Mathematical Sciences, Pondicherry University, Pondicherry−605014. The content of this thesis, in full or in parts, has not been submitted to any other Institute or University for the award of any degree or diploma.

I wish him all success in his life.

Date:

Place: Puducherry

Dr.Syeda Noor Fathima

DECLARATION

I hereby declare that the thesis entitled "**A Study of Some Topics in q-Series and Continued Fractions**" to Pondicherry University, for the award of the degree of the **Doctoral of Philosophy** in Mathematics is the result of the research work carried out by me in the Department of Mathematics, Ramanujan School of Mathematical Sciences, Pondicherry University, Pondicherry under the guidance of **Dr. S. N. Fathima**, Assistant Professor, Department of Mathematics, Ramanujan School of Mathematical Sciences, Pondicherry University, Pondicherry, during the period August, 2013-December, 2016.

I further declare that the results of this research work have not been submitted previously to this or any other University for any degree, diploma or fellowship.

Date:

Place: Puducherry

Yudhisthira Jamudulia

Research Fellow, Roll No. R34010

ACKNOWLEDGEMENT

I would have never been able to finish this study without the support, patients, co-operation and guidance of the following people and agency. It is to them that I owe my deepest gratitude.

To my guide, ***Dr.Syeda Noor Fathima***, Assistant Professor, Department of Mathematics, Ramanujan School of Mathematical Sciences, Pondicherry University, Pondicheery-605014. Her timely direction, complete co-operation and minute observation have made my research work fruitful rewarding.

I would like to express my thanks to CSIR-HRDG, CSIR Complex, Library Avenue, Pusa, New Delhi, for JRF and SRF (Sr. No. 1061041004, Ref. No. 20-06/2010(i)EU-IV, File No. 09/087(0686)/2011-EMR-1 and 09/559(0103)/2011-EMR-1) awarded to me for the period July, 2011 to July, 2016 which was crucial to the successful completion of thesis.

I am highly indebted to Prof. K. M. Tamizhmani, Professor and Dean, Ramanujan School of Mathematical Sciences, Pondicherry University, Pondicherry-605014, for accepting me as a Ph.D. Research Scholar in the Department and providing all necessary infrastructure facilities in the Department.

I am highly indebted to Prof. Chandrashekara Adiga, Department of Studies in Mathematics, University of Mysore, Manasagangotri, India and Prof. M. S. Mahadeva Naika, Department of Mathematics, Central College Campus, Bangalore University, Bengaluru for their valuable suggestions throughout my tenure.

My special thanks to Teaching and Non-Teaching staff of Department of Mathematics, Ramanujan School of Mathematical Sciences, Pondicherry University, Pondicherry for their support and encouragement.

I am thankful to all my fellow research friends of the Department of Mathematics, Ramanujan School of Mathematical Sciences, Pondicherry University, Pondicherry for their help throughout my tenure. I also thank to all my friends who came to my help whenever sought and kept me cheerful throughout to produce this thesis.

Lastly, my higher education and career was a dream of my parents and family members. I am indebted to all of them for their good will, love and affection. My sincere thanks to one and all who have supported directly and indirectly during the period. In particular, I pray for the blessings of my parents without whom I could not have dreamt anything in my life.

Date: ***Yudhisthira Jamudulia***

Place: Puducherry Research Fellow, Roll No. R34010

LIST OF PAPER PUBLICATIONS

1. **Six-Variable Generalization of Reciprocity Theorem Found in Ramanujan's Lost Notebook**, Adv. Stud. Contemp. Math.(Kyungshang) **26(4)**(2016), pp. 693 – 701. (With S. N. Fathima)

2. **A Transformation Formula for Basic Bilateral Hypergeometric** $_2\psi_2$ **Series**, Arab. j. math. (Communicated). (With S. N. Fathima)

3. **On Some Circular Summation Formulas for Theta Functions**, Presented by S. N. Fathima in The 28th International Conference of The Jangjeon Mathematical Society, held in Sherwood Club Kemer Hotel, Antalya-Turkey, Dated- 15 – 19 May, 2015, (To Appear) (With S. N. Fathima)

4. **On A Continued Fraction Identity From Ramanujan's Notebook**, Innovative J. (AJCEM), **3:1**(2014), 391 – 399. (With S. N. Fathima)

5. **An Interesting q-Continued Fractions of Ramanujan**, Palestine Journal of Mathematics, **4(1)**(2015), 198 – 205. (With S. N. Fathima and T. Kathiravan) (Not included in the Thesis)

LIST OF PAPER PRESENTED IN CONFERENCE

1. **Some Explicit Evaluation of Ramanujan's Cubic Continued Fraction**, Presented in the International Conference Recent Developments on Statistical Theory and Practice, $14-16$, December 2013, Department of Statistics, Pondicherry University R.V. Nagar, Kalapet, Puducherry – 605014, India. (With S. N. Fathima)

2. **A Note on Ramanujan's Remarkable Product of Theta-Functions**, Presented in the National Conference on Recent Advances in Numerical Methods and Its Application (RANMA-2014), $27-28$, January 2014, Department of Mathematics, Pondicherry University, R. V. Nagar, Kalapet, Puducherry-605014, India. (With S. N. Fathima)

3. **A Note on Explicit Evaluation of Ramanujan's Cubic Continued Fraction using Theta Function Identities**, Presented in the National Conference on Advances in Mathematics and its Applications to Sciences and Engineering ($AMASE-2016$), $22nd$, January 2016, Department of Mathematics, University College of Engineering Pattukkottai, Rajamadam, Thanjavur– 614701, Tamil Nadu, India. (With S. N. Fathima)

4. **An Analogue of Ramanujan's Remarkable Product of Theta Functions and its Applications**, Presented in the International Conference on Mathematic, Physics and Allied Sciences 2016, during $03-05$ March, 2016, Carmel College, Nuvem, Goa, India. (With S. N. Fathima)

ABSTRACT

A careful and constant attention was paid for the development of this thesis. Enough attempt has been taken to make this thesis comprehended easily. The main novel feature of this thesis is its increased emphasis on special topics. For completeness of this thesis, references of papers and books have been inserted explicitly.

The table of content gives neat and clear information of the subject matter of thesis. The thesis splits naturally into five chapters. Section 1 of Chapter I of this thesis begins with introduction and subsequent development of Ramanujan's notebooks along with few customary definitions we make use in the sequel. In Section 2, we present some Ramanujan's terminology with some important results. Section 3 contains some basic well known hypergeometric transformations. Section 4 deals with a remarkable result of Ramanujan- circular summation formula and in final Section of this Chapter, we deals with Ramanujan's continued fractions.

Chapter II contains three Sections. Section 2.1 begins with Ramanujan's reciprocity theorem and its generalization obtained by several mathematicians. Section 2.2 contains our main result, a neat and different six variable generalization of Ramanujan's reciprocity theorem. We conclude this Chapter by obtaining several beta, gamma identities and eta function identities for our main identity.

In Chapter III, we begin with basic terminology and important results related to basic hypergeometric series. In Section 3.2, we obtain our main result- ${}_2\psi_2$ transformation formula for ${}_2\psi_2$ bilateral series. As applications to our main result,

we obtain Fourier series related to theta functions.

Chapter IV begins with a short introduction on circular summation formula and contributions on the same by many researchers in Section 4.1.. We discuss new circular summation formula in Section 4.2. In Section 4.3, we also obtain new circular summation formula by applying Jacobi imaginary transformation formula. In Section 4.4, we obtain new circular summation formula by applying difference of theta function on the formulae obtained in Section 4.2. In the final Section, as application, we obtain some special cases as identity of theta functions.

In the concluding Chapter of this thesis, we study a nice continued fraction analogous to Ramanujan's continued fractions.

TABLE OF CONTENTS

CHAPTER I

INTRODUCTION

CHAPTER I

INTRODUCTION

SECTION 1

Srinivasa Aiyangar Ramanujan (1887 – 1920), the mathematical legendary genius ranks among the all time great Euler, Gauss and Jacobi. In his three notebooks, perhaps influenced by the style of Carr's Book, recorded thousands of original results without proof, which has greatly influenced some of the best research work in mathematics. In 1957, the Tata Institute of Fundamental Research, Bombay brought out the facsimile edition of these notebooks in two volumes.

In 1929, Watson and B. M. Wilson had undertaken the task of editing Ramanujan's notebooks, but were unable to complete their task partly due to premature death of Wilson in 1935. Professor Bruce C. Berndt of University of Illinois, USA, deserves special credit for his major role in editing Ramanujan's notebooks. Today the wider circle of mathematicians the world over have five edited volumes – Ramanujan Notebooks Parts I - V – which contain proofs of the theorems stated by Ramanujan or the references to the proofs. The five volumes contain 3254 results.

In addition to the three notebooks of Ramanujan, G. E. Andrews in 1976 while looking through a box of Watson's material in the library of Trinity college, Cambridge University, came across over 600 formulas in about 140 sheet of pages handwritten by Ramanujan. Some of the sheets contained claims of Ramanujan about mock theta functions. Sincere thanks to Prof. Bruce Carl Berndt and Prof. George E. Andrews for giving mathematical fraternity so far four edited volumes − Ramanujan Lost notebook Parts I - IV.

The research work presented in this thesis − for the most part is based on and motivated by the works of Ramanujan.

Throughout the thesis we employ the following notations and definitions:

$$(a)_\infty := (a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n)$$

and

$$(a)_n := (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k) = \frac{(a)_\infty}{(aq^n)_\infty}, \quad n: \text{ any integer}, \tag{1.1.1}$$

where a and q are complex number with $|q| < 1$. In particular, if n is a positive integer

$$(a)_{-n} = \frac{(-1)^n\, q^{n(n+1)/2}}{a^n\, (q/a)_n}, \qquad a \neq 0. \tag{1.1.2}$$

We shall define ${}_rF_s$, the generalized hypergeometric series by

$${}_rF_s\left[\begin{matrix} a_1, & a_2, & \cdots, & a_r & ;z \\ b_1, & b_2, & \cdots, & b_s & \end{matrix}\right]$$

$$= \sum_{n=0}^{\infty} \frac{a_1(a_1+1)\cdots(a_1+n-1)\cdots a_r(a_r+1)\cdots(a_r+n-1)}{n!\, b_1(b_1+1)\cdots(b_1+n-1)\cdots b_s(b_s+1)\cdots(b_s+n-1)}\, z^n \cdot \qquad (1.1.3)$$

By the ratio test, the series on right hand side of (1.1.3) converges absolutely for all z if $r \leq s$, and for $|z| < 1$ if $r = s+1$.

The generalized basic hypergeometric series ${}_{r+1}\phi_r$ is defined by

$$ {}_{r+1}\phi_r \left[\begin{matrix} a_1, & a_2, & \cdots, & a_{r+1} & ;q & ;z \\ b_1, & b_2, & \cdots, & b_r & & \end{matrix} \right] = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_{r+1})_n}{(q)_n (b_1)_n (b_2)_n \cdots (b_r)_n}\, z^n , \qquad (1.1.4)$$

where $a_1, a_2, \cdots, a_{r+1}, b_1, b_2, \cdots, b_r$ are arbitrary, except that of course, $(b_j)_n \neq 0$, $1 \leq j \leq r$, $0 \leq n < \infty$ and $(a)_n$ is as in (1.1.1). For $0 < |q| < 1$, the series on the right hand side of (1.1.4) converges absolutely for $|z| < 1$.

The basic bilateral hypergeometric series ${}_r\psi_r$ is defined by

$$ {}_r\psi_r \left[\begin{matrix} a_1, & a_2, & \cdots, & a_r & ;q & ;z \\ b_1, & b_2, & \cdots, & b_r & & \end{matrix} \right] = \sum_{n=-\infty}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_r)_n}{(b_1)_n (b_2)_n \cdots (b_r)_n}\, z^n , \qquad (1.1.5)$$

where $(a)_n$ and $(a)_{-n}$ are as defined in (1.1.1) and (1.1.2) respectively and the denominator factors are never zero. For $0 < |q| < 1$, the series converges absolutely in the annulus $\left| \frac{b_1 \cdots b_r}{a_1 \cdots a_r} \right| < |z| < 1$.

The eta-function, also known as the Dedekind eta function is defined by

$$\begin{aligned} \eta(\tau) &:= e^{\pi i\tau/12}\prod_{n=1}^{\infty}(1-e^{2\pi in\tau}), \qquad \text{Im } \tau > 0, \\ &:= q^{1/24}(q;q)_{\infty}, \qquad \text{where } e^{2\pi i\tau} = q. \end{aligned} \tag{1.1.6}$$

The q-difference operator and the q- shifted operator ζ are defined by [80]

$$D_q f(a) = \frac{1}{a}(f(a) - f(aq)) \qquad \text{and} \qquad \zeta\{f(a)\} = f(aq)$$

respectively.

An operator θ is defined by $\theta = \zeta^{-1}D_q$ and the operator $E(b\theta)$ is defined as

$$E(b\theta) = \sum_{n=0}^{\infty} \frac{(b\theta)^n q^{n(n-1)/2}}{(q;q)_n}.$$

Then the operator identities are [80, Theorem 1]:

$$E(b\theta)\{(at;q)_{\infty}\} = (at, bt;q)_{\infty}.$$

$$E(b\theta)\{(as, at;q)_{\infty}\} = \frac{(as, at, bs, bt;q)_{\infty}}{(abst/q;q)_{\infty}}, \qquad \left|\frac{abst}{q}\right| < 1,$$

where as usual,

$$(a_1, a_2, a_3 \cdots a_k;q)_n = (a_1;q)_n(a_2;q)_n(a_3;q)_n \cdots (a_k;q)_n,$$

where n is an integer or infinity.

The q- gamma function $\Gamma_q(x)$ is defined as

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, \qquad 0 < q < 1. \tag{1.1.7}$$

q- gamma function is a q-analogue of Euler's gamma function which was introduced by J. Thomae [109] and later by Jackson [71]. The definition of q-gamma function can be extended to $|q| < 1$ by using the principal values of q^x and $(1-q)^{1-x}$, from (1.1.7)

$$\begin{aligned}\Gamma_q(x+1) &= \frac{(q;q)_\infty}{(q^{x+1};q)_\infty}(1-q)^{-x} \\ &= \prod_{n=1}^{\infty} \frac{(1-q^n)(1-q^{n+1})^x}{(1-q^{n+x})(1-q^n)^x}.\end{aligned}$$

Hence

$$\begin{aligned}\lim_{q\to 1^-} \Gamma_q(x+1) &= \prod_{n=1}^{\infty} \frac{n}{n+x}\left(\frac{n+1}{n}\right)^x \\ &= x\left[x^{-1}\prod_{n=1}^{\infty}\left(1+\frac{x}{n}\right)^{-1}\left(1+\frac{1}{n}\right)^x\right] \\ &= \Gamma(x+1).\end{aligned}$$

Thus $\Gamma_q(x) \to \Gamma(x)$ as $q \to 1$, the ordinary gamma function.

Askey [21] defined the q-beta function as

$$B_q(x,y) = (1-q)\sum_{n=0}^{\infty} q^{nx}\frac{(q^{n+1})_\infty}{(q^{n+y})_\infty}. \tag{1.1.8}$$

He also proved that

$$B_q(x,y) = \frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)}. \tag{1.1.9}$$

The classical Jacobi's theta functions $\theta_i(z|\tau), i = 1,2,3,4$ for $q = e^{\pi i\tau}$, are defined as follows:

$$\begin{aligned}
\theta_1(z|\tau) &= -iq^{1/4}\sum_{m=-\infty}^{\infty}(-1)^m q^{m(m+1)}e^{(2m+1)iz}, && (1.1.10)\\
\theta_2(z|\tau) &= q^{1/4}\sum_{m=-\infty}^{\infty}(-1)^m q^{m(m+1)}e^{(2m+1)iz}, && (1.1.11)\\
\theta_3(z|\tau) &= \sum_{m=-\infty}^{\infty} q^{m^2}e^{2miz}, && (1.1.12)\\
\theta_4(z|\tau) &= \sum_{m=-\infty}^{\infty}(-1)^m q^{m^2}e^{2miz}. && (1.1.13)
\end{aligned}$$

Employing (1.1.10)-(1.1.13), we get the following properties of $\theta_i(z|\tau)$, $i = 1,2,3,4$

$$\theta_1(z+\pi|\tau) = -\theta_1(z|\tau), \quad \theta_1(z+\pi\tau|\tau) = -q^{-1}e^{-2iz}\theta_1(z|\tau), \tag{1.1.14}$$

$$\theta_2(z+\pi|\tau) = -\theta_2(z|\tau), \quad \theta_2(z+\pi\tau|\tau) = q^{-1}e^{-2iz}\theta_2(z|\tau), \tag{1.1.15}$$

$$\theta_3(z+\pi|\tau) = \theta_3(z|\tau), \quad \theta_3(z+\pi\tau|\tau) = q^{-1}e^{-2iz}\theta_3(z|\tau), \tag{1.1.16}$$

$$\theta_4(z+\pi|\tau) = \theta_4(z|\tau), \quad \theta_4(z+\pi\tau|\tau) = -q^{-1}e^{-2iz}\theta_4(z|\tau). \tag{1.1.17}$$

Applying induction on (1.1.14)-(1.1.17), we obtain

$$\begin{aligned}
\theta_1(z+n\pi\tau|\tau) &= (-1)^n q^{-n^2} e^{-2niz}\theta_1(z|\tau)\,, && (1.1.18)\\
\theta_2(z+n\pi\tau|\tau) &= q^{-n^2} e^{-2niz}\theta_2(z|\tau)\,, && (1.1.19)\\
\theta_3(z+n\pi\tau|\tau) &= q^{-n^2} e^{-2niz}\theta_3(z|\tau)\,, && (1.1.20)\\
\theta_4(z+n\pi\tau|\tau) &= (-1)^n q^{-n^2} e^{-2niz}\theta_4(z|\tau)\,. && (1.1.21)
\end{aligned}$$

The well-known Jacobi transformation formulas for theta functions are given by

$$\begin{aligned}
\theta_1\Big(\frac{z}{\tau}\Big|-\frac{1}{\tau}\Big) &= -i\sqrt{-i\tau}e^{iz^2/\pi\tau}\theta_1(z|\tau)\,, && (1.1.22)\\
\theta_2\Big(\frac{z}{\tau}\Big|-\frac{1}{\tau}\Big) &= \sqrt{-i\tau}e^{iz^2/\pi\tau}\theta_4(z|\tau)\,, && (1.1.23)\\
\theta_3\Big(\frac{z}{\tau}\Big|-\frac{1}{\tau}\Big) &= \sqrt{-i\tau}e^{iz^2/\pi\tau}\theta_3(z|\tau)\,, && (1.1.24)\\
\theta_4\Big(\frac{z}{\tau}\Big|-\frac{1}{\tau}\Big) &= \sqrt{-i\tau}e^{iz^2/\pi\tau}\theta_2(z|\tau)\,. && (1.1.25)
\end{aligned}$$

We use the notation

$$\frac{a_1}{b_1}\;+\;\frac{a_2}{b_2}\;+\;\frac{a_3}{b_3}\;+\cdots, \qquad (1.1.26)$$

for the continued fraction

$$\cfrac{a_1}{b_1+\cfrac{a_2}{b_2+\cfrac{a_3}{b_3+\cdots}}}\,.$$

We let A_n denote the n^{th} numerator and and B_n denote the n^{th} denominator, for (1.1.26). Thus, for $n \geq 1$

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \cdots + \frac{a_n}{b_n} = \frac{A_n}{B_n},$$

where

$$\begin{aligned} A_n &= b_n A_{n-1} + a_n A_{n-2}, \\ B_n &= b_n B_{n-1} + a_n B_{n-2}, \\ A_{-1} &= 1 = B_0 \qquad \text{and} \qquad A_0 = 0 = B_{-1}. \end{aligned}$$

The set of natural numbers is denoted by $\mathbf{N}$, the set of integers by $\mathbf{Z}$, the set of complex numbers by $\mathbf{C}$ and the set of real numbers by $\mathbf{R}$. We set $\hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ and $\hat{\mathbf{R}} = \mathbf{R} \cup \{\infty\}$.

If $a_{\mathbf{N}} = 0$, we say that the continued fraction (1.1.26) terminates, and we assign to it the value

$$f := \frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \cdots + \frac{a_{\mathbf{N}-1}}{b_{\mathbf{N}-1}} = \frac{A_{\mathbf{N}-1}}{B_{\mathbf{N}-1}},$$

if $a_n \neq 0,\ 1 \leq n < \mathbf{N}$. If $a_n \neq 0,\ 1 \leq n < \infty$, then the continued fraction (1.1.26) converges if $\lim_{n\to\infty}\left(\frac{A_n}{B_n}\right)$ exists in $\hat{\mathbf{C}}$. Its value is given by

$$f = \lim_{n\to\infty}\left(\frac{A_n}{B_n}\right),$$

and we write

$$f = \frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \cdots. \tag{1.1.27}$$

If $\lim_{n\to\infty}\left(\frac{A_n}{B_n}\right)$ does not exist in $\hat{\mathbf{C}}$, (and $a_n \neq 0,\ 1 \leq n < \infty$), we say that (1.1.26) diverges.

SECTION 2

In this Chapter 16 of his Second Notebook [3][26][91], Ramanujan develops two closely related topics: q-series and theta-functions. The first 17 sections are devoted to q-series, while the later 22 sections constitute a very thorough development of the theory of theta functions. We emphasize that topic on q-series ends with a most beautiful formula called Ramanujan's ${}_1\psi_1$ summation [91, Ch. 16, Entry 17]

$$1+\sum_{n=1}^{\infty}\frac{(1/\alpha;q^2)_n(-\alpha q)^n}{(\beta q^2;q^2)_n}\,z^n+\sum_{n=1}^{\infty}\frac{(1/\beta;q^2)_n(-\beta q)^n}{(\alpha q^2;q^2)_n}\,z^{-n}$$

$$=\frac{(-qz;q^2)_\infty(-q/z;q^2)_\infty(q^2;q^2)_\infty(\alpha\beta q^2;q^2)_\infty}{(-\alpha qz;q^2)_\infty(-\beta q/z;q^2)_\infty(\alpha q^2;q^2)_\infty(\beta q^2;q^2)_\infty}\,, \qquad (1.2.1)$$

where $|q|<1,\ \ |\beta q|<|z|<1/|\alpha q|$.

Hardy [64, pp. 222-223] has described (1.2.1) as "a remarkable formula with many parameters". Hardy didn't supply a proof but indicated that a proof could be constructed from the q-binomial theorem. Among the mathematicians who contributed to the proofs of (1.2.1) are W. Hahn [61], M. Jackson [73], Andrews [8][13], Andrews and R. Askey [17], Askey [20], S. Corteel and J Lovejoy [49], N. J. Fine [56, pp. 19-20], M. E. H. Ismail [70], K. Mimachi [83], K. Venkatachaliengar [110], A. J. Yee [113]. Of these proofs, the proof given by Venkatachalienger is elementary and self-contained.

Soon after the ${}_1\psi_1$ summation, Ramanujan [91, Ch. 16] defines the theta function as

$$f(a,b) := 1+\sum_{n=1}^{\infty}(ab)^{n(n-1)/2}(a^n+b^n) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2}b^{n(n-1)/2}, \tag{1.2.2}$$

where $|ab|<1$. One of the most important identities in the classical theory of theta functions is the Jacobi's triple product identity

$$\sum_{n=-\infty}^{\infty} q^{n^2}z^n = (-qz;q^2)_\infty(-q/z;q^2)_\infty(q^2;q^2)_\infty\,, \qquad z\neq 0, \tag{1.2.3}$$

which is a special case (with $\alpha = 0 = \beta$) of (1.2.1). The first published proof of (1.2.3) was given by C. G. J. Jacobi [74]. However, C. F. Gauss [59, pp. 464] proved it earlier, since the identity was recorded in his posthumous manuscript. The proof using the theory of Jacobi's theta functions can be found in [46, pp. 67-71]. Additional proofs of (1.2.3) have since been given by Andrews [14], J. A. Ewell [55], D. Foata and G.-N. Han [57], R. P. Lewis [78], Hardy and E. M. Wright [65, pp. 282-283].

If we now set $qz = a$, $q/z = b$ in (1.2.3), we obtain

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{1.2.4}$$

Identity (1.2.4) is the Jacobi's triple product identity in Ramanujan's notation

[91, Ch. 16, Entry 19]. It follows from (1.2.2) and (1.2.4) that [91, Ch. 16, Entry 22]

$$\varphi(q) := f(q,q) = 1+2\sum_{n=1}^{\infty} q^{n^2} = \frac{(-q;q^2)_\infty (q^2;q^2)_\infty}{(q;q^2)_\infty(-q^2;q^2)_\infty}, \tag{1.2.5}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} \tag{1.2.6}$$

and

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty. \tag{1.2.7}$$

The following results can be found in [26, Entry 21, pp. 36] and [26, Entry 26(ii), pp. 39] respectively:

$$\log(-a;q)_\infty = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}a^n}{n(1-q^n)}, \qquad |q|<1, \qquad |a|<1. \tag{1.2.8}$$

$$f^3(-q) = \sum_{n=0}^{\infty} (-1)^{n-1}(2n+1) q^{n(n+1)/2}. \tag{1.2.9}$$

In his lost notebook [93], Ramanujan gave a beautiful reciprocity theorem:

$$\rho(a,b)-\rho(b,a) = \left(\frac{1}{b}-\frac{1}{a}\right)\frac{(aq/b)_\infty (bq/a)_\infty (q)_\infty}{(-aq)_\infty(-bq)_\infty}, \tag{1.2.10}$$

where

$$\rho(a,b) := 1 + \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2} a^n\, b^{-n-1}}{(-aq)_{n+1}}, \qquad a, b \neq -q^{-n}.$$

The first proof of (1.2.10) was given by Andrews [15] who used considerably heavy machinery. He then employed (1.2.10) in a later paper [10] to prove two beautiful entries from Ramanujan's lost notebook related to Euler's famous theorem on partitions. For other proofs one may see the works of Berndt, S. H. Chan, B. P. Yeap and Yee [29].

In Chapter II of the present thesis, we give six variable generalization of Ramanujan's reciprocity theorem (1.2.10).

SECTION 3

The theory of basic hypergeometric series embodies many well-known summation and transformation formulas. In fact these series arose initially in combinatorics and classical analysis and interacts with number theory, physics and representation theory of quantum lie algebras. For more details one may see the book of Andrews [12].

Formulas for basic bilateral hypergeometric series were not discovered until 1907 when J. Dougall [54], using residue calculus, derived summations for the bilateral ${}_2H_2$ and very-well-poised ${}_5H_5$ series. Ramanujan extended q-binomial theorem by finding a summation formula for the bilateral ${}_1\psi_1$ series (1.2.1) which was brought before the mathematical world by Hardy [64]. Later, W. N. Bailey [24][25] carried out systematic investigations on bilateral basic hypergeometric series. Further significant contributions were made by L. J. Slater [104][105], a student of Bailey.

Some of the interesting and explicitly used transformations of ${}_2\phi_1(a,b;c;q,z)$ in literature are

$$\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(q)_n(c)_n}z^n = \frac{(b)_\infty(az)_\infty}{(c)_\infty(z)_\infty}\sum_{n=0}^{\infty}\frac{(c/b)_n(z)_n}{(q)_n(az)_n}b^n, \tag{1.3.1}$$

$$= \frac{(c/b)_\infty(bz)_\infty}{(c)_\infty(z)_\infty}\sum_{n=0}^{\infty}\frac{(abz/c)_n(b)_n}{(q)_n(bz)_n}\left(\frac{c}{b}\right)^n, \tag{1.3.2}$$

$$= \frac{(abz/c)_\infty}{(z)_\infty}\sum_{n=0}^{\infty}\frac{(c/a)_n(c/b)_n}{(q)_n(c)_n}\left(\frac{abz}{c}\right)^n, \tag{1.3.3}$$

$$= \frac{(az)_\infty}{(z)_\infty} \sum_{n=0}^{\infty} \frac{(a)_n (c/b)_n}{(q)_n (c)_n (az)_n} (bz)^n , \tag{1.3.4}$$

$$= \frac{(az)_\infty (bz)_\infty}{(c)_\infty (z)_\infty} \sum_{n=0}^{\infty} \frac{(z)_n (abz/c)_n}{(q)_n (az)_n (bz)_n} c^n \tag{1.3.5}$$

and

$$= \frac{(abz/c)_\infty}{(bz/c)_\infty} \sum_{n=0}^{\infty} \frac{(a)_n (c/b)_n}{(q)_n (c)_n (cq/bz)_n} q^n . \tag{1.3.6}$$

Transformation (1.3.1) is due to E. Heine; (1.3.2) is the iterate of Heine's transformation and (1.3.3) is the q-analogue of Euler's transformation. For these equations one may see Andrews, Askey and R. Roy [18, pp. 521-524], G. Gasper and M. Rahman [58, pp. 9-10]. Equation (1.3.4) is Jackson's q-analogue of Pfaff-Kummer transformation, which can also be found in Andrews, Askey and Roy [18, Eq.(10.10.12)], Gasper and Rahman [58, Eq. (1.5.4)]. Equation (1.3.5) appears in D. B. Sears [96] as the function $Y(1,6)$ in Table II.A. Equation (1.3.6) is due to F. H. Jackson and can be found in Gasper and Rahman [58, pp. 241, (III.5)].

In Chapter III of the present thesis, we obtain transformation formulas for ${}_2\psi_2$ basic bilateral series, and deduce therefrom some interesting eta-function identities.

SECTION 4

Several mathematicians such as L. Euler, C. F. Gauss, E. Heine, F. H. Jackson, L. J. Rogers and S. Ramanujan played pivotal role in the establishment of q-series. Among all, Ramanujan's contribution is undoubtedly more to the development of q-series either before or after his time. One of his most important theorem, stated without proof in [93, pp. 54] (see also [9, pp. 337]) is the circular summation formula:

Theorem 1.4.1. *For any positive integer $n \geq 2$, if*

$$U_r = a^{r(r+1)/2n} b^{r(r-1)/2n} \text{ and } V_r = a^{r(r-1)/2n} b^{r(r+1)/2n},$$

then

$$\sum_{r=0}^{n-1} U_r^n f^n \left(\frac{U_{n+r}}{U_r}, \frac{V_{n-r}}{U_r} \right) = f(a,b) F_n(ab),$$

where

$$F_n(q) = 1 + 2nq^{(n-1)/2} + \cdots, n > 3.$$

Ramanujan's circular summation can be restated in terms of classical theta function $\theta_3(z|\tau)$ defined by (1.1.12).

Theorem 1.4.2. *For any positive integer $n \geq 2$,*

$$\sum_{k=0}^{n-1} q^{k^2} e^{2kiz} \theta_3^n (z + k\pi\tau | n\tau) = F_n(\tau) \theta_3(z|\tau),$$

where

$$F_n(\tau) = 1 + 2nq^{n-1} + \cdots.$$

The first proof of Theorem 1.4.1 was given by S. S. Rangachari in [94], by using Mumford's theory of theta functions [84] and few results on weight polynomials in coding theory. Later, Son [107] gave much elementary proof of Theorem 1.4.1. Recently, Xu [114] has given a very elementary and neat proof of the circular summation formula. In Chapter IV of this thesis, we obtain new Ramanujan's summation summation for four theta functions employing elliptic functions.

SECTION 5

Ramanujan, a pioneer in the theory of continued fractions has recorded scores of results in his notebooks. In fact Chapter 12 of his second notebook [91] is devoted to continued fractions. One may see traces of continued fractions in Chapter 16 and about 60 results on continued fractions in the unorganized pages of second and third notebooks [91]. Ramanujan in his historic letter to Hardy [90, pp. xxvii, xxviii], [28, pp. 21-30, 53-62] communicated the following celebrated continued fraction, subsequently named Rogers-Ramanujan continued fraction,

$$R(q) := \frac{1}{1} \, {}_{+} \, \frac{q}{1} \, {}_{+} \, \frac{q^2}{1} \, {}_{+\cdots+} \, \frac{q^n}{1} \, {}_{+\cdots}, \qquad |q| < 1,$$

along with several of its evaluations. The continued fraction $R(q)$ is best known for its connection with the famous Rogers-Ramanujan identities

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty},$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty},$$

given by

$$R(q) = \frac{\displaystyle\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n}}{\displaystyle\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q)_n}}. \tag{1.5.1}$$

The Rogers-Ramanujan identities were first proved by L. J. Rogers [95] in his 1894 paper that was completely ignored, but became famous after these were rediscovered and published by Ramanujan [92]. The function $R(q)$ possesses a very beautiful and extensive theory, almost all of which was found by Ramanujan. For proofs of many of these theorems one may see papers by Berndt, S.-S.Huang, J. Sohn and S. H. Son [27] and S. Y. Kang [76] [77]. In fact, the first five chapters of the first volume by Andrews and Berndt [19] on Ramanujan's lost notebook are devoted to the Rogers-Ramanujan continued fraction.

Ramanujan [91, Ch. 16, Entries 15, 16] gave more general continued fractions that contains (1.5.1) as particular case, namely

$$\frac{1}{1}\,+\,\frac{aq}{1}\,+\,\frac{aq^2}{1}\,+\cdots+\,\frac{aq^n}{1}\,+\cdots = \frac{\displaystyle\sum_{n=0}^{\infty}\frac{a^n q^{n^2+n}}{(q)_n}}{\displaystyle\sum_{n=0}^{\infty}\frac{a^n q^{n^2}}{(q)_n}}, \qquad |q|<1 \tag{1.5.2}$$

and

$$\frac{1}{1}\,+\,\frac{aq}{1+bq}\,+\,\frac{aq^2}{1+bq^2}\,+\cdots+\,\frac{aq^n}{1+bq^n}\,+\cdots = \frac{\displaystyle\sum_{n=0}^{\infty}\frac{a^n q^{n^2+n}}{(q)_n(-bq)_n}}{\displaystyle\sum_{n=0}^{\infty}\frac{a^n q^{n^2}}{(q)_n(-bq)_n}}, \qquad |q|<1. \tag{1.5.3}$$

Identity (1.5.2) has been established earlier by Rogers [95] and then later by Watson [112]. Proof of (1.5.3) can be found in the works of V. Ramamani [87] and Andrews [16]. Ramanujan in his lost notebook [93], also recorded several continued fraction identities equivalent to and more general than (1.5.3). Some of these are

$$\frac{\displaystyle\sum_{n=0}^{\infty}\frac{\lambda^n q^{n^2+n}}{(q)_n(-bq)_n}}{\displaystyle\sum_{n=0}^{\infty}\frac{\lambda^n q^{n^2}}{(q)_n(-bq)_n}} = \frac{1}{1}\,+\,\frac{\lambda q}{1}\,+\,\frac{bq+\lambda q^2}{1}\,+\cdots+\,\frac{\lambda q^{2n+1}}{1}\,+\,\frac{bq^{n+1}+\lambda q^{2n+2}}{1}\,+\cdots,$$

$$= \frac{1}{1}\,+\,\frac{\lambda q}{1+bq}\,+\,\frac{\lambda q^2}{1+bq^2}\,+\cdots+\,\frac{\lambda q^n}{1+bq^n}\,+\cdots,$$

$$= \frac{1}{1-b}\,+\,\frac{b+\lambda q}{1-b}\,+\,\frac{b+\lambda q^2}{1-b}\,+\cdots+\,\frac{b+\lambda q^n}{1-b}\,+\cdots.$$

Some more of the elegant and fascinating continued fraction identities mentioned in his lost notebook [93] are

$$\frac{G(aq,\lambda q,b,q)}{G(a,\lambda,b,q)}$$

$$= \frac{1}{1}\,+\,\frac{aq+\lambda q}{1}\,+\,\frac{bq+\lambda q^2}{1}\,+\cdots+\,\frac{aq^{n+1}+\lambda q^{2n+1}}{1}\,+\,\frac{bq^{n+1}+\lambda q^{2n+2}}{1}\,+\cdots,$$

$$= \frac{1}{1}\,+\,\frac{aq+\lambda q}{1-aq+bq}\,+\cdots+\,\frac{aq+\lambda q^n}{1-aq+bq^n}\,+\cdots,$$

$$= \frac{1}{1+aq}\,+\,\frac{\lambda q-abq^2}{1+(aq+b)}\,+\cdots+\,\frac{\lambda q^n-abq^{2n}}{1+q^n(aq+b)}\,+\cdots,$$

$$= \frac{1}{a+c}\,-\,\frac{ab}{a+b+cq}\,-\cdots-\,\frac{ab}{a+b+cq^n}\,-\cdots,$$

$$= \frac{1}{c-b+c} + \frac{bc}{c-b+(a/q)} + \cdots + \frac{bc}{c-b+(a/q^n)} + \cdots,$$

where

$$G(a,\lambda,b,q) = \sum_{n=0}^{\infty} \frac{q^{(n^2+n)/2}(-\lambda/a)_n a^n}{(q)_n(-bq)_n} \cdot$$

This part of Ramanujan's work has been treated and developed consequently by several authors including Andrews [16], Hirschhorn [69], L. Carlitz [40], B. Gordon [60], W. A. Al-Salam and Ismail [7], K. G. Ramanathan [88] [89], R. Y. Denis [50] [51] [52], Bhargava and Adiga [32] [33], Bhargava, Adiga and Somashekara [34] [35], Adiga and Somashekara [6], A. Verma, Denis and K. Srinivasa Rao [111], S. N. Singh [102] and N. A Bhagirathi [30].

Motivated by these works in Chapter V, we derive several identities involving the Ramanujan continued Fraction $A(q)$ given by,

$$A(q) := \frac{1}{1-q^2} + \frac{q^2(1+q^2)^2}{1-q^6} + \frac{q^4(1+q^4)^2}{1-q^{10}} + \frac{q^6(1+q^6)^2}{1-q^{14}} + \cdots, \quad |q| < 1.$$

CHAPTER II

A SIX-VARIABLE GENERALIZATION OF RECIPROCITY THEOREM

CHAPTER II

A SIX-VARIABLE GENERALIZATION OF RECIPROCITY THEOREM

2.1 INTRODUCTION

In his lost notebook [93], Ramanujan stated several q-series identities and in [15], Andrews explicated these Ramanujan's discoveries in great detail. The same paper [15] also includes the first proof for the two-variable reciprocity theorem of Ramanujan [93, pp. 40].

Theorem 2.1.1. *If* $a, b \neq -q^{-n}, n \in \mathbb{Z}^+$ *and* $|q| < 1$, *then*

$$\rho(a,b) - \rho(b,a) = \left(\frac{1}{b} - \frac{1}{a}\right) \frac{(aq/b)_\infty (bq/a)_\infty (q)_\infty}{(-aq)_\infty (-bq)_\infty}. \tag{2.1.1}$$

where

$$\rho(a,b) = \left(1 + \frac{1}{b}\right) \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2} a^n b^{-n}}{(-aq)_n}.$$

In the recent past several mathematician to mention a few B. C. Berndt, S. H. Chan, B. P. Yeap and A. J. Yee [29], Adiga and Anitha [2], Adiga and Guruprasad [4], Somashekara and Mamta [106], Kang [75] have contributed to the proofs of (2.1.1) and in the process interesting generalization of (2.1.1) have been obtained.

In [75], Kang proved the following three variable generalization of (2.1.1).

Theorem 2.1.2. *If* $|c| < |a| < 1$ *and* $|c| < |b| < 1$, *then*

$$\rho(a,b,c) - \rho(b,a,c) = \left(\frac{1}{b} - \frac{1}{a}\right) \frac{(c)_\infty (aq/b)_\infty (bq/a)_\infty (q)_\infty}{(-c/a)_\infty (-c/b)_\infty (-aq)_\infty (-bq)_\infty}. \qquad (2.1.2)$$

where

$$\rho(a,b,c) = \left(1 + \frac{1}{b}\right) \sum_{n=0}^{\infty} \frac{(c)_n (-1)^n q^{n(n+1)/2} a^n b^{-n}}{(-aq)_n (-c/b)_{n+1}}, \qquad a, \frac{c}{b} \neq -q^{-n}.$$

A six-variable generalization of Ramanujan's reciprocity theorem (2.1.1) has been recently given by Ma [81] using Shukla's very well posed ${}_8\psi_8$ summation formula [101, (4.1)] and Watson's transformation formula [58, 2.5.1, pp. 43]

Theorem 2.1.3. *Let* a, b, c/x, d/x, $e/x (x = aq, bq)$ *be any complex numbers other than of the form* $-q^{-n}$, $n \geq 1$ *and* $w/y (y = a, b)$ *be not of the form* $-q^m$, $m \in \mathbb{Z}$, $|cde| < |abq^2|$, *there holds*

$$\rho(a,b,c,d,e,w) - \rho(b,a,c,d,e,w) = \Xi_0(a,b,c,d,e) \times \left\{ \frac{(wq-ab)(w-1)(cde-abq^2) + abw(c-q)(d-q)(e-q)}{cde - abq^2} \right\}, \qquad (2.1.3)$$

where

$$\Xi_0(a,b,c,d,e) = \left(\frac{1}{b} - \frac{1}{a}\right) \times \frac{(q, aq/b, bq/a, c, d, e, cd/(ab), ce/(ab), de/(ab); q)_\infty}{(-aq, -bq, -c/a, -c/b, -d/a, -d/b, -e/a, -e/b, cde/(abq); q)_\infty}.$$

and

$$\rho(a,b,c,d,e,w) = wb\sum_{k=0}^{\infty}\left(1-\frac{aq^{2k+1}}{b}\right)\frac{(-1/b,-a/w,-wq/b;q)_{k+1}}{(-aq,-wq/b,-a/w;q)_k}$$
$$\times\frac{(-aq/c,-aq/d,-aq/e;q)_k}{(-c/b,-d/b,-e/b;q)_{k+1}}\left(\frac{cde}{abq^2}\right)^k.$$

In Section 2.2, we derive a six variable generalization of reciprocity theorem which is a new and different from identity (2.1.3).

In Section 2.3, we deduce beta, gamma identities and eta function identities as application of our main identity.

2.2 MAIN RESULT

In this Section, we employ the technique of parametric augmentation [80] on three variable reciprocity theorem (2.1.2) and derive the following six variable generalization of reciprocity theorem:

Theorem 2.2.1. *For $ab \neq 0$, we have*

$$\rho(a,b,c,d,e,f) - \rho(b,a,c,d,e,f) = \left(\frac{1}{b} - \frac{1}{a}\right) \times \frac{(abcd, ef/(ab), cf, df, ce, de, aq/b, bq/a, q; q)_\infty}{(cdef/q, acq, bcq, ad, bd, fq/a, fq/b, e/a, e/b; q)_\infty}, \tag{2.2.1}$$

where

$$\rho(a,b,c,d,e,f) = \left(\frac{1}{b} - c\right)\left(1 - \frac{f}{a}\right)\sum_{n=0}^{\infty} \frac{(cf)_n(-1)^n q^{n(n+1)/2}(abcd)_n(ef/(ab))_n}{(acq)_n(ad)_{n+1}(fq/b)_n(e/b)_{n+1}} \left(\frac{a}{b}\right)^n.$$

ac, bc, f/a, $f/b \neq q^{-n}$, ad, bd, e/a, $e/b \neq q^{-n}$ and $cdef \neq q^{-n+1}$, $n \in \mathbb{Z}^+$.

Proof: Replacing a by $-ad$, b by $-ae$ and c by af in (2.1.2) and multiplying the resulting identity by $adef/(1-ad)(1-ae)$, we have

$$d\sum_{n=0}^{\infty} \frac{(af)_n(-1)^n q^{n(n+1)/2}}{(ad)_{n+1}(f/e)_{n+1}} \left(\frac{d}{e}\right)^n - e\sum_{n=0}^{\infty} \frac{(af)_n(-1)^n q^{n(n+1)/2}}{(ae)_{n+1}(f/d)_{n+1}} \left(\frac{e}{d}\right)^n = d\frac{(af, dq/e, e/d, q; q)_\infty}{(f/e, f/d, ad, ae; q)_\infty}. \tag{2.2.2}$$

Now (2.2.2) can be written as

$$
\begin{aligned}
&d\sum_{n=0}^{\infty}(af)_n(-1)^n q^{n(n+1)/2}(ae, adq^{n+1};q)_\infty\left(\frac{f}{d},\frac{f}{e}q^{n+1};q\right)_\infty\left(\frac{d}{e}\right)^n \\
&-e\sum_{n=0}^{\infty}(af)_n(-1)^n q^{n(n+1)/2}(ad, aeq^{n+1};q)_\infty\left(\frac{f}{e},\frac{f}{d}q^{n+1};q\right)_\infty\left(\frac{e}{d}\right)^n \\
&= d(af, dq/e, e/d, q;q)_\infty. \qquad (2.2.3)
\end{aligned}
$$

Applying $E(b\theta)$ to both sides of (2.2.3) with respect to the variable a, we obtain

$$
\begin{aligned}
&d\sum_{n=0}^{\infty}(af)_n(-1)^n q^{n(n+1)/2}\frac{(ae, adq^{n+1}, be, bdq^{n+1};q)_\infty}{(abdeq^n;q)_\infty}\left(\frac{f}{d},\frac{f}{e}q^{n+1};q\right)_\infty\left(\frac{d}{e}\right)^n \\
&-e\sum_{n=0}^{\infty}(af)_n(-1)^n q^{n(n+1)/2}\frac{(ad, aeq^{n+1}, bd, beq^{n+1};q)_\infty}{(abdeq^n;q)_\infty}\left(\frac{f}{e},\frac{f}{d}q^{n+1};q\right)_\infty\left(\frac{e}{d}\right)^n \\
&= d(af, bf, dq/e, e/d, q;q)_\infty. \qquad (2.2.4)
\end{aligned}
$$

Again applying $E(c\theta)$ to both sides of (2.2.4) with respect to variable f, we obtain

$$
\begin{aligned}
&d\sum_{n=0}^{\infty}(af)_n(-1)^n q^{n(n+1)/2}\frac{(ae, adq^{n+1}, be, bdq^{n+1};q)_\infty}{(abdeq^n;q)_\infty}\frac{\left(\frac{f}{d},\frac{f}{e}q^{n+1},\frac{c}{d},\frac{c}{e}q^{n+1};q\right)_\infty}{(cf/(de)q^n;q)_\infty}\left(\frac{d}{e}\right)^n \\
&-e\sum_{n=0}^{\infty}(af)_n(-1)^n q^{n(n+1)/2}\frac{(ad, aeq^{n+1}, bd, beq^{n+1};q)_\infty}{(abdeq^n;q)_\infty}\frac{\left(\frac{f}{e},\frac{f}{d}q^{n+1},\frac{c}{e},\frac{c}{d}q^{n+1};q\right)_\infty}{(cf/(de)q^n;q)_\infty}\left(\frac{e}{d}\right)^n \\
&= d\frac{(af, bf, ac, bc, dq/e, e/d, q;q)_\infty}{(abcf/q;q)_\infty}. \qquad (2.2.5)
\end{aligned}
$$

Multiplying both sides of (2.2.5) by

$$
\frac{(abde, cf/(de))_\infty}{(ad, ae, bd, be, f/d, f/e, c/d, c/e)_\infty},
$$

we obtain

$$
\begin{aligned}
&d\sum_{n=0}^{\infty}\frac{(af)_n(-1)^n q^{n(n+1)/2}(abde)_n(cf/(de))_n}{(ad,bd,f/e,c/e)_{n+1}}\left(\frac{d}{e}\right)^n\\
&-e\sum_{n=0}^{\infty}\frac{(af)_n(-1)^n q^{n(n+1)/2}(abde)_n(cf/(de))_n}{(ae,be,f/d,c/d)_{n+1}}\left(\frac{e}{d}\right)^n\\
&=d\frac{(abde,cf/(de),af,bf,ac,bc,dq/e,e/d,q;q)_\infty}{(abcf/q,ad,ae,bd,be,f/d,f/e,c/d,c/e;q)_\infty}. \qquad (2.2.6)
\end{aligned}
$$

(2.2.6) can be written as

$$
\begin{aligned}
&d(1-ae)(1-\frac{f}{d})\sum_{n=0}^{\infty}\frac{(af)_n(-1)^n q^{n(n+1)/2}(abde)_n(cf/(de))_n}{(adq)_n(bd)_{n+1}(fq/e)_n(c/e)_{n+1}}\left(\frac{d}{e}\right)^n\\
&-e(1-ad)(1-\frac{f}{e})\sum_{n=0}^{\infty}\frac{(af)_n(-1)^n q^{n(n+1)/2}(abde)_n(cf/(de))_n}{(aeq)^n(be)_{n+1}(fq/d)_n(c/d)_{n+1}}\left(\frac{e}{d}\right)^n\\
&=d\frac{(abde,cf/(de),af,bf,ac,bc,dq/e,e/d,q;q)_\infty}{(abcf/q,adq,aeq,bd,be,fq/d,fq/e,c/d,c/e;q)_\infty}. \qquad (2.2.7)
\end{aligned}
$$

Now by making substitutions $a\mapsto c$, $b\mapsto d$, $c\mapsto e$, $d\mapsto a$, $e\mapsto b$ in (2.2.7), we obtain (2.2.1). □

Corollary 2.2.2. *(Two variable reciprocity theorem)*

Substituting $c=-1$, $d=e=f=0$ *in* (2.2.1), *we easily obtain* (2.1.1).

Corollary 2.2.3. *(Jacobi triple product)*

Substituting $a=1$, $b=-z$, $c=d=e=f=0$ *in* (2.2.1), *we obtain (1.2.3).*

The identity (2.2.1) is the generalization of Ramanujan's reciprocity theorem. Again it turn out to be a generalization of Jacobi triple product (1.2.3).

2.3 CONNECTION WITH THE q-GAMMA AND q-BETA FUNCTIONS

In this Section, we deduce q-gamma, q-beta and eta function identities from (2.2.1).

1. For $0 < q < 1$ and $0 < x < 1$, we have

$$\begin{aligned}
&\frac{\beta_q(x,x)\beta_q(3x,2x)}{\beta_q(x,6x)\beta_q(x,x+1)} \\
&= q^x(q^{2x}-1)\sum_{n=0}^{\infty}\frac{(q^{4x})_n(-1)^n q^{n(n+1)/2}(q^{6x})_n(q^{x+1})_n}{(q^{3x+1})_n(q^{4x})_{n+1}(q^{2x+1})_n(q)_{n+1}}q^{xn} \\
&- (q^{3x}-1)(q^x+1)\sum_{n=0}^{\infty}\frac{(q^{4x})_n(-1)^n q^{n(n+1)/2}(q^{6x})_n(q^{x+1})_n}{(q^{2x+1})_n(q^{3x})_{n+1}(q^{x+1})_n(q^{1-x})_{n+1}}q^{-xn}. \qquad (2.3.1)
\end{aligned}$$

Proof: Putting $a = q^{2x}$, $b = q^x$, $c = q^x$, $d = q^{2x}$, $e = q^{x+1}$ and $f = q^{3x}$ in (2.2.1), we obtain

$$\begin{aligned}
&\frac{(q^{6x})_\infty(q^{x+1})_\infty(q^{5x})_\infty}{(q^{7x})_\infty(q^{2x+1})_\infty(q^{3x})_\infty} \\
&= q^x(q^{2x}-1)\sum_{n=0}^{\infty}\frac{(q^{4x})_n(-1)^n q^{n(n+1)/2}(q^{6x})_n(q^{x+1})_n}{(q^{3x+1})_n(q^{4x})_{n+1}(q^{2x+1})_n(q)_{n+1}}q^{xn} \\
&- (q^{3x}-1)(q^x+1)\sum_{n=0}^{\infty}\frac{(q^{4x})_n(-1)^n q^{n(n+1)/2}(q^{6x})_n(q^{x+1})_n}{(q^{2x+1})_n(q^{3x})_{n+1}(q^{x+1})_n(q^{1-x})_{n+1}}q^{-xn}. \qquad (2.3.2)
\end{aligned}$$

Employing (1.1.7), we obtain

$$\begin{aligned}
&\frac{\Gamma_q(7x)\Gamma_q(2x+1)\Gamma_q(3x)}{\Gamma_q(6x)\Gamma_q(x+1)\Gamma_q(5x)} \\
&= q^x(q^{2x}-1)\sum_{n=0}^{\infty}\frac{(q^{4x})_n(-1)^n q^{n(n+1)/2}(q^{6x})_n(q^{x+1})_n}{(q^{3x+1})_n(q^{4x})_{n+1}(q^{2x+1})_n(q)_{n+1}}q^{xn}
\end{aligned}$$

$$-(q^{3x}-1)(q^x+1)\sum_{n=0}^{\infty}\frac{(q^{4x})_n(-1)^n q^{n(n+1)/2}(q^{6x})_n(q^{x+1})_n}{(q^{2x+1})_n(q^{3x})_{n+1}(q^{x+1})_n(q^{1-x})_{n+1}}q^{-xn}. \tag{2.3.3}$$

Again employing (1.1.9), we obtain (2.3.1). □

2. For $0 < q < 1$ and $0 < x < 1$, we have

$$\frac{\beta_q(2x,2x+1)\beta_q(3x+1,3x+1)}{\beta_q(2x,4x+1)\beta_q(5x,x+2)}$$
$$=\frac{(1-q^{3x})^2}{1-q^x}\sum_{n=0}^{\infty}\frac{(q^{3x})_n(-1)^n q^{n(n+1)/2}(q^{4x+1})_n}{(q^{2x+1})_n(q^{x+1})_{n+1}(q^{q^{2x+1}})_n(q^{1-x})_{n+1}}q^{-xn}$$
$$-\frac{q^x(1-q^{2x})^2}{1-q^x}\sum_{n=0}^{\infty}\frac{(q^{3x})_n(-1)^n q^{n(n+1)/2}(q^{4x+1})_n}{(q^{3x+1})_n(q^{2x+1})_{n+1}(q^{q^{3x+1}})_n(q)_{n+1}}q^{xn}. \tag{2.3.4}$$

Proof: Putting $a = q^x$, $b = q^{2x}$, $c = q^x$, $d = q$, $e = q^{x+1}$ and $f = q^{4x}$ in (2.2.1), we obtain

$$\frac{(q^{4x+1})_\infty^2(q^{5x})_\infty(q^{x+2})_\infty}{(q^{3x+1})_\infty^2(q^{6x+1})_\infty(q^{2x+1})_\infty}$$
$$=\frac{(1-q^{3x})^2}{1-q^x}\sum_{n=0}^{\infty}\frac{(q^{3x})_n(-1)^n q^{n(n+1)/2}(q^{4x+1})_n}{(q^{2x+1})_n(q^{x+1})_{n+1}(q^{q^{2x+1}})_n(q^{1-x})_{n+1}}q^{-xn}$$
$$-\frac{q^x(1-q^{2x})^2}{1-q^x}\sum_{n=0}^{\infty}\frac{(q^{3x})_n(-1)^n q^{n(n+1)/2}(q^{4x+1})_n}{(q^{3x+1})_n(q^{2x+1})_{n+1}(q^{q^{3x+1}})_n(q)_{n+1}}q^{xn}. \tag{2.3.5}$$

Employing (1.1.7) in (2.3.5), we obtain

$$\frac{\Gamma_q(3x+1)^2\Gamma_q(6x+1)\Gamma_q(2x+1)}{\Gamma_q(4x+1)^2\Gamma_q(5x)\Gamma_q(x+2)}$$
$$=\frac{(1-q^{3x})^2}{1-q^x}\sum_{n=0}^{\infty}\frac{(q^{3x})_n(-1)^n q^{n(n+1)/2}(q^{4x+1})_n}{(q^{2x+1})_n(q^{x+1})_{n+1}(q^{q^{2x+1}})_n(q^{1-x})_{n+1}}q^{-xn}$$
$$-\frac{q^x(1-q^{2x})^2}{1-q^x}\sum_{n=0}^{\infty}\frac{(q^{3x})_n(-1)^n q^{n(n+1)/2}(q^{4x+1})_n}{(q^{3x+1})_n(q^{2x+1})_{n+1}(q^{q^{3x+1}})_n(q)_{n+1}}q^{xn}. \tag{2.3.6}$$

Again employing (1.1.9) and with some simplification, we obtain (2.3.4). □

3. For $0 < q < 1$ and $0 < x < y < 1$, we have

$$\frac{\beta_q(2x+1,4y)\beta_q(x+y,y-x+1)}{\beta_q(2y+1,2x+2y)\beta_q(3y-x,x-y+1)}$$
$$= (1-q^{x+y})(1+q^{y-x})\sum_{n=0}^{\infty}\frac{(q^{2y})_n(-1)^n q^{n(n+1)/2}(q^{2x+2y})_n(q^{2y-2x+1})_n}{(q^{2x+1})_n(q^{x+y})_{n+1}(q^{y-x+1})_n(q)_{n+1}}q^{(x-y)n}$$
$$- q^y(1-q^{2x})\sum_{n=0}^{\infty}\frac{(q^{2y})_n(-1)^n q^{n(n+1)/2}(q^{2x+2y})_n(q^{2y-2x+1})_n}{(q^{x+y+1})_n(q^{2y})_{n+1}(q^{2y-2x+1})_n(q^{y-x+1})_{n+1}}q^{(y-x)n}. \quad (2.3.7)$$

Proof: Putting $a = q^x$, $b = q^y$, $c = q^x$, $d = q^y$, $e = q^{y+1}$ and $f = q^{2y-x}$ in (2.2.1), we obtain

$$\frac{(q^{2x+2y})_\infty(q^{3y-x})_\infty(q^{2y+1})_\infty(q^{x-y+1})_\infty}{(q^{4y})_\infty(q^{x+y})_\infty(q^{2x+1})_\infty(q^{y-x+1})_\infty}$$
$$= (1-q^{x+y})(1+q^{y-x})\sum_{n=0}^{\infty}\frac{(q^{2y})_n(-1)^n q^{n(n+1)/2}(q^{2x+2y})_n(q^{2y-2x+1})_n}{(q^{2x+1})_n(q^{x+y})_{n+1}(q^{y-x+1})_n(q)_{n+1}}q^{(x-y)n}$$
$$- q^y(1-q^{2x})\sum_{n=0}^{\infty}\frac{(q^{2y})_n(-1)^n q^{n(n+1)/2}(q^{2x+2y})_n(q^{2y-2x+1})_n}{(q^{x+y+1})_n(q^{2y})_{n+1}(q^{2y-2x+1})_n(q^{y-x+1})_{n+1}}q^{(y-x)n}. \quad (2.3.8)$$

Applying (1.1.7) in (2.3.8), we obtain

$$\frac{\Gamma_q(4y)\Gamma_q(2x+1)\Gamma_q(x+y)\Gamma_q(y-x+1)}{\Gamma_q(2x+2y)\Gamma_q(2y+1)\Gamma_q(3y-x)\Gamma_q(x-y+1)}$$
$$= (1-q^{x+y})(1+q^{y-x})\sum_{n=0}^{\infty}\frac{(q^{2y})_n(-1)^n q^{n(n+1)/2}(q^{2x+2y})_n(q^{2y-2x+1})_n}{(q^{2x+1})_n(q^{x+y})_{n+1}(q^{y-x+1})_n(q)_{n+1}}q^{(x-y)n}$$
$$- q^y(1-q^{2x})\sum_{n=0}^{\infty}\frac{(q^{2y})_n(-1)^n q^{n(n+1)/2}(q^{2x+2y})_n(q^{2y-2x+1})_n}{(q^{x+y+1})_n(q^{2y})_{n+1}(q^{2y-2x+1})_n(q^{y-x+1})_{n+1}}q^{(y-x)n}. \quad (2.3.9)$$

Again applying (1.1.9) in (2.3.9), we easily obtain (2.3.7). □

4. For $0 < q < 1$, we have

$$\frac{\eta^2(2\tau)}{\eta(\tau)\tau(3\eta)} = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q^2)_n}. \tag{2.3.10}$$

Proof: Putting $a = -1$, $b = 1$, $c = d = e = f = q$ in (2.2.1), we obtain

$$\frac{(q^2)_\infty^2}{(q)_\infty(q^3)_\infty} = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q^2)_n}. \tag{2.3.11}$$

Employing (1.1.6) in (2.3.11), we obtain (2.3.10). □

CHAPTER III

A BILATERAL BASIC HYPERGEOMETRIC TRANSFORMATION FORMULA

CHAPTER III

A BILATERAL BASIC HYPERGEOMETRIC TRANSFORMATION FORMULA

3.1 INTRODUCTION

The "basic hypergeometric series" or "Eulerian series", founded by Euler and studied by Heine [67] first systematically. Other mathematicians such as Gauss, Jacobi, Bailey [23] contributed in their own way to this field. Similarly, plentiful references of basic hypergeometric series can be found in [22], Hardy and Wright [66], MacMohan [82]. Later, Hahn [61] [62] [63] and Sears [96] [97] [98] developed the theory systematically. For complete references and detail expositions of this theory, one may refer [11] [12]. Heine [68], F. H. Jackson [72], R. Y. Denis [53], S. P. Singh [103] have developed the theory of transformations of hypergeometric series and basic hypergeometric series which are extremely useful in the theory of partitions.

The most fundamental summation formula of basic hypergeometric series is due to Cauchy [41, pp. 45], the q- binomial theorem:

$$\sum_{k=0}^{\infty} \frac{(a)_k}{(q)_k} z^k = \frac{(az)_\infty}{(z)_\infty}, \qquad |z| < 1. \tag{3.1.1}$$

The Heine's transformation for ${}_2\phi_1$ series as in Gasper and Rahman [58, Eq. III.1, III.3, pp. 359]:

$$ {}_2\phi_1\left[\begin{matrix} a, b \\ c \end{matrix} ; q, z\right] = \frac{(b, z; q)_\infty}{(c, z; q)_\infty} {}_2\phi_1\left[\begin{matrix} c/b, z \\ az \end{matrix} ; q, b\right], \tag{3.1.2} $$

$$ = \frac{(abz/c; q)_\infty}{(z; q)_\infty} {}_2\phi_1\left[\begin{matrix} c/a, c/b \\ c \end{matrix} ; q, \frac{abz}{c}\right]. \tag{3.1.3} $$

also play a prominent role in q-series.

S. Bhargava and C. Adiga [31] proved by method of Ismail [70] the following ${}_2\psi_2$ summation formula:

$$ {}_2\psi_2\left[\begin{matrix} q/a, b \\ d, bq \end{matrix} ; q, a\right] = \frac{(d/b)_\infty (ab)_\infty (q)_\infty^2}{(q/b)_\infty (d)_\infty (a)_\infty (bq)_\infty}, \qquad |a| < 1, |d| < 1. \tag{3.1.4} $$

In Section 3.2 of this Chapter, we obtain transformation formula for ${}_2\psi_2$ bilateral series. In Section 3.3, we obtain Fourier series related to theta functions.

3.2 NEW TRANSFORMATION FORMULA FOR ${}_2\psi_2$

Theorem 3.2.1. *For* $\left|\dfrac{abzq}{cd}\right| < 1$

$$
\begin{aligned}
&{}_2\psi_2\left[\begin{matrix} a, b \\ c, d \end{matrix}; q, z\right] \\
&\quad = \frac{(q, c/a, c/b, q/d, abz/d, dq/abz; q)_\infty}{(c, q/a, q/b, c/d, z, cd/abz; q)_\infty} {}_2\phi_1\left[\begin{matrix} d/a, d/b \\ dq/c \end{matrix}; q, \frac{abzq}{cd}\right] \\
&\quad + \frac{(q, d/a, d/b, q/c, abz/c, cq/abz; q)_\infty}{(d, q/a, q/b, d/c, z, cd/abz; q)_\infty} {}_2\phi_1\left[\begin{matrix} c/a, c/b \\ cq/d \end{matrix}; q, \frac{abzq}{cd}\right].
\end{aligned}
\tag{3.2.1}
$$

Proof: As in [58, (III.9), pp. 359] and [58, (III.33), pp. 364], we have

$$
{}_3\phi_2\left[\begin{matrix} a, b, c \\ d, e \end{matrix}; q, \frac{de}{abc}\right] = \frac{(e/a, de/bc; q)_\infty}{(e, de/abc; q)_\infty} {}_3\phi_2\left[\begin{matrix} a, d/b, d/c \\ d, de/bc \end{matrix}; q, \frac{e}{a}\right], \tag{3.2.2}
$$

$|de/abc| < 1$, $|e/a| < 1$ and

$$
\begin{aligned}
&{}_3\phi_2\left[\begin{matrix} a, b, c \\ d, e \end{matrix}; q, \frac{de}{abc}\right] \\
&= \frac{(e/b, e/c, cq/a, q/d; q)_\infty}{(e, cq/d, q/a, e/bc; q)_\infty} {}_3\phi_2\left[\begin{matrix} c, d/a, cq/e \\ cq/a, bcq/e \end{matrix}; q, \frac{bq}{d}\right]
\end{aligned}
$$

$$-\frac{(q/d,eq/d,b,c,d/a,de/bcq,bcq^2/de;q)_\infty}{(d/q,e,bq/d,cq/d,q/a,e/bc,bcq/e;q)_\infty}{}_3\phi_2\left[\begin{matrix}aq/d,bq/d,cq/d\\ q^2/d,eq/d\end{matrix};q,\frac{de}{abc}\right], \tag{3.2.3}$$

$|de/abc| < 1$, $|bq/d| < 1$, respectively.

Set b to d/b, c to d/c and e to de/bc in (3.2.3) and then substituting the resulting identity in the right hand side of (3.2.2), we obtain

$$\begin{aligned}&{}_3\phi_2\left[\begin{matrix}a,b,c\\ d,e\end{matrix};q,\frac{de}{abc}\right]\\ &=\frac{(e/a,e/b,e/c,q/d,,dq/ac;q)_\infty}{(e,q/a,q/c,e/d,de/abc;q)_\infty}{}_3\phi_2\left[\begin{matrix}d/a,d/c,bq/e\\ dq/ac,dq/e\end{matrix};q,\frac{q}{b}\right]\\ &-\frac{(d/a,d/b,d/c,q/d,e/a,qe/bc,e/q,q^2/e;q)_\infty}{(e,d/q,q/a,q/b,q/c,e/d,dq/e,de/abc;q)_\infty}{}_3\phi_2\left[\begin{matrix}q/b,q/c,aq/d\\ q^2/d,eq/bc\end{matrix};q,\frac{e}{a}\right],\end{aligned} \tag{3.2.4}$$

where $|de/abc| < 1$, $|q/b| < 1$ and $|e/a| < 1$.

Shifting the index of summation on the left hand side of (3.2.4) by m such that the new sum runs from $-m$ to infinity and then replacing a, c, d, e by aq^{-m}, cq^{-m}, dq^{-m}, eq^{-m} respectively, we get

$$\sum_{k=-m}^{\infty} \frac{(a, c, bq^m; q)_k}{(d, e, q^{m+1}; q)_k} \left(\frac{de}{abc}\right)^k$$

$$= \frac{(e/a, e/b, e/c, q/d, dq/ac; q)_\infty}{(e, q/a, q/c, e/d, de/abc; q)_\infty} \frac{(q, bq/e; q)_m}{(b, dq/ac; q)_m} {}_3\phi_2 \left[\begin{matrix} d/a, d/c, bq^{m+1}/e \\ dq^{m+1}/ac, dq/e \end{matrix} ; q, \frac{q}{b} \right]$$

$$- \frac{(d/a, d/b, d/c, q/d, e/a, qe/bc, e/q, q^2/e; q)_\infty}{(e, d/q, q/a, q/b, q/c, e/d, dq/e, de/abc; q)_\infty} \frac{(q, bq/d, q^2/e; q)_m}{(b, q^2/e, q^2/d; q)_m} \times$$

$${}_3\phi_2 \left[\begin{matrix} q^{m+1}/c, q/b, aq/d \\ q^{m+2}/d, eq/bc \end{matrix} ; q, \frac{e}{a} \right], \tag{3.2.5}$$

where $|de/abc| < 1$, $|q/b| < 1$ and $|e/a| < 1$.

Letting $m \longrightarrow \infty$ and assuming $|b| < 1$ in (3.2.5), Tannery's theorem [36] enables us to interchange the limits and the summation. Thus, we get

$$\sum_{k=-\infty}^{\infty} \frac{(a, c; q)_k}{(d, e; q)_k} \left(\frac{de}{abc}\right)^k = \frac{(q, bq/e, e/a, e/b, e/c, q/d; q)_\infty}{(b, e, q/a, q/c, e/d, de/abc; q)_\infty} {}_2\phi_1 \left[\begin{matrix} d/a, d/c \\ dq/e \end{matrix} ; q, \frac{q}{b} \right]$$

$$- \frac{(q, bq/d, d/a, d/b, d/c, q/d, e/a, qe/bc, e/q, q^2/e; q)_\infty}{(b, e, d/q, q/a, q/b, q/c, e/d, dq/e, de/abc, q^2/d; q)_\infty} {}_2\phi_1 \left[\begin{matrix} q/b, aq/d \\ eq/bc \end{matrix} ; q, \frac{e}{a} \right], \tag{3.2.6}$$

where $|de/abc| < 1$, $|q/b| < 1$ and $|e/a| < 1$.

By the substitution $b \longrightarrow de/abz$, $c \longrightarrow b$ and $e \longrightarrow c$ in (3.2.6) and then applying

Heine's transformation (1.3.1)

$$\sum_{k=-\infty}^{\infty} \frac{(a,b;q)_k}{(c,d;q)_k} z^k = \frac{(q,c/a,c/b,q/d,abz/d,dq/abz;q)_\infty}{(c,z,q/a,q/b,c/d,cd/abz;q)_\infty} {}_2\phi_1 \left[\begin{matrix} d/a, d/b \\ dq/c \end{matrix} ; q, \frac{abzq}{cd} \right]$$

$$- \frac{(q,d/a,d/b,q/d,c/q,abz/c,cq/abz,q^2/c,cq/d;q)_\infty}{(c,z,q/a,q/b,c/d,d/q,dq/c,cd/abz,q^2/d;q)_\infty} {}_2\phi_1 \left[\begin{matrix} c/a, c/b \\ cq/d \end{matrix} ; q, \frac{abzq}{cd} \right],$$

where $|abzq/cd| < 1$ and $|z| < 1$. Hence the proof of Theorem 3.2.1. □

As consequence of Theorem 3.2.1, we can derive the following results:

Corollary 3.2.2.

On setting d to q in Theorem 3.2.1, we obtain Heine's transformation (1.3.3).

Corollary 3.2.3.

On setting b to d and c to b in Theorem 3.2.1, we obtain Ramanujan's summation formula (1.2.1).

Corollary 3.2.4.

On setting a to q/a, c to bq and z to a in Theorem 3.2.1 and using q-binomial Theorem (3.1.1), *we obtain ${}_2\psi_2$ summation formula* (3.1.4).

3.3 FOURIER-SERIES DEVELOPMENT RELATED TO THE THETA FUNCTIONS

In this Section, as an application of (3.2.1), we obtain some generalization of Fourier series development related to theta functions given by Ramanujan:

Theorem 3.3.1. *For $|q| < 1$, $r > 0$ and θ real, the following identity hold:*

$$\log\left[\frac{1+2\sum_{k=1}^{\infty} q^{\frac{rk^2}{2}}\cos(\frac{rk\theta}{2})}{f(-q^r)}\right] = 2\sum_{k=1}^{\infty}\frac{(-1)^{k-1}q^{\frac{rk}{2}}}{k(1-q^{rk})}\cos\left(\frac{kr\theta}{2}\right), \quad (3.3.1)$$

$$\frac{1}{4}\log\left[\frac{\sum_{k=1}^{\infty}\frac{(-1)^{k-1}q^{rk(k-1)/2}\sin(2k-1)r\theta}{\sin r\theta}}{\sum_{k=1}^{\infty}(-1)^{k-1}(2k-1)q^{rk(k-1)/2}}\right] = \sum_{k=1}^{\infty}\frac{q^{rk}\sin^2(rk\theta)}{k(1-q^{rk})}, \quad (3.3.2)$$

$$1+4\sum_{k=1}^{\infty}\frac{q^{\frac{rk}{2}}\cos(kr\theta/2)}{1+q^{rk}} = \varphi^2(-q^r)\frac{1+2\sum_{k=1}^{\infty}q^{\frac{rk^2}{2}}\cos(\frac{rk\theta}{2})}{1+2\sum_{k=1}^{\infty}(-1)^k q^{\frac{rk^2}{2}}\cos(\frac{rk\theta}{2})}, \quad (3.3.3)$$

$$\log\left[\frac{\varphi^2(q^{\frac{r}{2}})}{1+4\cos\frac{r\theta}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}q^{rk-\frac{r}{2}}\cos(r(k-\frac{1}{2}))\theta}{1-q^{rk-\frac{r}{2}}}}\right] = 4\sum_{k=1}^{\infty}\frac{(-1)^{k-1}q^{\frac{rk}{2}}\sin^2(\frac{rk\theta}{2})}{k(1+q^{\frac{rk}{2}})}, \quad (3.3.4)$$

$$\log\left[\frac{\varphi^2(q^{\frac{r}{2}})}{1+4\sum_{k=1}^{\infty}\frac{q^{\frac{rk}{2}}\cos(rk\theta)}{1+q^{rk}}}\right] = \sum_{k=1}^{\infty}\frac{q^{r(k-\frac{1}{2})}\sin^2(r\theta(k-\frac{1}{2}))\theta}{(2k-1)(1-q^{r(2k-1)})}. \quad (3.3.5)$$

Proof: On setting a to d, z to $-\frac{e^{\frac{ir\theta}{2}}q^{\frac{r}{2}}}{b}$, q to q^r in Theorem 3.2.1, we obtain

$$\sum_{k=-\infty}^{\infty} \frac{(b;q^r)_k}{(c;q^r)_k}(-1)^k \frac{q^{kr/2}e^{irk\theta/2}}{b^k} = \frac{(q^r, c/b, -e^{ir\theta/2}q^{r/2}, -e^{-ir\theta/2}q^{r/2}; q^r)_\infty}{(c, q^r/b, -e^{ir\theta/2}q^{r/2}/b, -ce^{-ir\theta/2}q^{-r/2}; r^r)_\infty}. \tag{3.3.6}$$

On letting $b \longrightarrow \infty$ and $c \longrightarrow 0$ and using (1.2.7) in identity (3.3.6), we obtain

$$\frac{1 + 2\sum_{k=1}^{\infty} q^{\frac{rk^2}{2}} \cos(\frac{rk\theta}{2})}{f(-q^r)} = (-q^{r/2}e^{ir\theta/2}; q^r)_\infty(-q^{r/2}e^{-ir\theta/2}; q^r)_\infty. \tag{3.3.7}$$

Taking logarithms on both sides of (3.3.7) and employing identity (1.2.8), we complete the proof of identity (3.3.1).

On setting a to d, z to $q^r e^{2ir\theta}/b$ and q to q^r in Theorem 3.2.1, we obtain

$$\sum_{k=-\infty}^{\infty} \frac{(b;q^r)_k}{(c;q^r)_k} \frac{q^{rk}e^{2irk\theta}}{b^k} = \frac{(q^r, c/b, q^r e^{2ir\theta}, e^{-2ir\theta}; q^r)_\infty}{(c, q^r/b, e^{2ir\theta}q^r/b, cq^{-r}e^{-2ir\theta}; q^r)_\infty}. \tag{3.3.8}$$

Now letting $b \longrightarrow \infty$ and $c \longrightarrow 0$ in (3.3.8), we obtain

$$(q^r;q^r)_\infty(q^r e^{2ir\theta};q^r)_\infty(e^{-2ir\theta};q^r)_\infty = \sum_{k=-\infty}^{\infty}(-1)^k q^{rk(k+1)/2}e^{2irk\theta}. \tag{3.3.9}$$

Hence, we have

$$(q^r;q^r)_\infty(q^r e^{2ir\theta};q^r)_\infty(q^r e^{-2ir\theta};q^r)_\infty = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}q^{rk(k-1)/2}\sin(2k-1)r\theta}{\sin r\theta}.$$

Again employing (1.2.7) and (1.2.9), we obtain

$$(q^r;q^r)_\infty^3 = \sum_{k=1}^{\infty}(-1)^{k-1}(2k-1)q^{rk(k-1)/2}. \tag{3.3.10}$$

Employing (3.3.10), we deduce that

$$\frac{\sum\limits_{k=1}^{\infty}\frac{(-1)^{k-1}q^{rk(k-1)/2}\sin(2k-1)r\theta}{\sin r\theta}}{\sum\limits_{k=1}^{\infty}(-1)^{k-1}(2k-1)q^{rk(k-1)/2}} = \frac{(e^{2ir\theta}q^r;q^r)_\infty(e^{-2ir\theta}q^r;q^r)_\infty}{(q^r;q^r)_\infty^2}. \tag{3.3.11}$$

Taking the logarithm of both sides of (3.3.11) and employing (1.2.8), we complete the proof of identity (3.3.2).

On setting a and d to $q^{r/2}e^{-ir\theta/2}$, b to -1, c to $-q^r$, z to $q^{r/2}e^{ir\theta/2}$ and q to q^r in Theorem 3.2.1, we obtain

$$\sum_{k=-\infty}^{\infty}\frac{(-1;q^r)_k}{(-q^r;q^r)_k}q^{rk/2}e^{irk\theta/2} = \frac{(-q^{r/2}e^{ir\theta/2};q^r)_\infty(-q^{r/2}e^{-ir\theta/2};q^r)_\infty(q^r;q^r)_\infty^2}{(q^{r/2}e^{ir\theta/2};q^r)_\infty(q^{r/2}e^{-ir\theta/2};q^r)_\infty(-q^r;q^r)_\infty^2}. \tag{3.3.12}$$

Employing (1.2.3) and (1.2.5) in (3.3.12), we get

$$1+2\sum_{k=1}^{\infty}\frac{q^{rk/2}}{1+q^{rk}}(e^{irk\theta/2}+e^{-irk\theta/2}) = \varphi^2(-q^r)\frac{f(q^{r/2}e^{ir\theta/2},q^{r/2}e^{-ir\theta/2})}{f(-q^{r/2}e^{ir\theta/2},-q^{r/2}e^{-ir\theta/2})}. \tag{3.3.13}$$

With some simplification, (3.3.13) reduces to identity (3.3.3).

On setting a to d, b to $q^{-r/2}$, c to $q^{3r/2}$, z to $-q^r e^{ri\theta}$ and q to q^r in Theorem 3.2.1, we obtain

$$\begin{aligned}\frac{(q^r, q^{2r}, -q^{r/2}e^{ri\theta}, -q^{r/2}e^{-ri\theta}; q^r)_\infty}{(q^{3r/2}, q^{3r/2}, -q^r e^{ri\theta}, -q^r e^{-ri\theta}; q^r)_\infty} &= \sum_{k=-\infty}^{\infty} \frac{(q^{-r/2}; q^r)_k}{(q^{3r/2}; q^r)_k}(-q^r e^{ri\theta})^k \\ &=1 + \sum_{k=1}^{\infty} \frac{(q^{-r/2}; q^r)_k (-1)^k q^{rk}}{(q^{3r/2}; q^r)_k}(e^{rik\theta} + e^{-rik\theta}).\end{aligned} \tag{3.3.14}$$

Hence R.H.S of (3.3.14)

$$\begin{aligned}&1 - \frac{(1-q^{-r/2})q^r}{1-q^{3r/2}}(e^{ri\theta} + e^{-ri\theta}) + \sum_{k=2}^{\infty} \frac{(1-q^{-r/2})(1-q^{r/2})(-1)^k q^{rk}}{(1-q^{rk-r/2})(1-q^{rk+r/2})}(e^{rik\theta} + e^{-rik\theta}) \\ &= \frac{1-q^{-r/2}}{1+q^{r/2}}\left\{1 + \sum_{k=1}^{\infty} \frac{(-1)^{k-1} q^{rk-r/2}(e^{(ri\theta)(k-1/2)} + e^{(ri\theta)(-k+1/2)})(e^{ri\theta/2} + e^{-ri\theta/2})}{1-q^{rk-r/2}}\right\}.\end{aligned} \tag{3.3.15}$$

With the use of (1.2.5) and (3.3.15), we may rewrite (3.3.14)

$$\varphi^2(q^{r/2})\frac{F(r,\theta)}{F(r,0)} = 1 + 4\cos(r\theta/2)\sum_{k=1}^{\infty} \frac{(-1)^{k-1} q^{rk-r/2}\cos(rk - r/2)\theta}{1-q^{rk-r/2}}. \tag{3.3.16}$$

where

$$F(r,\theta) = \frac{(-e^{ir\theta}q^{r/2}; q^r)_\infty(-q^{r/2}e^{-ir\theta}; q^r)_\infty}{(-e^{ir\theta}q^r; q^r)_\infty(-q^r e^{-ir\theta}; q^r)_\infty}.$$

Taking logarithms on both sides of (3.3.16) and employing (1.2.8) with simplification, we complete the proof of identity (3.3.4).

On setting a to d, b to -1, c to $-q^r$, z to $q^{r/2}e^{ri\theta}$ and q to q^r in Theorem 3.2.1, we obtain

$$1+\sum_{k=1}^{\infty}\frac{(-1;q^r)_k}{(-q^r;q^r)_k}q^{rk/2}(e^{rki\theta}+e^{-irk\theta})=\frac{(-q^{r/2}e^{ri\theta};q^r)_\infty(-q^{r/2}e^{-ri\theta};q^r)_\infty(q^r;q^r)_\infty^2}{(q^{r/2}e^{ri\theta};q^r)_\infty(q^{r/2}e^{-ri\theta};q^r)_\infty(-q^r;q^r)_\infty^2}. \tag{3.3.17}$$

Employing (1.2.5),

$$\begin{aligned} 1+4\sum_{k=1}^{\infty}\frac{q^{rk/2}\cos(rk\theta)}{1+q^{rk}} &= \frac{(-q^{r/2}e^{ri\theta};q^r)_\infty(-q^{r/2}e^{-ri\theta};q^r)_\infty(q^r;q^r)_\infty^2}{(q^{r/2}e^{ri\theta};q^r)_\infty(q^{r/2}e^{-ri\theta};q^r)_\infty(-q^r;q^r)_\infty^2} \\ &= \varphi^2(q^r)\frac{G(r,\theta)}{G(r,0)}, \end{aligned} \tag{3.3.18}$$

where

$$G(r,\theta)=\frac{(-zq;q^2)_\infty(-q/z;q^2)_\infty}{(zq;q^2)_\infty(q/z;q^2)_\infty}.$$

Taking logarithm on both sides of (3.3.18) and employing (1.2.8), we have the proof of identity (3.3.5). □

Remark. *For* $r=2$, (3.3.1), (3.3.3), (3.3.4) *and* (3.3.5) *reduces to [26, Entry 33(i)], [26, Entry 33(iii)], [26, Entry 34(i)] and [26, Entry 34(ii)] respectively and for* $r=1$, (3.3.2) *reduces to [26, Entry 33(ii)].*

CHAPTER IV

SOME NEW IDENTITIES ON CIRCULAR SUMMATION FORMULA

CHAPTER IV

SOME NEW IDENTITIES ON CIRCULAR SUMMATION FORMULA

4.1 INTRODUCTION

The lost notebook [93], contains mathematical works of Ramanujan's most profound discoveries, fall under the purview of q-series. These include mock theta functions, q- series transformation, continued fractions, partial theta functions, false theta functions, partition, combinatorics, congruences, integrals, theta type series involving indefinite quadratic forms, modular equations, modular relations and many more.

On page 54 of the same lost notebook, Ramanujan recorded a beautiful formula of q-series, which is now well known as Ramanujan's circular summation formula:

Theorem 4.1.1. *For any positive integer $n \geq 2$, if*

$$U_r = a^{r(r+1)/2n} b^{r(r-1)/2n} \text{ and } V_r = a^{r(r-1)/2n} b^{r(r+1)/2n},$$

then

$$\sum_{r=0}^{n-1} U_r^n f^n\left(\frac{U_{n+r}}{U_r}, \frac{V_{n-r}}{U_r}\right) = f(a,b) F_n(ab), \tag{4.1.1}$$

where

$$F_n(q) = 1 + 2nq^{(n-1)/2} + \cdots, n > 3.$$

Applying the theory of elliptic functions, H. H. Chan, Z. G. Liu and S. T. Ng proved a dual form of Ramanujan's circular summation in [45]. M. Boon, M. L. Glasser, J. Zak and I. J. Zucker [37] have proved an additive decomposition of θ_3. A general result that unifies the results of [37] and [45] is proved by Zeng [115]:

For a, b, n and k any positive integers with $k = a + b$,

$$\sum_{s=0}^{kn-1} \theta_3^a\Big(\frac{z}{kn} + \frac{y}{a} + \frac{\pi s}{kn}\Big|\frac{\tau}{kn^2}\Big)\theta_3^b\Big(\frac{z}{kn} - \frac{y}{b} + \frac{\pi s}{kn}\Big|\frac{\tau}{kn^2}\Big) = \mathscr{C}_{33}\Big(a, b; \frac{y}{ab}, \frac{\tau}{kn^2}\Big)\theta_3(z|\tau),$$

where

$$\mathscr{C}_{33}(a, b; y, \tau) = kn \sum_{\substack{m_1,\cdots m_a, n_1, \cdots n_b = -\infty \\ m_1+\cdots+m_a+n_1+\cdots+n_b=0}}^{+\infty} q^{m_1^2+\cdots m_a^2+n_1^2+\cdots n_b^2} e^{2k(m_1+\cdots+m_a)iy}.$$

For more recent works on Ramanujan's circular summation one may refer [38], [39], [43], [47], [48], [79], [85], [86], [99], [116], [117] and [118].

In Section 4.2 of this Chapter, we obtain new Ramanujan's summation for four theta functions employing elliptic functions. In Section 4.3, we deduce results applying Jacobi imaginary transformation formulas. Section 4.4 contains some new results by employing difference of theta functions. In final Section, as an application of our main result, we obtain some special cases.

4.2 NEW CIRCULAR SUMMATION FORMULA

Theorem 4.2.1. *Let a, b, c, l, m and n are positive integers with $a+b+c=m$. If a, b, c are even, then we have*

$$\sum_{s=0}^{lmn-1} \theta_1^a\Big(\frac{z}{lmn}+\frac{y}{a}+\frac{\pi s}{lmn}\Big|\frac{\tau}{lmn^2}\Big)\theta_2^b\Big(\frac{z}{lmn}+\frac{y}{b}+\frac{\pi s}{lmn}\Big|\frac{\tau}{lmn^2}\Big)\times$$
$$\theta_3^c\Big(\frac{z}{lmn}+\frac{y}{c}+\frac{\pi s}{lmn}\Big|\frac{\tau}{lmn^2}\Big) = \mathscr{F}_{123}\Big(a,b,c;\frac{y}{abc},\frac{\tau}{lmn^2}\Big)\theta_3(z|\tau), \tag{4.2.1}$$

where

$$\mathscr{F}_{123}(a,b,c;y,\tau) = lmni^a q^{\frac{a+b}{4}} \sum_{\substack{u_1,\cdots u_a,v_1,\cdots v_b,w_1,\cdots w_c=-\infty \\ 2(u_1+\cdots+u_a+v_1+\cdots+v_b+w_1+\cdots+w_c)+a+b=0}}^{+\infty} (-1)^{u_1+\cdots+u_a}$$
$$\times q^{u_1^2+\cdots u_a^2+v_1^2+\cdots v_b^2+w_1^2+\cdots w_c^2+u_1+\cdots u_a+v_1+\cdots+v_b}$$
$$\times e^{2\{bc(u_1+\cdots+u_a)+ac(v_1+\cdots+v_b)+ab(w_1+\cdots+w_c)+abc\}iy}. \tag{4.2.2}$$

Proof: Let $f(z)$ be the left side of (4.2.1). Then

$$f(z+\pi) =$$
$$\sum_{s=1}^{lmn-1} \theta_1^a\Big(\frac{z}{lmn}+\frac{y}{a}+\frac{\pi s}{lmn}\Big|\frac{\tau}{lmn^2}\Big)\theta_2^b\Big(\frac{z}{lmn}+\frac{y}{b}+\frac{\pi s}{lmn}\Big|\frac{\tau}{lmn^2}\Big)\times$$
$$\theta_3^c\Big(\frac{z}{lmn}+\frac{y}{c}+\frac{\pi s}{lmn}\Big|\frac{\tau}{lmn^2}\Big)+\theta_1^a\Big(\frac{z}{lmn}+\frac{y}{a}\Big|\frac{\tau}{lmn^2}\Big)\theta_2^b\Big(\frac{z}{lmn}+\frac{y}{b}\Big|\frac{\tau}{lmn^2}\Big)\times$$
$$\theta_3^c\Big(\frac{z}{lmn}+\frac{y}{c}\Big|\frac{\tau}{lmn^2}\Big). \tag{4.2.3}$$

$$f(z+\pi)=$$

$$\sum_{s=1}^{lmn-1}\theta_1^a\Big(\frac{z}{lmn}+\frac{y}{a}+\frac{\pi s}{lmn}\Big|\frac{\tau}{lmn^2}\Big)\theta_2^b\Big(\frac{z}{lmn}+\frac{y}{b}+\frac{\pi s}{lmn}\Big|\frac{\tau}{lmn^2}\Big)\times$$

$$\theta_3^c\Big(\frac{z}{lmn}+\frac{y}{c}+\frac{\pi s}{lmn}\Big|\frac{\tau}{lmn^2}\Big)+(-1)^{a+b}\theta_1^a\Big(\frac{z}{lmn}+\frac{y}{a}\Big|\frac{\tau}{lmn^2}\Big)\times$$

$$\theta_2^b\Big(\frac{z}{lmn}+\frac{y}{b}\Big|\frac{\tau}{lmn^2}\Big)\theta_3^c\Big(\frac{z}{lmn}+\frac{y}{c}\Big|\frac{\tau}{lmn^2}\Big). \tag{4.2.4}$$

Since a and b are even, we have from (4.2.3) and (4.2.4)

$$f(z+\pi)=f(z). \tag{4.2.5}$$

Again from (4.2.3) and a, b, c even, we have

$$f(z+\pi\tau)=$$

$$\sum_{s=0}^{lmn-1}\theta_1^a\Big(\frac{z}{lmn}+\frac{y}{a}+\frac{\pi s}{lmn}+n\pi\frac{\tau}{lmn^2}\Big|\frac{\tau}{lmn^2}\Big)\times$$

$$\theta_2^b\Big(\frac{z}{lmn}+\frac{y}{b}+\frac{\pi s}{lmn}+n\pi\frac{\tau}{lmn^2}\Big|\frac{\tau}{lmn^2}\Big)\times\theta_3^c\Big(\frac{z}{lmn}+\frac{y}{c}+\frac{\pi s}{lmn}+n\pi\frac{\tau}{lmn^2}\Big|\frac{\tau}{lmn^2}\Big)$$

$$=q^{-1}e^{-2iz}\sum_{s=0}^{lmn-1}\theta_1^a\Big(\frac{z}{lmn}+\frac{y}{a}+\frac{\pi s}{lmn}\Big|\frac{\tau}{lmn^2}\Big)\theta_2^b\Big(\frac{z}{lmn}+\frac{y}{b}+\frac{\pi s}{lmn}\Big|\frac{\tau}{lmn^2}\Big)$$

$$\theta_3^c\Big(\frac{z}{lmn}+\frac{y}{c}+\frac{\pi s}{lmn}\Big|\frac{\tau}{lmn^2}\Big)=q^{-1}e^{-2iz}f(z). \tag{4.2.6}$$

By (4.2.5) and (4.2.6), we have constructed an elliptic function $f(z)/\theta_3(z|\tau)$ with double periods π and $\pi\tau$ and only have a simple pole at $z=\pi/2+\pi\tau/2$ in the period parallelogram. Hence the function $f(z)/\theta_3(z|\tau)$ is a constant, say

$F_{123}(a,b,c;y,\tau)$, i.e.

$$\frac{f(z)}{\theta_3(z|\tau)} = F_{123}(a,b,c;y,\tau)\theta_3(z|\tau),$$

or, equivalently

$$f(z) = F_{123}(a,b,c;y,\tau)\theta_3(z|\tau).$$

Hence

$$\sum_{s=0}^{lmn-1} \theta_1^a\Big(\frac{z}{lmn}+\frac{y}{a}+\frac{\pi s}{lmn}\Big|\frac{\tau}{lmn^2}\Big)\theta_2^b\Big(\frac{z}{lmn}+\frac{y}{b}+\frac{\pi s}{lmn}\Big|\frac{\tau}{lmn^2}\Big)\times$$
$$\theta_3^c\Big(\frac{z}{lmn}+\frac{y}{c}+\frac{\pi s}{lmn}\Big|\frac{\tau}{lmn^2}\Big) = F_{123}(a,b,c;y,\tau)\theta_3(z|\tau). \quad (4.2.7)$$

Employing (1.1.10), (1.1.11) and (1.1.13), we obtain

$$F_{123}(a,b,c;y,\tau)\sum_{m=-\infty}^{\infty} q^{m^2}e^{2miz} = (-1)^a i^a q^{\frac{a+b}{4lmn^2}}$$
$$\sum_{s=0}^{lmn-1}\ \sum_{u_1,\cdots u_a,v_1,\cdots v_b,w_1,\cdots,w_c=-\infty}^{\infty} (-1)^{u_1+\cdots+u_c} q^{\frac{u_1^2+\cdots+u_a^2+v_1^2+\cdots+v_a^2+w_1^2+\cdots+w_a^2+u_1+\cdots+u_a+v_1+\cdots+v_b}{lnm^2}}$$
$$\times e^{\frac{\{2(u_1+\cdots+u_a+v_1+\cdots+v_b+w_1+\cdots+w_c)+a+b\}}{lmn}iz}$$
$$\times e^{\left\{\frac{2(u_1+\cdots+u_a)+a}{a}+\frac{2(v_1+\cdots+v_a)+b}{b}+\frac{w_1+\cdots+w_a}{c}\right\}iy}$$
$$\times e^{\frac{\{2(u_1+\cdots+u_a+v_1+\cdots+v_b+w_1+\cdots+w_c)+a+b\}}{lmn}i\pi s}. \quad (4.2.8)$$

The constant term on both sides of (4.2.8), we have

$$F_{123}(a,b,c;y,\tau) = lmni^a q^{\frac{a+b}{4lmn^2}} \sum_{\substack{u_1,\cdots u_a,v_1,\cdots v_b,w_1,\cdots w_c=-\infty \\ 2(u_1+\cdots+u_a+v_1+\cdots+v_b+ \\ w_1+\cdots w_c)+a+b=0}}^{+\infty} (-1)^{u_1+\cdots+u_c} \times q^{\frac{u_1^2+\cdots+u_a^2+v_1^2+\cdots+v_a^2+w_1^2+\cdots+w_a^2+u_1+\cdots+u_a+v_1+\cdots+v_b}{lnm^2}} \times e^{\frac{2\{bc(u_1+\cdots+u_a)+ac(v_1+\cdots+v_a)+ab(w_1+\cdots+w_a)+abc\}}{abc}iy}.$$

It is clear that

$$F_{123}(a,b,c;y,\tau) = \mathscr{F}_{123}\Big(a,b,c;\frac{y}{abc},\frac{\tau}{lmn^2}\Big),$$

where

$$\mathscr{F}_{123}(a,b,c;y,\tau) = lmni^a q^{\frac{a+b}{4}} \sum_{\substack{u_1,\cdots u_a,v_1,\cdots v_b,w_1,\cdots w_c=-\infty \\ 2(u_1+\cdots+u_a+v_1+\cdots+v_b+w_1+\cdots+w_c)+a+b=0}}^{+\infty} (-1)^{u_1+\cdots+u_a} \times q^{u_1^2+\cdots u_a^2+v_1^2+\cdots v_b^2+w_1^2+\cdots w_c^2+u_1+\cdots u_a+v_1+\cdots+v_b} \times e^{2\{bc(u_1+\cdots+u_a)+ac(v_1+\cdots+v_b)+ab(w_1+\cdots+w_c)+abc\}iy}.$$

This complete the proof of Theorem 4.2.1. □

The proof of following Theorems 4.2.2-4.2.5 can be proved in similar way.

Theorem 4.2.2. *Let a, b, c, l, m and n are positive integers with $a+b+c=m$. If a, b, c are even, then we have*

$$\sum_{s=0}^{lmn-1}\theta_1^a\Big(\frac{z}{lmn}+\frac{y}{a}+\frac{\pi s}{lmn}\Big|\frac{\tau}{lmn^2}\Big)\theta_2^b\Big(\frac{z}{lmn}+\frac{y}{b}+\frac{\pi s}{lmn}\Big|\frac{\tau}{lmn^2}\Big)$$
$$\theta_4^c\Big(\frac{z}{lmn}+\frac{y}{c}+\frac{\pi s}{lmn}\Big|\frac{\tau}{lmn^2}\Big)=\mathscr{F}_{124}\Big(a,b,c;\frac{y}{abc},\frac{\tau}{lmn^2}\Big)\theta_3(z|\tau), \tag{4.2.9}$$

where

$$\mathscr{F}_{124}(a,b,c;y,\tau)=lmni^a q^{\frac{a+b}{4}}\times \sum_{\substack{u_1,\cdots u_a,v_1,\cdots,v_b,w_1,\cdots w_c=-\infty\\ 2(u_1+\cdots+u_a+v_1+\cdots+v_b+w_1+\cdots+w_c)+a+b=0}}^{+\infty}(-1)^{u_1+\cdots+u_a+w_1+\cdots+w_c}$$
$$\times q^{u_1^2+\cdots u_a^2+v_1^2+\cdots v_b^2+w_1^2+\cdots w_c^2+u_1+\cdots u_a+v_1+\cdots+v_b}$$
$$\times e^{2\{bc(u_1+\cdots+u_a)+ac(v_1+\cdots+v_b)+ab(w_1+\cdots+w_c)+abc\}iy}. \tag{4.2.10}$$

Theorem 4.2.3. *Let a, b, c, l, m and n are positive integers with $a+b+c=m$. If a, b, c are even, then we have*

$$\sum_{s=0}^{lmn-1}\theta_1^a\Big(\frac{z}{lmn}+\frac{y}{a}+\frac{\pi s}{lmn}\Big|\frac{\tau}{lmn^2}\Big)\theta_3^b\Big(\frac{z}{lmn}+\frac{y}{b}+\frac{\pi s}{lmn}\Big|\frac{\tau}{lmn^2}\Big)$$
$$\theta_4^c\Big(\frac{z}{lmn}+\frac{y}{c}+\frac{\pi s}{lmn}\Big|\frac{\tau}{lmn^2}\Big)=\mathscr{F}_{134}\Big(a,b,c;\frac{y}{abc},\frac{\tau}{lmn^2}\Big)\theta_3(z|\tau), \tag{4.2.11}$$

where

$$\mathscr{F}_{134}(a,b,c;y,\tau) = lmni^a q^{\frac{a}{4}} \sum_{\substack{u_1,\cdots u_a,v_1,\cdots v_b,w_1,\cdots w_c=-\infty \\ 2(u_1+\cdots+u_a+v_1+\cdots+v_b+w_1+\cdots+w_c)+a=0}}^{+\infty} (-1)^{u_1+\cdots+u_a+w_1+\cdots+w_c}$$
$$\times q^{u_1^2+\cdots u_a^2+v_1^2+\cdots v_b^2+w_1^2+\cdots+w_c^2+u_1+\cdots+u_a}$$
$$\times e^{[2\{bc(u_1+\cdots+u_a)+ac(v_1+\cdots+v_b)+ab(w_1+\cdots+w_c)\}+abc]iy}. \tag{4.2.12}$$

Theorem 4.2.4. *Let a, b, c, l, m and n are positive integers with $a+b+c=m$. If a, b, c are even, then we have*

$$\sum_{s=0}^{lmn-1} \theta_2^a\Big(\frac{z}{lmn}+\frac{y}{a}+\frac{\pi s}{lmn}\Big|\frac{\tau}{lmn^2}\Big)\theta_3^b\Big(\frac{z}{lmn}+\frac{y}{b}+\frac{\pi s}{lmn}\Big|\frac{\tau}{lmn^2}\Big)$$
$$\theta_4^c\Big(\frac{z}{lmn}+\frac{y}{c}+\frac{\pi s}{lmn}\Big|\frac{\tau}{lmn^2}\Big) = \mathscr{F}_{234}\Big(a,b,c;\frac{y}{abc},\frac{\tau}{lmn^2}\Big)\theta_3(z|\tau), \tag{4.2.13}$$

where

$$\mathscr{F}_{234}(a,b,c;y,\tau) = lmnq^{\frac{a}{4}} \sum_{\substack{u_1,\cdots u_a,v_1,\cdots v_b,w_1,\cdots w_c=-\infty \\ 2(u_1+\cdots+u_a+v_1+\cdots+v_b+w_1+\cdots+w_c)+a=0}}^{+\infty} (-1)^{w_1+\cdots+w_c}$$
$$\times q^{u_1^2+\cdots u_a^2+v_1^2+\cdots v_b^2+w_1^2+\cdots+w_c^2+u_1+\cdots+u_a}$$
$$\times e^{[2\{bc(u_1+\cdots+u_a)+ac(v_1+\cdots+v_b)+ab(w_1+\cdots+w_c)\}+abc]iy}. \tag{4.2.14}$$

Theorem 4.2.5. *Let a, b, c, d, k, l, m and n are positive integers with $a+b+c+d=m$. If a, b, c, d are even, then we have*

$$\sum_{s=0}^{klmn-1} \theta_1^a\Big(\frac{z}{klmn}+\frac{y}{a}+\frac{\pi s}{klmn}\Big|\frac{\tau}{klmn^2}\Big)\theta_2^b\Big(\frac{z}{klmn}+\frac{y}{b}+\frac{\pi s}{klmn}\Big|\frac{\tau}{klmn^2}\Big)$$
$$\theta_3^c\Big(\frac{z}{klmn}+\frac{y}{c}+\frac{\pi s}{klmn}\Big|\frac{\tau}{klmn^2}\Big)\theta_4^d\Big(\frac{z}{klmn}+\frac{y}{d}+\frac{\pi s}{klmn}\Big|\frac{\tau}{klmn^2}\Big)$$
$$=\mathscr{F}_{1234}\Big(a,b,c,d;\frac{y}{abcd},\frac{\tau}{klmn^2}\Big)\theta_3(z|\tau), \quad (4.2.15)$$

where

$$\mathscr{F}_{1234}(a,b,c,d;y,\tau)=klmni^a q^{\frac{a+b}{4}}$$
$$\times \sum_{\substack{u_1,\cdots u_a,v_1,\cdots v_b,w_1,\cdots w_c,x_1,\cdots x_d=-\infty \\ 2(u_1+\cdots+u_a+v_1+\cdots+v_b+w_1+\cdots w_c+x_1+\cdots+x_d)+a+b=0}}^{+\infty} (-1)^{u_1+\cdots+u_a+x_1+\cdots+x_d}$$
$$\times q^{u_1^2+\cdots u_a^2+v_1^2+\cdots v_b^2+w_1^2+\cdots+w_c^2+x_1^2+\cdots+x_d^2+u_1+\cdots+u_a+v_1+\cdots+v_b}$$
$$\times e^{[2\{bc(u_1+\cdots+u_a)+ac(v_1+\cdots+v_b)+ab(w_1+\cdots+w_c)\}+abc]iy}. \quad (4.2.16)$$

4.3 RESULTS OBTAINED BY APPLYING JACOBI IMAGINARY TRANSFORMATION FORMULAE

In this Section, we deduce results applying Jacobi imaginary transformation formulas (1.1.22)-(1.1.25).

Theorem 4.3.1. *For positive even integers l, m, n, a, b and c with $a+b+c=m$, we have*

$$\sum_{s=0}^{lmn-1} q^{\frac{s^2\pi^2\tau^2 nm+6s\pi\tau(zmn+y)+6zy+z^2mn-z^2lmn}{\pi^2\tau^2 lmn}}\theta_1^a\Big(nz+\frac{y}{a}+ns\pi\tau\Big|lmn^2\tau\Big)$$
$$\theta_4^b\Big(nz+\frac{y}{b}+ns\pi\tau\Big|lmn^2\tau\Big)\theta_3^c\Big(nz+\frac{y}{c}+ns\pi\tau\Big|lmn^2\tau\Big)=H_{123}(a,b,c;y,\tau)\theta_3(z|\tau), \tag{4.3.1}$$

where

$$H_{123}(a,b,c;y,\tau)$$
$$=\frac{i^{\frac{3-2a-2m}{2}}\tau^{\frac{1-m}{2}}}{(lmn^2)^{\frac{m}{2}}}q^{-\frac{y^2(ab+bc+ac)}{abclmn^2\pi^2\tau^2}}\mathscr{F}_{123}\Big(a,b,c;\frac{y}{abclmn^2\tau},-\frac{1}{lmn^2\tau}\Big)\theta_3(z|\tau) \tag{4.3.2}$$
$$=i^a\frac{\pi^2\tau^2l^2m^2n^3a+24zy+4z^2mn-4z^2lmn}{4\pi^2\tau^2lmn}$$
$$\times\sum_{s=0}^{m-1} q^{\frac{4s^2l^2n^2\pi^2\tau^2nm+24sln\pi\tau(zmn+y)}{4\pi^2\tau^2lmn}}$$
$$\times\sum_{\substack{d_1,\cdots d_a,f_1,\cdots f_b,g_1,\cdots,g_c=-\infty\\2(d_1+\cdots+d_a+f_1+\cdots+f_b+g_1+\cdots+g_c)+a=0}}^{+\infty}(-1)^{d_1+\cdots+d_a+f_1+\cdots+f_b}$$
$$\times q^{lmn^2(d_1^2+\cdots d_a^2+f_1^2+\cdots f_b^2+g_1^2+\cdots g_c^2+d_1+\cdots d_a)}$$
$$\times e^{\frac{[2\{bc(d_1+\cdots+d_a)+ac(f_1+\cdots+f_b)+ab(g_1+\cdots+g_c)\}+abc]}{abc}iy}. \tag{4.3.3}$$

Proof: Replacing τ by $\frac{-1}{lm\tau}$, then replacing z by $\frac{z}{\tau}$ and y by $\frac{y\pi}{lmn^2}$ in (4.2.1), we obtain

$$\sum_{s=0}^{lmn-1}\theta_1^a\Big(\frac{z}{lmn\tau}+\frac{y\pi}{almn^2}+\frac{\pi s}{lmn}\Big|-\frac{1}{lmn^2\tau}\Big)\theta_2^b\Big(\frac{z}{lmn\tau}+\frac{y\pi}{blmn^2}+\frac{\pi s}{lmn}\Big|-\frac{1}{lmn^2\tau}\Big)$$
$$\theta_3^c\Big(\frac{z}{lmn\tau}+\frac{y\pi}{clmn^2}+\frac{\pi s}{lmn}\Big|-\frac{1}{lmn^2\tau}\Big)=\mathscr{F}_{123}\Big(a,b,c;\frac{y\pi}{abclmn^2},-\frac{1}{lmn^2\tau}\Big)\theta_3(z|\tau).\tag{4.3.4}$$

Applying the Jacobi imaginary transformation formulas (1.1.22), (1.1.23) and (1.1.24) in (4.3.4), we obtain

$$\sum_{s=0}^{lmn-1}\left(-i\sqrt{-ilmn^2\tau}e^{\frac{i(nz+y\pi\tau/a+ns\pi\tau)^2}{lmn^2\pi\tau}}\right)^a\left(\sqrt{-ilmn^2\tau}e^{\frac{i(nz+y\pi\tau/b+ns\pi\tau)^2}{lmn^2\pi\tau}}\right)^b$$
$$\left(\sqrt{-ilmn^2\tau}e^{\frac{i(nz+y\pi\tau/c+ns\pi\tau)^2}{lmn^2\pi\tau}}\right)^c\theta_1^a\left(nz+\frac{y\pi\tau}{a}+ns\pi\tau|lmn^2\tau\right)$$
$$\theta_4^b\left(nz+\frac{y\pi\tau}{a}+ns\pi\tau|lmn^2\tau\right)\theta_3^c\left(nz+\frac{y\pi\tau}{a}+ns\pi\tau|lmn^2\tau\right)$$
$$=\sqrt{-i\tau}e^{iz^2/\pi\tau}\mathscr{F}_{123}\Big(a,b,c;\frac{y\pi}{abclmn^2},-\frac{1}{lmn^2\tau}\Big)\theta_3(z|\tau).\tag{4.3.5}$$

Simplifying (4.3.5), we obtain

$$\sum_{s=0}^{lmn-1}q^{\frac{s^2\pi^2\tau^2nm+6s\pi\tau znm+6sy\pi^2\tau^2+6zy\pi\tau+z^2nm-z^2lnm}{\pi^2\tau^2lnm}}\theta_1^a\left(nz+\frac{y\pi\tau}{a}+ns\pi\tau|lmn^2\tau\right)$$
$$\theta_4^b\left(nz+\frac{y\pi\tau}{a}+ns\pi\tau|lmn^2\tau\right)\theta_3^c\left(nz+\frac{y\pi\tau}{a}+ns\pi\tau|lmn^2\tau\right)$$
$$=\frac{i^{\frac{3-2a-2m}{2}}\tau^{\frac{1-m}{2}}}{(lmn^2)^{m/2}}\mathscr{F}_{123}\Big(a,b,c;\frac{y\pi}{abclmn^2},-\frac{1}{lmn^2\tau}\Big)\theta_3(z|\tau).\tag{4.3.6}$$

Replacing y by $\frac{y}{\pi\tau}$ in (4.3.6), we obtain (4.3.1) and (4.3.2) of Theorem 4.3.1.

Applying series expansion of (1.1.10), (1.1.12) and (1.1.13) in (4.3.1), we have

$$H_{123}(a,b,c;y,\tau)\sum_{w=-\infty}^{\infty} q^{w^2}e^{2wiz}$$

$$= i^a q^{\frac{24zy+4z^2mn-4z^2lmn+l^2m^2n^3\pi^2\tau^2a}{4\pi^2\tau^2lmn}} \times \sum_{s=0}^{lmn-1}\sum_{u_1,\cdots,u_a,v_1,\cdots v_b,w_1,\cdots,w_c=-\infty}^{+\infty}(-1)^{d_1+\cdots+d_a+f_1+\cdots+f_b}$$

$$\times e^{\{2(d_1+\cdots+d_a+f_1+\cdots+f_b+g_1+\cdots+g_c)+a\}inz+\{2(d_1+\cdots+d_a+f_1+\cdots+f_b+g_1+\cdots+g_c)+a\}ins\pi\tau}$$

$$\times e^{[2\{bc(d_1+\cdots+d_a)+ac(f_1+\cdots+f_b)+ab(g_1+\cdots+g_c)\}+abc]ins\pi\tau}$$

$$\times q^{\frac{4s^2\pi^2\tau^2nm+24s\pi\tau zmn+24s\pi\tau y}{4\pi^2\tau^2lmn}+lmn^2(d_1^2+\cdots+d_a^2+f_1^3+\cdots+f_b^2+g_1^2+\cdots+g_c^2+d_1+\cdots+d_a)}$$

$$= i^a q^{\frac{24zy+4z^2mn-4z^2lmn+l^2m^2n^3\pi^2\tau^2a}{4\pi^2\tau^2lmn}} \sum_{\beta=0}^{n-1}\sum_{\alpha=0}^{l-1}\sum_{s=0}^{m-1}\sum_{u_1,\cdots u_a,v_1,\cdots v_b,w_1,\cdots w_c=-\infty}^{+\infty}(-1)^{d_1+\cdots+d_a+f_1+\cdots+f_b}$$

$$\times e^{\{2(d_1+\cdots+d_a+f_1+\cdots+f_b+g_1+\cdots+g_c)+a\}inz+\{2(d_1+\cdots+d_a+f_1+\cdots+f_b+g_1+\cdots+g_c)+a\}in(sln+\alpha n+\beta)\pi\tau}$$

$$\times e^{[2\{bc(d_1+\cdots+d_a)+ac(f_1+\cdots+f_b)+ab(g_1+\cdots+g_c)\}+abc]in(sln+\alpha n+\beta)\pi\tau}$$

$$\times q^{lmn^2(d_1^2+\cdots+d_a^2+f_1^3+\cdots+f_b^2+g_1^2+\cdots+g_c^2+d_1+\cdots+d_a)}$$

$$\times q^{\frac{4(sln+\alpha n+\beta)^2\pi^2\tau^2nm+24(sln+\alpha n+\beta)\pi\tau zmn+24(sln+\alpha n+\beta)\pi\tau y}{4\pi^2\tau^2lmn}}$$

$$= i^a q^{\frac{24zy+4z^2mn-4z^2lmn+l^2m^2n^3\pi^2\tau^2a}{4\pi^2\tau^2lmn}} \sum_{\alpha=0}^{l-1}\sum_{s=0}^{m-1}\sum_{u_1,\cdots u_a,v_1,\cdots v_b,w_1,\cdots w_c=-\infty}^{+\infty}(-1)^{d_1+\cdots+d_a+f_1+\cdots+f_b}$$

$$\times e^{\{2(d_1+\cdots+d_a+f_1+\cdots+f_b+g_1+\cdots+g_c)+a\}inz+\{2(d_1+\cdots+d_a+f_1+\cdots+f_b+g_1+\cdots+g_c)+a\}in(sln+\alpha n)\pi\tau}$$

$$\times e^{[2\{bc(d_1+\cdots+d_a)+ac(f_1+\cdots+f_b)+ab(g_1+\cdots+g_c)\}+abc]in(sln+\alpha n)\pi\tau}$$

$$\times q^{lmn^2(d_1^2+\cdots+d_a^2+f_1^3+\cdots+f_b^2+g_1^2+\cdots+g_c^2+d_1+\cdots+d_a)}$$

$$\times q^{\frac{4(sln+\alpha n)^2\pi^2\tau^2nm+24(sln+\alpha n)\pi\tau zmn+24(sln+\alpha n)\pi\tau y}{4\pi^2\tau^2lmn}}$$

$$= i^a q^{\frac{24zy+4z^2mn-4z^2lmn+l^2m^2n^3\pi^2\tau^2a}{4\pi^2\tau^2lmn}} \sum_{s=0}^{m-1} \sum_{u_1,\cdots u_a,v_1,\cdots v_b,w_1,\cdots w_c=-\infty}^{+\infty} (-1)^{d_1+\cdots+d_a+f_1+\cdots+f_b}$$

$$\times e^{\{2(d_1+\cdots+d_a+f_1+\cdots+f_b+g_1+\cdots+g_c)+a\}inz}$$

$$\times e^{\{2(d_1+\cdots+d_a+f_1+\cdots+f_b+g_1+\cdots+g_c)+a\}inslnπ\tau}$$

$$\times e^{[2\{bc(d_1+\cdots+d_a)+ac(f_1+\cdots+f_b)+ab(g_1+\cdots+g_c)\}+abc]inslnπ\tau}$$

$$\times q^{lmn^2(d_1^2+\cdots+d_a^2+f_1^3+\cdots+f_b^2+g_1^2+\cdots+g_c^2+d_1+\cdots+d_a)+\frac{4s^2l^2n^2\pi^2\tau^2nm+24slnπ\tau zmn+24slnπ\tau y}{4\pi^2\tau^2lmn}}. \quad (4.3.7)$$

Comparing the constant term on both sides of (4.3.7), we easily get (4.3.3). Hence the proof of Theorem 4.3.1. □

Similarly, we can prove the following Theorems 4.3.2-4.3.5

Theorem 4.3.2. *For positive even integers l, m, n, a, b and c with $a+b+c=m$, we have*

$$\sum_{s=0}^{lmn-1} q^{\frac{s^2\pi^2\tau^2nm+6s\pi\tau(zmn+y)+6zy+z^2mn-z^2lmn}{\pi^2\tau^2lmn}} \theta_1^a\Big(nz+\frac{y}{a}+ns\pi\tau\Big|lmn^2\tau\Big)$$
$$\theta_4^b\Big(nz+\frac{y}{b}+ns\pi\tau\Big|lmn^2\tau\Big)\theta_2^c\Big(nz+\frac{y}{c}+ns\pi\tau\Big|lmn^2\tau\Big) = H_{124}(a,b,c;y,\tau)\theta_3(z|\tau),$$
$$(4.3.8)$$

where

$$
\begin{aligned}
&H_{124}(a,b,c;y,\tau)\\
&=\frac{i^{\frac{3-2a-2m}{2}}\tau^{\frac{1-m}{2}}}{(lmn^2)^{\frac{m}{2}}}q^{-\frac{y^2(ab+bc+ac)}{abclmn^2\pi^2\tau^2}}\mathscr{F}_{124}\Big(a,b,c;\frac{y}{abclmn^2\tau},-\frac{1}{lmn^2\tau}\Big)\theta_3(z|\tau) \qquad (4.3.9)\\
&=i^a\frac{\pi^2\tau^2l^2m^2n^3(a+b)+24zy+4z^2mn-4z^2lmn}{4\pi^2\tau^2lmn}\\
&\times\sum_{s=0}^{m-1}q^{\frac{4s^2l^2\pi^2\tau^2n^3m+24slnn\pi\tau(zmn+y)}{4\pi^2\tau^2lmn}}\sum_{\substack{d_1,\cdots d_a,f_1,\cdots f_b,g_1,\cdots,g_c=-\infty\\(d_1+\cdots+d_a+f_1+\cdots+f_b+g_1+\cdots+g_c)+\frac{a+b}{2}=0}}^{+\infty}(-1)^{d_1+\cdots+d_a+g_1+\cdots+g_c}\\
&\times q^{lmn^2(d_1^2+\cdots d_a^2+f_1^2+\cdots f_b^2+g_1^2+\cdots g_c^2+d_1+\cdots d_a+f_1+\cdots+f_b)}\\
&\times e^{\frac{[2\{bc(d_1+\cdots+d_a)+ac(f_1+\cdots+f_b)+ab(g_1+\cdots+g_c)\}+abc]}{abc}iy}.
\end{aligned}
\tag{4.3.10}
$$

Theorem 4.3.3. *For positive even integers l, m, n, a, b and c with $a+b+c=m$, we have*

$$
\begin{aligned}
&\sum_{s=0}^{lmn-1}q^{\frac{s^2\pi^2\tau^2nm+6s\pi\tau(zmn+y)+6zy+z^2mn-z^2lmn}{\pi^2\tau^2lmn}}\theta_1^a\Big(nz+\frac{y}{a}+ns\pi\tau\Big|lmn^2\tau\Big)\\
&\theta_3^b\Big(nz+\frac{y}{b}+ns\pi\tau\Big|lmn^2\tau\Big)\theta_2^c\Big(nz+\frac{y}{c}+ns\pi\tau\Big|lmn^2\tau\Big)=H_{134}(a,b,c;y,\tau)\theta_3(z|\tau),
\end{aligned}
\tag{4.3.11}
$$

where

$$\begin{aligned}
&H_{134}(a,b,c;y,\tau)\\
&=\frac{i^{\frac{3-2a-2m}{2}}\tau^{\frac{1-m}{2}}}{(lmn^2)^{\frac{m}{2}}}q^{-\frac{y^2(ab+bc+ac)}{abclmn^2\pi^2\tau^2}}\mathscr{F}_{134}\Big(a,b,c;\frac{y}{abclmn^2\tau},-\frac{1}{lmn^2\tau}\Big)\theta_3(z|\tau) \qquad (4.3.12)\\
&=i^a\frac{\pi^2\tau^2l^2m^2n^3(a+c)+24zy+4z^2mn-4z^2lmn}{4\pi^2\tau^2lmn}\\
&\times\sum_{s=0}^{m-1}q^{\frac{4s^2l^2\pi^2\tau^2n^3m+24sln\pi\tau(zmn+y)}{4\pi^2\tau^2lmn}}\sum_{\substack{d_1,\cdots d_a,f_1,\cdots f_b,g_1,\cdots,g_c=-\infty\\(d_1+\cdots+d_a+f_1+\cdots+f_b+g_1+\cdots+g_c)+\frac{a+c}{2}=0}}^{+\infty}(-1)^{d_1+\cdots+d_a}\\
&\qquad\times q^{lmn^2(d_1^2+\cdots d_a^2+g_1^2+\cdots g_c^2+g_1^2+\cdots g_c^2+g_1+\cdots g_c)}\\
&\qquad\times e^{\frac{[2\{bc(d_1+\cdots+d_a)+ac(f_1+\cdots+f_b)+ab(g_1+\cdots+g_c)\}+abc]}{abc}iy}.
\end{aligned}$$

(4.3.13)

Theorem 4.3.4. *For positive even integers l, m, n, a, b and c with $a+b+c=m$, we have*

$$\begin{aligned}
&\sum_{s=0}^{lmn-1}q^{\frac{s^2\pi^2\tau^2nm+6s\pi\tau(zmn+y)+6zy+z^2mn-z^2lmn}{\pi^2\tau^2lmn}}\theta_4^a\Big(nz+\frac{y}{a}+ns\pi\tau\Big|lmn^2\tau\Big)\\
&\theta_3^b\Big(nz+\frac{y}{b}+ns\pi\tau\Big|lmn^2\tau\Big)\theta_2^c\Big(nz+\frac{y}{c}+ns\pi\tau\Big|lmn^2\tau\Big)=H_{234}(a,b,c;y,\tau)\theta_3(z|\tau),
\end{aligned}$$

(4.3.14)

where

$$\begin{aligned}
&H_{234}(a,b,c;y,\tau)\\
&=\frac{(-i\tau)^{\frac{1-m}{2}}}{(lmn^2)^{\frac{m}{2}}}q^{-\frac{y^2(ab+bc+ac)}{abclmn^2\pi^2\tau^2}}\mathscr{F}_{234}\Big(a,b,c;\frac{y}{abclmn^2\tau},-\frac{1}{lmn^2\tau}\Big)\theta_3(z|\tau) \qquad (4.3.15)\\
&=\frac{\pi^2\tau^2l^2m^2n^3c+24zy+4z^2mn-4z^2lmn}{4\pi^2\tau^2lmn}
\end{aligned}$$

$$\times \sum_{s=0}^{m-1} q^{\frac{4s^2l^2\pi^2\tau^2n^3m+24sln\pi\tau(zmn+y)}{4\pi^2\tau^2lmn}} \sum_{\substack{d_1,\cdots d_a,f_1,\cdots f_b,g_1,\cdots,g_c=-\infty \\ (d_1+\cdots+d_a+f_1+\cdots+f_b+g_1+\cdots+g_c)+\frac{c}{2}=0}}^{+\infty} (-1)^{d_1+\cdots+d_a}$$

$$\times q^{lmn^2(d_1^2+\cdots d_a^2+g_1^2+\cdots g_c^2+g_1^2+\cdots g_c^2+d_1+\cdots d_a+f_1+\cdots+f_b)}$$

$$\times e^{\frac{[2\{bc(d_1+\cdots+d_a)+ac(f_1+\cdots+f_b)+ab(g_1+\cdots+g_c)\}+abc]}{abc}iy}. \tag{4.3.16}$$

Theorem 4.3.5. *For positive even integers k, l, m, n, a, b, c and d with $a+b+c+d=m$, we have*

$$\sum_{s=0}^{klmn-1} q^{\frac{s^2\pi^2\tau^2nm+8ys\pi\tau+z^2mn+8zny+2zs\pi\tau mn-z^2klmn}{\pi^2\tau^2lmn}} \theta_1^a\Big(nz+\frac{y}{a}+ns\pi\tau\Big|klmn^2\tau\Big)$$
$$\theta_4^b\Big(nz+\frac{y}{b}+ns\pi\tau\Big|klmn^2\tau\Big)\theta_3^c\Big(nz+\frac{y}{c}+ns\pi\tau\Big|klmn^2\tau\Big)$$
$$\theta_2^d\Big(nz+\frac{y}{d}+ns\pi\tau\Big|klmn^2\tau\Big) = H_{1234}(a,b,c,d;y,\tau)\theta_3(z|\tau), \tag{4.3.17}$$

where

$$H_{1234}(a,b,c,d;y,\tau)$$
$$= \frac{(-i)^{\frac{1-2a-m}{2}}\tau^{\frac{1-m}{2}}}{(klmn^2)^{\frac{m}{2}}} q^{-\frac{y^2(abc+bcd+acd+abd)}{abcdklmn^2\pi^2\tau^2}} \mathscr{F}_{1234}\Big(a,b,c,d;\frac{y}{abcdklmn^2\tau},-\frac{1}{klmn^2\tau}\Big)\theta_3(z|\tau) \tag{4.3.18}$$

$$= \frac{\pi^2\tau^2k^2l^2m^2n^3(a+b)+8z\pi\tau mn+4z^2mn+32zny-4z^2klmn}{4\pi^2\tau^2klmn}$$
$$\times \sum_{s=0}^{m-1} q^{\frac{4s^2k^2l^2\pi^2\tau^2n^3m+32y\pi\tau skln+skln}{4\pi^2\tau^2klmn}} \sum_{\substack{d_1,\cdots d_a,f_1,\cdots f_b,g_1,\cdots,g_c,h_1,\cdots,h_d=-\infty \\ (d_1+\cdots+d_a+f_1+\cdots+f_b+g_1+\cdots+g_c+h_1+\cdots+h_d)+\frac{a+b}{2}=0}}^{+\infty}$$
$$\times (-1)^{d_1+\cdots+d_a+h_1+\cdots+h_d}$$

$$
\times\, q^{klmn^2(d_1^2+\cdots d_a^2+f_1^2+\cdots f_b^2+g_1^2+\cdots g_c^2+h_1^2+\cdots h_d^2+d_1+\cdots d_a+f_1+\cdots+f_b)}
$$

$$
\times\, e^{\frac{2\{bcd(d_1+\cdots+d_a)+acd(f_1+\cdots+f_b)+abd(g_1+\cdots+g_c)+abc(h_1+\cdots+h_d)+abcd\}}{abcd}iy}. \tag{4.3.19}
$$

4.4 RESULTS OBTAINED AS DIFFERENCE OF JACOBI THETA FUNCTIONS

L. C. Shen has obtained the Fourier series expansion of triple product of Jacobi theta functions in [100]. These expansion can be converted into difference of theta functions as follows:

Lemma 4.4.1. *From* (1.1.10), (1.1.11), (1.1.12) *and* (1.1.13), *we have*

$$\theta_1(z|\tau)\theta_2(z|\tau)\theta_3(z|\tau) = \\ -iq^{3/2}(q^2;q^2)_\infty^2\{e^{41z}\theta_4(3z+2\pi\tau|3\tau) - e^{-4iz}\theta_4(3z-2\pi\tau|3\tau)\}\,, \tag{4.4.1}$$

$$\theta_1(z|\tau)\theta_2(z|\tau)\theta_4(z|\tau) = \\ iq^{3/2}(q^2;q^2)_\infty^2\{e^{41z}\theta_3(3z+2\pi\tau|3\tau) - e^{-4iz}\theta_3(3z-2\pi\tau|3\tau)\}\,, \tag{4.4.2}$$

$$\theta_1(z|\tau)\theta_3(z|\tau)\theta_4(z|\tau) = \\ iq^{5/4}(q^2;q^2)_\infty^2\{e^{21z}\theta_2(3z+\pi\tau|3\tau) - e^{-2iz}\theta_2(3z-\pi\tau|3\tau)\}\,, \tag{4.4.3}$$

$$\theta_2(z|\tau)\theta_3(z|\tau)\theta_4(z|\tau) = \\ -iq^{1/2}(q^2;q^2)_\infty^2\{e^{21z}\theta_1(3z+\pi\tau|3\tau) - e^{-2iz}\theta_1(3z-\pi\tau|3\tau)\}\,. \tag{4.4.4}$$

Proof: We have from [100, Proposition 2.1]

$$\begin{aligned}\theta_1(z|\tau)\theta_2(z|\tau)\theta_3(z|\tau) &= 2q^{3/2}(q^2;q^2)_\infty^2\sum_{n=-\infty}^{\infty}(-1)^n q^{3n^2+4n}\sin(6n+4)z \\ &= -iq^{3/2}(q^2;q^2)_\infty^2\sum_{n=-\infty}^{\infty}(-1)^n q^{3n^2+4n}\Big(e^{(6n+4)iz}-e^{-(6n+4)iz}\Big) \\ &= -iq^{3/2}(q^2;q^2)_\infty^2\Big(e^{4iz}\sum_{n=-\infty}^{\infty}(-1)^n q^{3n^2}e^{2ni(2\pi\tau+3z)}\end{aligned}$$

$$- e^{-4iz} \sum_{n=-\infty}^{\infty} (-1)^n q^{3n^2} e^{2ni(2\pi\tau-3z)}\Big). \tag{4.4.5}$$

We obtain (4.4.1) by replacing n by $-n$ in second summation and by the definition of $\theta_4(z|\tau)$ (1.1.13). Similarly, (4.4.2) and (4.4.4) can be proved. Now

$$\begin{aligned}
\theta_1(z|\tau)\theta_3(z|\tau)\theta_4(z|\tau) &= 2q^{1/4}(q^2;q^2)_\infty^2 \sum_{n=-\infty}^{\infty} q^{3n^2+4n}\sin(6n+1)z \\
&= iq^{1/4}(q^2;q^2)_\infty^2\Big(\sum_{n=-\infty}^{\infty} q^{3n^2+n}e^{-(6n+1)iz} - \sum_{n=-\infty}^{\infty} q^{3n^2+n}e^{(6n+1)iz}\Big) \\
&= iq^{5/4}(q^2;q^2)_\infty^2\Big(e^{2iz}\sum_{n=-\infty}^{\infty} q^{3n^2-3n}e^{i(2n-1)(3z+\pi\tau)} \\
&- e^{-2iz}\sum_{n=-\infty}^{\infty} q^{3n^2-3n}e^{i(2n-1)(3z-\pi\tau)}\Big).
\end{aligned} \tag{4.4.6}$$

(4.4.3) is achieved by replacing n by $-n$ in the first summation and n by $n-1$ in the second summation and employing (1.1.11). □

In this Section, we obtain some new results by employing lemma 4.4.1.

Theorem 4.4.2. *Setting $a = b = c = t$ in (4.2.1) and employing (4.4.1), we obtain*

$$\sum_{s=0}^{lmn-1}\begin{pmatrix} e^{4i(\frac{z}{lmn}+\frac{y}{t}+\frac{\pi s}{lmn})}\theta_4\Big(3(\frac{z}{lmn}+\frac{y}{t}+\frac{\pi s}{lmn})+\frac{2\pi\tau}{lmn^2}\big|\frac{3\tau}{lmn^2}\Big) \\ -e^{-4i(\frac{z}{lmn}+\frac{y}{t}+\frac{\pi s}{lmn})}\theta_4\Big(3(\frac{z}{lmn}+\frac{y}{t}+\frac{\pi s}{lmn})-\frac{2\pi\tau}{lmn^2}\big|\frac{3\tau}{lmn^2}\Big)\end{pmatrix}^t$$
$$= \mathscr{K}_{123}\Big(t,t,t;\frac{y}{t^3},\frac{\tau}{lmn^2}\Big)\theta_3(z|\tau), \tag{4.4.7}$$

where

$$\mathscr{K}_{123}(t,t,t;y,\tau) = \frac{lmn}{q^t(q^2;q^2)_\infty^{2t}} \sum_{\substack{u_1,\cdots u_t,v_1,\cdots v_t,w_1,\cdots w_t=-\infty \\ u_1+\cdots+u_t+v_1+\cdots+v_t+w_1+\cdots+w_t+t=0}}^{+\infty} (-1)^{u_1+\cdots+u_t+w_1+\cdots+w_t}$$
$$\times q^{u_1^2+\cdots+u_t^2+v_1^2+\cdots+v_t^2+w_1^2+\cdots w_t^2+u_1+\cdots+u_t+v_1+\cdots+v_t}$$
$$\times e^{2\{t^2(u_1+\cdots+u_t+v_1+\cdots+v_t+w_1+\cdots+w_t)+t\}iy}. \tag{4.4.8}$$

Theorem 4.4.3. *Setting* $a=b=c=r$ *in* (4.2.9) *and employing* (4.4.2), *we obtain*

$$\sum_{s=0}^{lmn-1}\left(\begin{array}{c} e^{4i(\frac{z}{lmn}+\frac{y}{r}+\frac{\pi s}{lmn})}\theta_3\Big(3(\frac{z}{lmn}+\frac{y}{r}+\frac{\pi s}{lmn})+\frac{2\pi\tau}{lmn^2}\big|\frac{3\tau}{lmn^2}\Big) \\ -e^{-4i(\frac{z}{lmn}+\frac{y}{r}+\frac{\pi s}{lmn})}\theta_3\Big(3(\frac{z}{lmn}+\frac{y}{r}+\frac{\pi s}{lmn})-\frac{2\pi\tau}{lmn^2}\big|\frac{3\tau}{lmn^2}\Big)\end{array}\right)^r$$
$$= \mathscr{K}_{124}\Big(r,r,r;\frac{y}{r^3},\frac{\tau}{lmn^2}\Big)\theta_3(z|\tau), \tag{4.4.9}$$

where

$$\mathscr{K}_{124}(r,r,r;y,\tau) =$$
$$\frac{lmn}{q^{r/2}(q^2;q^2)_\infty^{2r}} \sum_{\substack{u_1,\cdots u_r,v_1,\cdots v_r,w_1,\cdots w_r=-\infty \\ u_1+\cdots+u_r+v_1+\cdots+v_r+w_1+\cdots+w_r+1=0}}^{+\infty} (-1)^{u_1+\cdots+u_r+w_1+\cdots+w_r}$$
$$\times q^{u_1^2+\cdots+u_r^2+v_1^2+\cdots+v_r^2+w_1^2+\cdots w_r^2+u_1+\cdots u_r+v_1+\cdots+v_r}$$
$$\times e^{2\{r^2(u_1+\cdots+u_r+v_1+\cdots+v_r+w_1+\cdots+w_r)+r\}iy}. \tag{4.4.10}$$

Theorem 4.4.4. *Setting $a = b = c = p$ in (4.2.11) and employing (4.4.3), we obtain*

$$\sum_{s=0}^{lmn-1}\begin{pmatrix} e^{2i(\frac{z}{lmn}+\frac{y}{p}+\frac{\pi s}{lmn})}\theta_2\Big(3(\frac{z}{lmn}+\frac{y}{p}+\frac{\pi s}{lmn})+\frac{\pi\tau}{lmn^2}\big|\frac{3\tau}{lmn^2}\Big) \\ -e^{-2i(\frac{z}{lmn}+\frac{y}{p}+\frac{\pi s}{lmn})}\theta_2\Big(3(\frac{z}{lmn}+\frac{y}{p}+\frac{\pi s}{lmn})-\frac{\pi\tau}{lmn^2}\big|\frac{3\tau}{lmn^2}\Big)\end{pmatrix}^p$$
$$= \mathscr{K}_{134}\Big(p,p,p;\frac{y}{p^3},\frac{\tau}{lmn^2}\Big)\theta_3(z|\tau), \qquad (4.4.11)$$

where

$$\mathscr{K}_{134}(p,p,p;y,\tau)$$
$$= \frac{lmn}{q^p(q^2;q^2)_\infty^{2p}} \sum_{\substack{u_1,\cdots u_p,v_1,\cdots v_p,w_1,\cdots w_p=-\infty \\ u_1+\cdots+u_p+v_1+\cdots+v_p+w_1+\cdots+w_p+\frac{p}{2}=0}}^{+\infty} (-1)^{u_1+\cdots+u_p+w_1+\cdots+w_p}$$
$$\times q^{u_1^2+\cdots u_p^2+v_1^2+\cdots+v_p^2+w_1^2+\cdots w_p^2+u_1+\cdots u_p+v_1+\cdots+v_p}$$
$$\times e^{2\{p^2(u_1+\cdots+u_p+v_1+\cdots+v_p+w_1+\cdots+w_p)+p\}iy}. \qquad (4.4.12)$$

Theorem 4.4.5. *Setting $a = b = c = x$ in (4.2.13) and employing (4.4.4), we obtain*

$$\sum_{s=0}^{lmn-1}\begin{pmatrix} e^{2i(\frac{z}{lmn}+\frac{y}{x}+\frac{\pi s}{lmn})}\theta_1\Big(3(\frac{z}{lmn}+\frac{y}{x}+\frac{\pi s}{lmn})+\frac{\pi\tau}{lmn^2}\big|\frac{3\tau}{lmn^2}\Big) \\ -e^{-2i(\frac{z}{lmn}+\frac{y}{x}+\frac{\pi s}{lmn})}\theta_1\Big(3(\frac{z}{lmn}+\frac{y}{x}+\frac{\pi s}{lmn})-\frac{\pi\tau}{lmn^2}\big|\frac{3\tau}{lmn^2}\Big)\end{pmatrix}^x$$
$$= \mathscr{K}_{234}\Big(x,x,x;\frac{y}{x^3},\frac{\tau}{lmn^2}\Big)\theta_3(z|\tau), \qquad (4.4.13)$$

where

$$
\begin{aligned}
&\mathscr{K}_{234}(x,x,x;y,\tau) \\
&= \frac{lmn}{i^x(q^2;q^2)_\infty^{2x}} \sum_{\substack{u_1,\cdots,u_x,v_1,\cdots v_x,w_1,\cdots,w_x=-\infty \\ u_1+\cdots+u_x+v_1+\cdots+v_x+w_1+\cdots+w_x+\frac{x}{2}=0}}^{+\infty} (-1)^{u_1+\cdots+u_x+w_1+\cdots+w_x} \\
&\qquad \times q^{u_1^2+\cdots u_x^2+v_1^2+\cdots+v_x^2+w_1^2+\cdots w_x^2+u_1+\cdots u_x+v_1+\cdots+v_x} \\
&\qquad \times e^{2\{x^2(u_1+\cdots+u_x+v_1+\cdots+v_x+w_1+\cdots+w_x)+x\}iy}. \qquad (4.4.14)
\end{aligned}
$$

4.5 APPLICATION

In this Section, we gives some special cases of Theorems 4.2.1-4.2.5 and obtain some interesting identities of theta functions.

Corollary 4.5.1. *For l and n positive even integers, we have*

$$\sum_{s=0}^{6ln-1} \theta_1^2\Big(\frac{z}{6ln}+\frac{y}{2}+\frac{\pi s}{6ln}\Big|\frac{\tau}{6ln^2}\Big)\theta_2^2\Big(\frac{z}{6ln}+\frac{y}{2}+\frac{\pi s}{6ln}\Big|\frac{\tau}{6ln^2}\Big)$$
$$\times\theta_3^2\Big(\frac{z}{6ln}+\frac{y}{2}+\frac{\pi s}{6ln}\Big|\frac{\tau}{6ln^2}\Big) = 6ln\theta_1^2\Big(0\Big|\frac{\tau}{6ln^2}\Big)\theta_2^2\Big(0\Big|\frac{\tau}{6ln^2}\Big)\theta_3^2\Big(0|\frac{\tau}{6ln^2}\Big)\theta_3(z|\tau) \tag{4.5.1}$$

Proof: Setting $a=b=c=2$ in (4.2.1)

$$\begin{aligned}&\mathscr{F}_{123}(2,2,2;y,\tau)\\ &= -6lnq \sum_{\substack{u_1,u_2,v_1,v_2,w_1,w_2=-\infty\\ u_1+u_2+v_1+v_2+w_1+w_2+2=0}}^{+\infty} (-1)^{u_1+u_a} q^{u_1^2+u_2^2+v_1^2+v_2^2+w_1^2+w_2^2+u_1+u_2+v_1+v_2}\\ &\qquad\times e^{8\{(u_1+u_1)+(v_1+v_2)+(w_1+w_2)+2\}iy}\\ &= -6lnq \sum_{u_1,u_2,v_1,v_2,w_1,w_2=-\infty}^{+\infty} (-1)^{u_1+u_a} q^{u_1^2+u_2^2+v_1^2+v_2^2+w_1^2+w_2^2+u_1+u_2+v_1+v_2}\\ &= 6ln\theta_1^2(0|\tau)\theta_2^2(0|\tau)\theta_3^2(0|\tau).\end{aligned} \tag{4.5.2}$$

Changing τ by $\frac{\tau}{6ln^2}$ in (4.5.2), we obtain Corollary 4.5.1. □

Setting $l=n=1$ in Corollary 4.5.1,

$$\sum_{s=0}^{5} \theta_1^2\Big(\frac{z}{6}+\frac{y}{2}+\frac{\pi s}{6}\Big|\frac{\tau}{6}\Big)\theta_2^2\Big(\frac{z}{6}+\frac{y}{2}+\frac{\pi s}{6}\Big|\frac{\tau}{6}\Big)\theta_3^2\Big(\frac{z}{6}+\frac{y}{2}+\frac{\pi s}{6}\Big|\frac{\tau}{6}\Big)$$

$$= 6\theta_1^2\Big(0\Big|\frac{\tau}{6}\Big)\theta_2^2\Big(0\Big|\frac{\tau}{6}\Big)\theta_3^2\Big(0|\frac{\tau}{6}\Big)\theta_3(z|\tau). \tag{4.5.3}$$

Setting $z \longmapsto 6z$, $y \longmapsto 2y$ and $\tau \longmapsto 6\tau$ in (4.5.3), we obtain

$$\begin{aligned}
&\theta_1^2(z+y|\tau)\theta_2^2(z+y|\tau)\theta_3^2(z+y|\tau) + \theta_1^2\Big(z+y+\frac{\pi}{6}\Big|\tau\Big)\theta_2^2\Big(z+y+\frac{\pi}{6}\Big|\tau\Big)\\
&\theta_3^2\Big(z+y+\frac{\pi}{6}\Big|\tau\Big) + \theta_1^2\Big(z+y+\frac{\pi}{3}\Big|\tau\Big)\theta_2^2\Big(z+y+\frac{\pi}{3}\Big|\tau\Big)\theta_3^2\Big(z+y+\frac{\pi}{3}\Big|\tau\Big)+\\
&\theta_1^2\Big(z+y+\frac{\pi}{2}\Big|\tau\Big)\theta_2^2\Big(z+y+\frac{\pi}{2}\Big|\tau\Big)\theta_3^2\Big(z+y+\frac{\pi}{2}\Big|\tau\Big) + \theta_1^2\Big(z+y+\frac{2\pi}{3}\Big|\tau\Big)\\
&\theta_2^2\Big(z+y+\frac{2\pi}{3}\Big|\tau\Big)\theta_3^2\Big(z+y+\frac{2\pi}{3}\Big|\tau\Big) + \theta_1^2\Big(z+y+\frac{5\pi}{6}\Big|\tau\Big)\theta_2^2\Big(z+y+\frac{5\pi}{6}\Big|\tau\Big)\\
&\theta_3^2\Big(z+y+\frac{5\pi}{6}\Big|\tau\Big) = 6\theta_1^2(0|\tau)\theta_2^2(0|\tau)\theta_3^2(0|\tau)\theta_3(6z|6\tau).
\end{aligned} \tag{4.5.4}$$

$$\begin{aligned}
&\theta_1^2(z+y|\tau)\theta_2^2(z+y|\tau)\theta_3^2(z+y|\tau) + \theta_1^2\Big(z+y+\frac{\pi}{6}\Big|\tau\Big)\theta_2^2\Big(z+y+\frac{\pi}{6}\Big|\tau\Big)\\
&\theta_3^2\Big(z+y+\frac{\pi}{6}\Big|\tau\Big) + \theta_1^2\Big(z+y+\frac{\pi}{3}\Big|\tau\Big)\theta_2^2\Big(z+y+\frac{\pi}{3}\Big|\tau\Big)\theta_3^2\Big(z+y+\frac{\pi}{3}\Big|\tau\Big)+\\
&\theta_1^2(z+y|\tau)\theta_2^2(z+y|\tau)\theta_4^2(z+y|\tau) + \theta_1^2\Big(z+y-\frac{\pi}{3}\Big|\tau\Big)\theta_2^2\Big(z+y-\frac{\pi}{3}\Big|\tau\Big)\\
&\theta_3^2\Big(z+y-\frac{\pi}{3}\Big|\tau\Big) + \theta_1^2\Big(z+y-\frac{\pi}{6}\Big|\tau\Big)\theta_2^2\Big(z+y-\frac{\pi}{6}\Big|\tau\Big)\theta_3^2\Big(z+y-\frac{\pi}{6}\Big|\tau\Big)\\
&= 6\theta_1^2(0|\tau)\theta_2^2(0|\tau)\theta_3^2(0|\tau)\theta_3(6z|6\tau).
\end{aligned} \tag{4.5.5}$$

$$\theta_1^2(z+y|\tau)\theta_2^2(z+y|\tau)\theta_3^2(z+y|\tau)+\theta_1^2(z+y|\tau)\theta_2^2(z+y|\tau)\theta_4^2(z+y|\tau)$$
$$+\theta_1^2\Big(z+y-\frac{\pi}{3}\Big|\tau\Big)\theta_2^2\Big(z+y-\frac{\pi}{3}\Big|\tau\Big)\theta_4^2\Big(z+y-\frac{\pi}{3}\Big|\tau\Big)+\theta_1^2\Big(z+y+\frac{\pi}{3}\Big|\tau\Big)$$
$$\theta_2^2\Big(z+y+\frac{\pi}{3}\Big|\tau\Big)\theta_3^2\Big(z+y+\frac{\pi}{3}\Big|\tau\Big)+\theta_1^2\Big(z+y-\frac{\pi}{3}\Big|\tau\Big)\theta_2^2\Big(z+y-\frac{\pi}{3}\Big|\tau\Big)$$
$$\theta_3^2\Big(z+y-\frac{\pi}{3}\Big|\tau\Big)+\theta_1^2\Big(z+y-\frac{\pi}{6}\Big|\tau\Big)\theta_2^2\Big(z+y-\frac{\pi}{6}\Big|\tau\Big)\theta_3^2\Big(z+y-\frac{\pi}{6}\Big|\tau\Big)$$
$$=6\theta_1^2(0|\tau)\theta_2^2(0|\tau)\theta_3^2(0|\tau)\theta_3(6z|6\tau). \tag{4.5.6}$$

Setting $y=0$ in (4.5.5) and (4.5.6), we obtain

$$\theta_1^2\Big(z+\frac{\pi}{6}\Big|\tau\Big)\theta_2^2\Big(z+\frac{\pi}{6}\Big|\tau\Big)\theta_3^2\Big(z+\frac{\pi}{6}\Big|\tau\Big)+\theta_1^2\Big(z-\frac{\pi}{6}\Big|\tau\Big)\theta_2^2\Big(z-\frac{\pi}{6}\Big|\tau\Big)\theta_3^2\Big(z-\frac{\pi}{6}\Big|\tau\Big)$$
$$=\theta_1^2\Big(z-\frac{\pi}{3}\Big|\tau\Big)\theta_2^2\Big(z-\frac{\pi}{3}\Big|\tau\Big)\theta_4^2\Big(z-\frac{\pi}{3}\Big|\tau\Big)+\theta_1^2\Big(z-\frac{\pi}{3}\Big|\tau\Big)\theta_2^2\Big(z-\frac{\pi}{3}\Big|\tau\Big)\theta_4^2\Big(z-\frac{\pi}{3}\Big|\tau\Big). \tag{4.5.7}$$

Taking $a=b=c=2$ in (4.2.9), we obtain

Corollary 4.5.2. *For l and n positive even integers, we have*

$$\sum_{s=0}^{6ln-1}\theta_1^2\Big(\frac{z}{6ln}+\frac{y}{2}+\frac{\pi s}{6ln}\Big|\frac{\tau}{6ln^2}\Big)\theta_2^2\Big(\frac{z}{6ln}+\frac{y}{2}+\frac{\pi s}{6ln}\Big|\frac{\tau}{6ln^2}\Big)\theta_4^2\Big(\frac{z}{6ln}+\frac{y}{2}+\frac{\pi s}{6ln}\Big|\frac{\tau}{6ln^2}\Big)$$
$$=6ln\theta_1^2\Big(0\Big|\frac{\tau}{6ln^2}\Big)\theta_2^2\Big(0\Big|\frac{\tau}{6ln^2}\Big)\theta_4^2\Big(0|\frac{\tau}{6ln^2}\Big)\theta_3(z|\tau). \tag{4.5.8}$$

Setting $l = n = 1$ in Corollary 4.5.2 and then $z \longmapsto 6z$, $y \longmapsto 2y$ and $\tau \longmapsto 6\tau$, we obtain

$$\theta_1^2(z+y|\tau)\theta_2^2(z+y|\tau)\theta_4^2(z+y|\tau) + \theta_1^2\Big(z+y+\frac{\pi}{6}\Big|\tau\Big)\theta_2^2\Big(z+y+\frac{\pi}{6}\Big|\tau\Big)$$
$$\theta_4^2\Big(z+y+\frac{\pi}{6}\Big|\tau\Big) + \theta_1^2\Big(z+y+\frac{\pi}{3}\Big|\tau\Big)\theta_2^2\Big(z+y+\frac{\pi}{3}\Big|\tau\Big)\theta_4^2\Big(z+y+\frac{\pi}{3}\Big|\tau\Big)+$$
$$\theta_1^2\Big(z+y+\frac{\pi}{2}\Big|\tau\Big)\theta_2^2\Big(z+y+\frac{\pi}{2}\Big|\tau\Big)\theta_4^2\Big(z+y+\frac{\pi}{2}\Big|\tau\Big) + \theta_1^2\Big(z+y+\frac{2\pi}{3}\Big|\tau\Big)$$
$$\theta_2^2\Big(z+y+\frac{2\pi}{3}\Big|\tau\Big)\theta_4^2\Big(z+y+\frac{2\pi}{3}\Big|\tau\Big) + \theta_1^2\Big(z+y+\frac{5\pi}{6}\Big|\tau\Big)\theta_2^2\Big(z+y+\frac{5\pi}{6}\Big|\tau\Big)$$
$$\theta_4^2\Big(z+y+\frac{5\pi}{6}\Big|\tau\Big) = 6\theta_1^2(0|\tau)\theta_2^2(0|\tau)\theta_4^2(0|\tau)\theta_3(6z|6\tau). \tag{4.5.9}$$

$$\theta_1^2(z+y|\tau)\theta_2^2(z+y|\tau)\theta_4^2(z+y|\tau) + \theta_1^2(z+y|\tau)\theta_2^2(z+y|\tau)\theta_3^2\Big(z+y|\tau\Big)$$
$$+\theta_1^2\Big(z+y-\frac{\pi}{3}\Big|\tau\Big)\theta_2^2\Big(z+y-\frac{\pi}{3}\Big|\tau\Big)\theta_3^2\Big(z+y-\frac{\pi}{3}\Big|\tau\Big) + \theta_1^2\Big(z+y+\frac{\pi}{3}\Big|\tau\Big)$$
$$\theta_2^2\Big(z+y+\frac{\pi}{3}\Big|\tau\Big)\theta_4^2\Big(z+y+\frac{\pi}{3}\Big|\tau\Big) + \theta_1^2\Big(z+y-\frac{\pi}{3}\Big|\tau\Big)\theta_2^2\Big(z+y-\frac{\pi}{3}\Big|\tau\Big)$$
$$\theta_4^2\Big(z+y-\frac{\pi}{3}\Big|\tau\Big) + \theta_1^2\Big(z+y-\frac{\pi}{6}\Big|\tau\Big)\theta_2^2\Big(z+y-\frac{\pi}{6}\Big|\tau\Big)\theta_4^2\Big(z+y-\frac{\pi}{6}\Big|\tau\Big)$$
$$= 6\theta_1^2(0|\tau)\theta_2^2(0|\tau)\theta_4^2(0|\tau)\theta_3(6z|6\tau). \tag{4.5.10}$$

Setting $y = 0$ in (4.5.10), we obtain

$$\theta_1^2(z|\tau)\theta_2^2(z|\tau)\theta_4^2(z|\tau) + \theta_1^2(z|\tau)\theta_2^2(z|\tau)\theta_3^2\Big(z|\tau\Big)$$
$$+\theta_1^2\Big(z-\frac{\pi}{3}\Big|\tau\Big)\theta_2^2\Big(z-\frac{\pi}{3}\Big|\tau\Big)\theta_3^2\Big(z-\frac{\pi}{3}\Big|\tau\Big) + \theta_1^2\Big(z+\frac{\pi}{3}\Big|\tau\Big)\theta_2^2\Big(z+\frac{\pi}{3}\Big|\tau\Big)\theta_4^2\Big(z+\frac{\pi}{3}\Big|\tau\Big)$$
$$+\theta_1^2\Big(z-\frac{\pi}{3}\Big|\tau\Big)\theta_2^2\Big(z-\frac{\pi}{3}\Big|\tau\Big)\theta_4^2\Big(z-\frac{\pi}{3}\Big|\tau\Big) + \theta_1^2\Big(z-\frac{\pi}{6}\Big|\tau\Big)\theta_2^2\Big(z-\frac{\pi}{6}\Big|\tau\Big)\theta_4^2\Big(z-\frac{\pi}{6}\Big|\tau\Big)$$
$$= 6\theta_1^2(0|\tau)\theta_2^2(0|\tau)\theta_4^2(0|\tau)\theta_3(6z|6\tau). \tag{4.5.11}$$

Taking $a=b=c=2$ in (4.2.11), we obtain

Corollary 4.5.3. *For l and n positive even integers, we have*

$$\sum_{s=0}^{6ln-1}\theta_1^2\Big(\frac{z}{6ln}+\frac{y}{2}+\frac{\pi s}{6ln}\Big|\frac{\tau}{6ln^2}\Big)\theta_3^2\Big(\frac{z}{6ln}+\frac{y}{2}+\frac{\pi s}{6ln}\Big|\frac{\tau}{6ln^2}\Big)\theta_4^2\Big(\frac{z}{6ln}+\frac{y}{2}+\frac{\pi s}{6ln}\Big|\frac{\tau}{6ln^2}\Big)$$
$$=6ln\theta_1^2\Big(0\Big|\frac{\tau}{6ln^2}\Big)\theta_3^2\Big(0\Big|\frac{\tau}{6ln^2}\Big)\theta_4^2\Big(0|\frac{\tau}{6ln^2}\Big)\theta_3(z|\tau). \tag{4.5.12}$$

Setting $l=n=1$ in Corollary 4.5.3 and then $z\longmapsto 6z$, $y\longmapsto 2y$ and $\tau\longmapsto 6\tau$, we obtain

$$\theta_1^2(z+y|\tau)\theta_3^2(z+y|\tau)\theta_4^2(z+y|\tau)+\theta_1^2\Big(z+y+\frac{\pi}{6}\Big|\tau\Big)\theta_3^2\Big(z+y+\frac{\pi}{6}\Big|\tau\Big)$$
$$\theta_4^2\Big(z+y+\frac{\pi}{6}\Big|\tau\Big)+\theta_1^2\Big(z+y+\frac{\pi}{3}\Big|\tau\Big)\theta_3^2\Big(z+y+\frac{\pi}{3}\Big|\tau\Big)\theta_4^2\Big(z+y+\frac{\pi}{3}\Big|\tau\Big)$$
$$+\theta_1^2\Big(z+y+\frac{\pi}{2}\Big|\tau\Big)\theta_3^2\Big(z+y+\frac{\pi}{2}\Big|\tau\Big)\theta_4^2\Big(z+y+\frac{\pi}{2}\Big|\tau\Big)+\theta_1^2\Big(z+y+\frac{2\pi}{3}\Big|\tau\Big)$$
$$\theta_3^2\Big(z+y+\frac{2\pi}{3}\Big|\tau\Big)\theta_4^2\Big(z+y+\frac{2\pi}{3}\Big|\tau\Big)+\theta_1^2\Big(z+y+\frac{5\pi}{6}\Big|\tau\Big)\theta_3^2\Big(z+y+\frac{5\pi}{6}\Big|\tau\Big)$$
$$\theta_4^2\Big(z+y+\frac{5\pi}{6}\Big|\tau\Big)=6\theta_1^2(0|\tau)\theta_3^2(0|\tau)\theta_4^2(0|\tau)\theta_3(6z|6\tau). \tag{4.5.13}$$

$$\theta_1^2(z+y|\tau)\theta_3^2(z+y|\tau)\theta_4^2(z+y|\tau)+\theta_2^2(z+y|\tau)\theta_3^2(z+y|\tau)\theta_4^2\Big(z+y|\tau)$$
$$+\theta_2^2\Big(z+y-\frac{\pi}{3}\Big|\tau\Big)\theta_3^2\Big(z+y-\frac{\pi}{3}\Big|\tau\Big)\theta_4^2\Big(z+y-\frac{\pi}{3}\Big|\tau\Big)+\theta_1^2\Big(z+y+\frac{\pi}{3}\Big|\tau\Big)$$
$$\theta_3^2\Big(z+y+\frac{\pi}{3}\Big|\tau\Big)\theta_4^2\Big(z+y+\frac{\pi}{3}\Big|\tau\Big)+\theta_1^2\Big(z+y-\frac{\pi}{3}\Big|\tau\Big)\theta_3^2\Big(z+y-\frac{\pi}{3}\Big|\tau\Big)$$
$$\theta_4^2\Big(z+y-\frac{\pi}{3}\Big|\tau\Big)+\theta_1^2\Big(z+y-\frac{\pi}{6}\Big|\tau\Big)\theta_3^2\Big(z+y-\frac{\pi}{6}\Big|\tau\Big)\theta_4^2\Big(z+y-\frac{\pi}{6}\Big|\tau\Big)$$
$$=6\theta_1^2(0|\tau)\theta_3^2(0|\tau)\theta_4^2(0|\tau)\theta_3(6z|6\tau). \tag{4.5.14}$$

Setting $y=0$ in (4.5.14), we obtain

$$\theta_1^2(z|\tau)\theta_3^2(z|\tau)\theta_4^2(z|\tau)+\theta_2^2(z|\tau)\theta_3^2(z|\tau)\theta_4^2\Big(z|\tau)$$
$$+\theta_2^2\Big(z-\frac{\pi}{3}\Big|\tau\Big)\theta_3^2\Big(z-\frac{\pi}{3}\Big|\tau\Big)\theta_4^2\Big(z-\frac{\pi}{3}\Big|\tau\Big)+\theta_1^2\Big(z+\frac{\pi}{3}\Big|\tau\Big)\theta_3^2\Big(z+\frac{\pi}{3}\Big|\tau\Big)\theta_4^2\Big(z+\frac{\pi}{3}\Big|\tau\Big)$$
$$+\theta_1^2\Big(z-\frac{\pi}{3}\Big|\tau\Big)\theta_3^2\Big(z-\frac{\pi}{3}\Big|\tau\Big)\theta_4^2\Big(z-\frac{\pi}{3}\Big|\tau\Big)+\theta_1^2\Big(z-\frac{\pi}{6}\Big|\tau\Big)\theta_3^2\Big(z-\frac{\pi}{6}\Big|\tau\Big)\theta_4^2\Big(z-\frac{\pi}{6}\Big|\tau\Big)$$
$$=6\theta_1^2(0|\tau)\theta_3^2(0|\tau)\theta_4^2(0|\tau)\theta_3(6z|6\tau). \tag{4.5.15}$$

Taking $a=b=c=2$ in (4.2.13), we obtain

Corollary 4.5.4. *For l and n positive even integers, we have*

$$\sum_{s=0}^{6ln-1}\theta_2^2\Big(\frac{z}{6ln}+\frac{y}{2}+\frac{\pi s}{6ln}\Big|\frac{\tau}{6ln^2}\Big)\theta_3^2\Big(\frac{z}{6ln}+\frac{y}{2}+\frac{\pi s}{6ln}\Big|\frac{\tau}{6ln^2}\Big)\theta_4^2\Big(\frac{z}{6ln}+\frac{y}{2}+\frac{\pi s}{6ln}\Big|\frac{\tau}{6ln^2}\Big)$$
$$=6ln\theta_2^2\Big(0\Big|\frac{\tau}{6ln^2}\Big)\theta_3^2\Big(0\Big|\frac{\tau}{6ln^2}\Big)\theta_4^2\Big(0|\frac{\tau}{6ln^2}\Big)\theta_3(z|\tau). \tag{4.5.16}$$

Setting $l=n=1$ in Corollary 4.5.4 and then $z\longmapsto 6z$, $y\longmapsto 2y$ and $\tau\longmapsto 6\tau$, we obtain

$$\theta_2^2(z+y|\tau)\theta_3^2(z+y|\tau)\theta_4^2(z+y|\tau)+\theta_2^2\Big(z+y+\frac{\pi}{6}\Big|\tau\Big)\theta_3^2\Big(z+y+\frac{\pi}{6}\Big|\tau\Big)$$
$$\theta_4^2\Big(z+y+\frac{\pi}{6}\Big|\tau\Big)+\theta_2^2\Big(z+y+\frac{\pi}{3}\Big|\tau\Big)\theta_3^2\Big(z+y+\frac{\pi}{3}\Big|\tau\Big)\theta_4^2\Big(z+y+\frac{\pi}{3}\Big|\tau\Big)+$$
$$\theta_2^2\Big(z+y+\frac{\pi}{2}\Big|\tau\Big)\theta_3^2\Big(z+y+\frac{\pi}{2}\Big|\tau\Big)\theta_4^2\Big(z+y+\frac{\pi}{2}\Big|\tau\Big)+\theta_2^2\Big(z+y+\frac{2\pi}{3}\Big|\tau\Big)$$
$$\theta_3^2\Big(z+y+\frac{2\pi}{3}\Big|\tau\Big)\theta_4^2\Big(z+y+\frac{2\pi}{3}\Big|\tau\Big)+\theta_2^2\Big(z+y+\frac{5\pi}{6}\Big|\tau\Big)\theta_3^2\Big(z+y+\frac{5\pi}{6}\Big|\tau\Big)$$
$$\theta_4^2\Big(z+y+\frac{5\pi}{6}\Big|\tau\Big)=6\theta_2^2(0|\tau)\theta_3^2(0|\tau)\theta_4^2(0|\tau)\theta_3(6z|6\tau). \tag{4.5.17}$$

$$\theta_2^2(z+y|\tau)\theta_3^2(z+y|\tau)\theta_4^2(z+y|\tau)+\theta_1^2(z+y|\tau)\theta_3^2(z+y|\tau)\theta_4^2\Big(z+y|\tau)$$
$$+\theta_1^2\Big(z+y-\frac{\pi}{3}\Big|\tau\Big)\theta_3^2\Big(z+y-\frac{\pi}{3}\Big|\tau\Big)\theta_4^2\Big(z+y-\frac{\pi}{3}\Big|\tau\Big)+\theta_2^2\Big(z+y+\frac{\pi}{3}\Big|\tau\Big)$$
$$\theta_3^2\Big(z+y+\frac{\pi}{3}\Big|\tau\Big)\theta_4^2\Big(z+y+\frac{\pi}{3}\Big|\tau\Big)+\theta_2^2\Big(z+y-\frac{\pi}{3}\Big|\tau\Big)\theta_3^2\Big(z+y-\frac{\pi}{3}\Big|\tau\Big)$$
$$\theta_4^2\Big(z+y-\frac{\pi}{3}\Big|\tau\Big)+\theta_2^2\Big(z+y-\frac{\pi}{6}\Big|\tau\Big)\theta_3^2\Big(z+y-\frac{\pi}{6}\Big|\tau\Big)\theta_4^2\Big(z+y-\frac{\pi}{6}\Big|\tau\Big)$$
$$=6\theta_2^2(0|\tau)\theta_3^2(0|\tau)\theta_4^2(0|\tau)\theta_3(6z|6\tau). \tag{4.5.18}$$

Setting $y=0$ in (4.5.18), we obtain

$$\theta_2^2(z|\tau)\theta_3^2(z|\tau)\theta_4^2(z|\tau)+\theta_1^2(z|\tau)\theta_3^2(z|\tau)\theta_4^2\Big(z|\tau)$$
$$+\theta_1^2\Big(z-\frac{\pi}{3}\Big|\tau\Big)\theta_3^2\Big(z-\frac{\pi}{3}\Big|\tau\Big)\theta_4^2\Big(z-\frac{\pi}{3}\Big|\tau\Big)+\theta_2^2\Big(z+\frac{\pi}{3}\Big|\tau\Big)\theta_3^2\Big(z+\frac{\pi}{3}\Big|\tau\Big)$$
$$\theta_4^2\Big(z+\frac{\pi}{3}\Big|\tau\Big)+\theta_2^2\Big(z-\frac{\pi}{3}\Big|\tau\Big)\theta_3^2\Big(z-\frac{\pi}{3}\Big|\tau\Big)\theta_4^2\Big(z-\frac{\pi}{3}\Big|\tau\Big)+\theta_2^2\Big(z-\frac{\pi}{6}\Big|\tau\Big)$$
$$\theta_3^2\Big(z-\frac{\pi}{6}\Big|\tau\Big)\theta_4^2\Big(z-\frac{\pi}{6}\Big|\tau\Big)=6\theta_2^2(0|\tau)\theta_3^2(0|\tau)\theta_4^2(0|\tau)\theta_3(6z|6\tau). \tag{4.5.19}$$

Taking $a=b=c=d=2$ in (4.2.15), we obtain

Corollary 4.5.5. *For k, l and n positive even integers, we have*

$$\sum_{s=0}^{8kln-1}\theta_1^2\Big(\frac{z}{8kln}+\frac{y}{2}+\frac{\pi s}{8kln}\Big|\frac{\tau}{8kln^2}\Big)\theta_2^2\Big(\frac{z}{8kln}+\frac{y}{2}+\frac{\pi s}{8kln}\Big|\frac{\tau}{8kln^2}\Big)$$
$$\theta_3^2\Big(\frac{z}{8kln}+\frac{y}{2}+\frac{\pi s}{8kln}\Big|\frac{\tau}{8kln^2}\Big)\theta_4^2\Big(\frac{z}{8kln}+\frac{y}{2}+\frac{\pi s}{8kln}\Big|\frac{\tau}{8kln^2}\Big)$$
$$=8kln\theta_1^2\Big(0\Big|\frac{\tau}{6ln^2}\Big)\theta_2^2\Big(0\Big|\frac{\tau}{6ln^2}\Big)\theta_3^2\Big(0\Big|\frac{\tau}{6ln^2}\Big)\theta_4^2\Big(0|\frac{\tau}{6ln^2}\Big)\theta_3(z|\tau). \tag{4.5.20}$$

Setting $k = l = n = 1$ in Corollary 4.5.5 and then $z \longmapsto 8z$, $y \longmapsto 2y$ and $\tau \longmapsto 8\tau$, we obtain

$$\begin{aligned}
&\theta_1^2(z+y|\tau)\theta_2^2(z+y|\tau)\theta_3^2(z+y|\tau)\theta_4^2(z+y|\tau)\\
&+\theta_1^2\Big(z+y+\frac{\pi}{8}\Big|\tau\Big)\theta_2^2\Big(z+y+\frac{\pi}{8}\Big|\tau\Big)\theta_3^2\Big(z+y+\frac{\pi}{8}\Big|\tau\Big)\theta_4^2\Big(z+y+\frac{\pi}{8}\Big|\tau\Big)\\
&+\theta_1^2\Big(z+y+\frac{\pi}{4}\Big|\tau\Big)\theta_2^2\Big(z+y+\frac{\pi}{4}\Big|\tau\Big)\theta_3^2\Big(z+y+\frac{\pi}{4}\Big|\tau\Big)\theta_4^2\Big(z+y+\frac{\pi}{4}\Big|\tau\Big)\\
&+\theta_1^2\Big(z+y+\frac{3\pi}{8}\Big|\tau\Big)\theta_2^2\Big(z+y+\frac{3\pi}{8}\Big|\tau\Big)\theta_3^2\Big(z+y+\frac{3\pi}{8}\Big|\tau\Big)\theta_4^2\Big(z+y+\frac{3\pi}{8}\Big|\tau\Big)\\
&+\theta_1^2\Big(z+y+\frac{\pi}{2}\Big|\tau\Big)\theta_2^2\Big(z+y+\frac{\pi}{2}\Big|\tau\Big)\theta_3^2\Big(z+y+\frac{\pi}{2}\Big|\tau\Big)\theta_4^2\Big(z+y+\frac{\pi}{2}\Big|\tau\Big)\\
&+\theta_1^2\Big(z+y+\frac{5\pi}{8}\Big|\tau\Big)\theta_2^2\Big(z+y+\frac{5\pi}{8}\Big|\tau\Big)\theta_3^2\Big(z+y+\frac{5\pi}{8}\Big|\tau\Big)\theta_4^2\Big(z+y+\frac{5\pi}{8}\Big|\tau\Big)\\
&+\theta_1^2\Big(z+y+\frac{6\pi}{8}\Big|\tau\Big)\theta_2^2\Big(z+y+\frac{6\pi}{8}\Big|\tau\Big)\theta_3^2\Big(z+y+\frac{6\pi}{8}\Big|\tau\Big)\theta_4^2\Big(z+y+\frac{6\pi}{8}\Big|\tau\Big)\\
&+\theta_1^2\Big(z+y+\frac{7\pi}{8}\Big|\tau\Big)\theta_2^2\Big(z+y+\frac{7\pi}{8}\Big|\tau\Big)\theta_3^2\Big(z+y+\frac{7\pi}{8}\Big|\tau\Big)\theta_4^2\Big(z+y+\frac{7\pi}{8}\Big|\tau\Big)\\
&=8\theta_1^2(0|\tau)\theta_2^2(0|\tau)\theta_3^2(0|\tau)\theta_4^2(0|\tau)\theta_3(8z|8\tau). \qquad (4.5.21)
\end{aligned}$$

$$\begin{aligned}
&\theta_1^2(z+y|\tau)\theta_2^2(z+y|\tau)\theta_3^2(z+y|\tau)\theta_4^2(z+y|\tau)\\
&+\theta_1^2\Big(z+y+\frac{\pi}{8}\Big|\tau\Big)\theta_2^2\Big(z+y+\frac{\pi}{8}\Big|\tau\Big)\theta_3^2\Big(z+y+\frac{\pi}{8}\Big|\tau\Big)\theta_4^2\Big(z+y+\frac{\pi}{8}\Big|\tau\Big)\\
&+\theta_1^2\Big(z+y+\frac{\pi}{4}\Big|\tau\Big)\theta_2^2\Big(z+y+\frac{\pi}{4}\Big|\tau\Big)\theta_3^2\Big(z+y+\frac{\pi}{4}\Big|\tau\Big)\theta_4^2\Big(z+y+\frac{\pi}{4}\Big|\tau\Big)\\
&+\theta_1^2\Big(z+y+\frac{3\pi}{8}\Big|\tau\Big)\theta_2^2\Big(z+y+\frac{3\pi}{8}\Big|\tau\Big)\theta_3^2\Big(z+y+\frac{3\pi}{8}\Big|\tau\Big)\theta_4^2\Big(z+y+\frac{3\pi}{8}\Big|\tau\Big)\\
&+\theta_1^2(z+y|\tau)\theta_2^2(z+y|\tau)\theta_3^2(z+y|\tau)\theta_4^2(z+y|\tau)\\
&+\theta_1^2\Big(z+y-\frac{3\pi}{8}\Big|\tau\Big)\theta_2^2\Big(z+y-\frac{3\pi}{8}\Big|\tau\Big)\theta_3^2\Big(z+y-\frac{3\pi}{8}\Big|\tau\Big)\theta_4^2\Big(z+y-\frac{3\pi}{8}\Big|\tau\Big)\\
&+\theta_1^2\Big(z+y-\frac{\pi}{4}\Big|\tau\Big)\theta_2^2\Big(z+y-\frac{\pi}{4}\Big|\tau\Big)\theta_3^2\Big(z+y-\frac{\pi}{4}\Big|\tau\Big)\theta_4^2\Big(z+y-\frac{\pi}{4}\Big|\tau\Big)
\end{aligned}$$

$$+\theta_1^2\Big(z+y-\frac{\pi}{8}\Big|\tau\Big)\theta_2^2\Big(z+y-\frac{\pi}{8}\Big|\tau\Big)\theta_3^2\Big(z+y-\frac{\pi}{8}\Big|\tau\Big)\theta_4^2\Big(z+y-\frac{\pi}{8}\Big|\tau\Big)$$
$$=8\theta_1^2(0|\tau)\theta_2^2(0|\tau)\theta_3^2(0|\tau)\theta_4^2(0|\tau)\theta_3(8z|8\tau). \tag{4.5.22}$$

Simplifying (4.5.22), we obtain

$$\theta_1^2(z+y|\tau)\theta_2^2(z+y|\tau)\theta_3^2(z+y|\tau)\theta_4^2(z+y|\tau)$$
$$+\theta_1^2\Big(z+y-\frac{\pi}{4}\Big|\tau\Big)\theta_2^2\Big(z+y-\frac{\pi}{4}\Big|\tau\Big)\theta_3^2\Big(z+y-\frac{\pi}{4}\Big|\tau\Big)\theta_4^2\Big(z+y-\frac{\pi}{4}\Big|\tau\Big)$$
$$+\theta_1^2\Big(z+y-\frac{\pi}{8}\Big|\tau\Big)\theta_2^2\Big(z+y-\frac{\pi}{8}\Big|\tau\Big)\theta_3^2\Big(z+y-\frac{\pi}{8}\Big|\tau\Big)\theta_4^2\Big(z+y-\frac{\pi}{8}\Big|\tau\Big)$$
$$+\theta_1^2\Big(z+y-\frac{3\pi}{8}\Big|\tau\Big)\theta_2^2\Big(z+y-\frac{3\pi}{8}\Big|\tau\Big)\theta_3^2\Big(z+y-\frac{3\pi}{8}\Big|\tau\Big)\theta_4^2\Big(z+y-\frac{3\pi}{8}\Big|\tau\Big)$$
$$=4\theta_1^2(0|\tau)\theta_2^2(0|\tau)\theta_3^2(0|\tau)\theta_4^2(0|\tau)\theta_3(8z|8\tau). \tag{4.5.23}$$

Setting $y=0$ in (4.5.23), we obtain

$$\theta_1^2(z|\tau)\theta_2^2(z|\tau)\theta_3^2(z|\tau)\theta_4^2(z|\tau)$$
$$+\theta_1^2\Big(z-\frac{\pi}{4}\Big|\tau\Big)\theta_2^2\Big(z-\frac{\pi}{4}\Big|\tau\Big)\theta_3^2\Big(z-\frac{\pi}{4}\Big|\tau\Big)\theta_4^2\Big(z-\frac{\pi}{4}\Big|\tau\Big)$$
$$+\theta_1^2\Big(z-\frac{\pi}{8}\Big|\tau\Big)\theta_2^2\Big(z-\frac{\pi}{8}\Big|\tau\Big)\theta_3^2\Big(z-\frac{\pi}{8}\Big|\tau\Big)\theta_4^2\Big(z-\frac{\pi}{8}\Big|\tau\Big)$$
$$+\theta_1^2\Big(z-\frac{3\pi}{8}\Big|\tau\Big)\theta_2^2\Big(z-\frac{3\pi}{8}\Big|\tau\Big)\theta_3^2\Big(z-\frac{3\pi}{8}\Big|\tau\Big)\theta_4^2\Big(z-\frac{3\pi}{8}\Big|\tau\Big)$$
$$=4\theta_2^2(0|\tau)\theta_3^2(0|\tau)\theta_4^2(0|\tau)\theta_3(6z|6\tau). \tag{4.5.24}$$

CHAPTER V

A BEAUTIFUL CONTINUED FRACTION IDENTITY FROM RAMANUJAN'S NOTEBOOK

CHAPTER V

A BEAUTIFUL CONTINUED FRACTION IDENTITY FROM RAMANUJAN'S NOTEBOOK

5.1 INTRODUCTION

Ramanujan recorded about 200 results on continued fractions in his notebooks [91] and lost notebook [93] without proof. The only result on continued fraction that he published [92] [90, pp. 214-215], is related to the now celebrated Roger-Ramanujan continued fraction defined by

$$R(q) := \frac{1}{1} \, {}_{+} \frac{q}{1} \, {}_{+} \frac{q^2}{1} \, {}_{+} \frac{q^3}{1} \, {}_{+ \cdots}, \qquad |q| < 1 ,$$

$$S(q) := -R(-q) ,$$

which was first introduced by L.J. Roger [95] and independently rediscovered by Ramanujan. In his first two letters to G.H. Hardy [90, pp. xxvii-xxviii], [28, pp. 21-30, 53-62], Ramanujan communicated several results concerning $R(q)$. In particular, he asserted that

$$R(e^{-2\pi}q) = \sqrt{\frac{5+\sqrt{5}}{2}} - \frac{\sqrt{5}+1}{2} ,$$

$$S(e^{-\pi}q) = \sqrt{\frac{5-\sqrt{5}}{2}} - \frac{\sqrt{5}-1}{2} .$$

which were first proved by G.N. Watson [112]. In his lost notebook [93, pp. 26], Ramanujan claims that

$$R(q) = \frac{\sqrt{5}-1}{2} \exp\left((-1/5)\int_q^1 \frac{(1-t)^5(1-t^2)^5\cdots}{(1-t^5)(1-t^{10})\cdots}\frac{dt}{t}\right), \tag{5.1.1}$$

$$= \frac{\sqrt{5}-1}{2} - \frac{\sqrt{5}}{1+\frac{3+\sqrt{5}}{2}} \exp\left((1/5)\int_q^1 \frac{(1-t)^5(1-t^2)^5\cdots}{(1-t^{1/5})(1-t^{2/5})\cdots}\frac{dt}{t^{4/5}}\right), \tag{5.1.2}$$

where $0 < q < 1$. The equality (5.1.1) was first proved by G.E. Andrews [16] and equality (5.1.2) was proved by S.H. Son [108]. On page 365 of his lost notebook [93], Ramanujan recorded five modular equations relating $R(q)$ with $R(-q)$, $R(q^2)$, $R(q^3)$ $R(q^4)$ and $R(q^5)$.

Motivated by these works, in this Chapter we study the Ramanujan continued fraction

$$A(q) := \frac{1}{1-q^2} + \frac{q^2(1+q^2)^2}{1-q^6} + \frac{q^4(1+q^4)^2}{1-q^{10}} + \frac{q^6(1+q^6)^2}{1-q^{14}} + \cdots, \quad |q| < 1$$

$$= \frac{(-q^8;q^8)_\infty^2}{(-q^4;q^8)_\infty^2}. \tag{5.1.3}$$

A similar continued fraction is been previously studied by C. Adiga and N. Anitha [1].

5.2 A q-IDENTITY RELATED TO $A(q)$

Theorem 5.2.1.

$$A(q) = \frac{(q^4;q^4)_\infty^2}{(q^8;q^8)_\infty^4} \sum_{n=0}^{\infty} \frac{q^{4n}}{1+q^{8n+4}}\,. \tag{5.2.1}$$

Proof: Changing q to q^8, then setting $a = -q^4$, $b = -q^{12}$ and $z = q^4$ in ${}_1\psi_1$ summation formula (1.2.1), we complete the proof of Theorem 5.2.1. □

5.3 PRODUCT REPRESENTATION FOR $A(q)$

Theorem 5.3.1. *Let $A(q)$ be defined by* (5.1.3)*. Then*

$$A(q) = \frac{\psi^2(q^8)}{\psi^2(q^4)}. \tag{5.3.1}$$

Proof: From [26, Ch. 16, Entry 11], for $|q| < 1$

$$\frac{(-a)_\infty(b)_\infty - (a)_\infty(-b)_\infty}{(-a)_\infty(b)_\infty + (a)_\infty(-b)_\infty} = \frac{a-b}{1-q+}\frac{(a-bq)(aq-b)}{1-q^3}\,+\,\frac{q(a-bq^2)(aq^2-b)}{1-q^5}\,+\cdots. \tag{5.3.2}$$

Rationalizing left hand side of (5.3.2) and then changing q to q^2, a to q and b to $-q$ in the resulting identity, we obtain

$$\frac{\{(-q;q^2)_\infty^2 - (q;q^2)_\infty^2\}^2}{(-q;q^2)_\infty^4 - (q;q^2)_\infty^4} = \frac{2q}{1-q^2}\,+\,\frac{q^2(1+q^2)^2}{1-q^6}\,+\,\frac{q^4(1+q^4)^2}{1-q^{10}}\,+\cdots. \tag{5.3.3}$$

Multiplying numerator and denominator of left hand side of (5.3.3) by $(q^2;q^2)_\infty$

and using (5.4.9), we obtain

$$\frac{\{f(q,q)-f(-q,-q)\}^2}{f^2(q,q)-f^2(-q,-q)} = \frac{2q}{1-q^2\ +}\ \frac{q^2(1+q^2)^2}{1-q^6\ \ +}\ \frac{q^4(1+q^4)^2}{1-q^{10}\ \ +\cdots}. \qquad (5.3.4)$$

Employing [26, Ch. 16, Entry 30 (iii) and (vi)] in (5.3.4) we obtain

$$\frac{q\,f^2(1,q^8)}{f(1,q^4)\psi(q^4)} = \frac{2q}{1-q^2\ +}\ \frac{q^2(1+q^2)^2}{1-q^6\ \ +}\ \frac{q^4(1+q^4)^2}{1-q^{10}\ \ +\cdots}. \qquad (5.3.5)$$

Finally applying [26, Ch. 16, Entry 18(ii)] and (1.2.6), we complete the proof (5.3.1). □

5.4 SOME IDENTITIES INVOLVING $A(q)$

We obtain several relations of $A(q)$ in terms of theta function $\varphi(q)$ and $\psi(q)$.

Theorem 5.4.1.

$$A(q) = \frac{\psi^2(q^4)}{\varphi^2(q^4)}, \tag{5.4.1}$$

$$A(q) = \frac{\psi(q^8)}{\varphi(q^4)}, \tag{5.4.2}$$

$$A(q) = \frac{\psi^4(-q^2)}{\varphi^2(-q^2)\varphi^2(q^4)}, \tag{5.4.3}$$

$$A(q) = \frac{\psi^2(q^4)\varphi(-q^4)}{\varphi^2(-q^8)\varphi(q^4)}, \tag{5.4.4}$$

$$\frac{A^{n+1}(q) + A^{n+1}(-q)}{A(q) + A(-q)} = A^n(q), \tag{5.4.5}$$

$$A(q)A(q^2)A(q^3)A(q^4)\cdots = \frac{1}{\psi^2(q^4)}, \tag{5.4.6}$$

$$A^2(q)A(q^2)A(q^3)A(q^4)\cdots = \frac{\psi^2(q^8)}{\psi^4(q^4)}. \tag{5.4.7}$$

Proof: Replacing q by q^4 in [26, Ch. 16, Entry 23(v)] and then squaring both sides, we obtain (5.4.1). On using [26, Ch. 16, Entry 25(iv)] in (5.4.1) we obtain (5.4.2).

Setting $a = b = q^2$ and $a = b = q^4$ in [26, Ch. 16, Entry 30(i)] and substituting in (5.3.1) we have

$$A(q) = \left(\frac{\psi(q^4)}{\psi(q^2)}\right)^4 \left(\frac{\varphi(q^2)}{\varphi(q^4)}\right)^2. \tag{5.4.8}$$

Using [26, Ch. 16, Entry 25(iii)] twice in (5.4.8), we obtain (5.4.3).

Using [26, Ch. 16, Entry 25(iii)] and [26, Ch. 16, Entry 25(iv)] in (5.3.1), we have

$$A(q) = \frac{\varphi(-q^4)\psi(q^8)}{\varphi^2(-q^8)}. \tag{5.4.9}$$

Employing [26, Ch. 16, Entry 25(ii)] in (5.4.9), we obtain

$$A(q) = \left(\frac{\varphi(-q^4)}{\varphi^2(-q^8)}\right)\left(\frac{\varphi(q) - \varphi(-q)}{4q}\right). \tag{5.4.10}$$

From [26, Ch. 16, Entry 25(i)] and [26, Ch. 16, Entry 25(v)], we have

$$\begin{aligned} \varphi(q) - \varphi(-q) &= \frac{\varphi^2(q) - \varphi^2(-q)}{\varphi(q) + \varphi(-q)} \\ &= 4q\frac{\psi^2(q^4)}{\varphi(q^4)}. \end{aligned} \tag{5.4.11}$$

Substituting (5.4.11) in (5.4.10), we obtain (5.4.4).

Proofs of (5.4.5), (5.4.6) and (5.4.7) follows directly from (5.3.1). □

We can also observer the following for $A(q)$

Theorem 5.4.2. *If $u = A(q)$, $v = A^{\frac{1}{2}}(q)$ and $w = A^{\frac{1}{4}}(q)$, then*

$$w^6 = uv\,, \tag{5.4.12}$$

$$\frac{u}{v} + \frac{v}{w} + \frac{w}{u} - \frac{1}{w^3} = v + w\,. \tag{5.4.13}$$

Proof: From (5.3.1) we have

$$uv = A(q)A^{\frac{1}{2}}(q) = \left(\frac{\psi(q^8)}{\psi(q^4)}\right)^3,$$

which completes the proof of (5.4.12).

Also from (5.3.1), we have

$$\begin{aligned}\frac{u}{v}+\frac{v}{w}+\frac{w}{u}-\frac{1}{w^3} &= \frac{\psi^2(q^8)}{\psi^2(q^4)}\frac{\psi(q^4)}{\psi(q^8)}+\frac{\psi(q^8)}{\psi(q^4)}\frac{\psi^{\frac{1}{2}}(q^4)}{\psi^{\frac{1}{2}}(q^8)}+\frac{\psi^{\frac{1}{2}}(q^8)}{\psi^{\frac{1}{2}}(q^4)}\frac{\psi^2(q^4)}{\psi^2(q^8)}-\frac{\psi^{\frac{3}{2}}(q^4)}{\psi^{\frac{3}{2}}(q^8)}\\ &= \frac{\psi(q^8)}{\psi(q^4)}+\frac{\psi^{\frac{1}{2}}(q^8)}{\psi^{\frac{1}{2}}(q^4)},\end{aligned}$$

which completes the proof of (5.4.13). □

5.5 INTEGRAL REPRESENTATION OF $A(q)$

Theorem 5.5.1. *For* $0 < |q| < 1$,

$$A(q) = \exp \int \left(\frac{\varphi^4(-q^4) - 1}{q} \right) dq \,, \tag{5.5.1}$$

where $\varphi(q)$ *is as defined in* (1.2.5).

Proof: Taking log on both sides of (5.3.1), we have

$$\log A(q) = 2 \log \psi(q^8) - 2 \log \psi(q^4) \,. \tag{5.5.2}$$

Employing [26, Ch.16, Entry 23(ii)] on right hand side of (5.5.2), we obtain

$$\log A(q) = 2 \sum_{n=1}^{\infty} \frac{(-1)^n q^{4n}}{n(1+q^{4n})} \,. \tag{5.5.3}$$

Differentiating (5.5.3) and simplifying, we have

$$\frac{d}{dq} \log A(q) = \frac{8}{q} \sum_{n=1}^{\infty} \frac{(-1)^n q^{4n}}{(1+q^{4n})^2} \,. \tag{5.5.4}$$

Using Jacobi's identity [26, Ch.16, Identity 33.5, pp. 54] and integrating both sides and finally exponentiating both sides of identity (5.5.4), we complete the proof of Theorem 5.5.1. □

5.6 RELATION BETWEEN $A(q)$ AND HYPERGEOMETRIC FUNCTION

In this Section, we deduce relations between $A(q)$ and hypergeometric function ${}_2F_1(a,b;c;x)$ where

$$ {}_2F_1(a,b;c;x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} x^k, \qquad |x| < 1. \tag{5.6.1} $$

Theorem 5.6.1. *If*

$$ \begin{aligned} x &= k^2 = 1 - \frac{\varphi^4(-q)}{\varphi^4(q)}, \\ q &= \exp\left(-\pi \frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-k^2)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;k^2)} \right), \end{aligned} $$

and

$$ z = {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;k^2\right), $$

then

(i) $A(q) = \dfrac{1}{q}$,

(ii) $A(q) + \dfrac{1}{2q} = \dfrac{1}{qx}\{1 + \sqrt{(1-x)} - \sqrt[4]{(1-x)} - \sqrt[4]{(1-x)^3}\}$.

Proof: From [26, Ch. 17, Entry 11(i), pp. 123], we have

$$ \psi(q) = \sqrt{\frac{1}{2} z} \left(\frac{x}{q}\right)^{\frac{1}{8}}. \tag{5.6.2} $$

Then employing (5.6.2) in (5.3.1) we obtain (i).

Again from [26, Ch. 17, Entry 11(iv),(v), pp. 123] and (5.3.1), we have

$$A(q) = \frac{1}{2q} \frac{(1 - (\sqrt[4]{1-x})^2}{1 - \sqrt{1-x}}. \tag{5.6.3}$$

Rationalizing (5.6.3) and simple manipulation completes the proof of (ii). □

5.7 MODULAR EQUATIONS OF DEGREE n

This Section contains new modular equations of $B(q)$, where $B(q) = 2qA(q)$.

We say modulus β has degree n over the modulus α when

$$n\frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-\alpha)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;\alpha)} = \frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-\beta)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;\beta)}. \tag{5.7.1}$$

where ${}_2F_1(a,b;c;x)$ is defined as in (5.6.1). A modular equation of degree n is an equation relating α and β induced by (5.7.1).

Theorem 5.7.1. *If*

$$q = \exp\left(-\pi\frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-\alpha)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;\alpha)}\right), \tag{5.7.2}$$

then

$$\alpha = 1 - \left(\frac{1-B(q)}{1+B(q)}\right)^4. \tag{5.7.3}$$

Proof: On employing [26, Ch. 16, Entry 25(ii)] and [26, Ch. 16, Entry 25(v)] in (5.3.1), we have

$$A(q) = \frac{1}{2q}\left(\frac{1-\dfrac{\varphi(-q)}{\varphi(q)}}{1+\dfrac{\varphi(-q)}{\varphi(q)}}\right).$$

Thus,

$$B(q) = \frac{1-\dfrac{\varphi(-q)}{\varphi(q)}}{1+\dfrac{\varphi(-q)}{\varphi(q)}}. \tag{5.7.4}$$

Also from [26, Ch. 17, Entry 5, pp. 100] and (5.7.2) it is implied that

$$\alpha = 1 - \frac{\varphi^4(-q)}{\varphi^4(q)}. \tag{5.7.5}$$

Using (5.7.5) in (5.7.4), we complete the proof of (5.7.3). □

Let q and α is related by (5.7.2). If β has degree n over α then from Theorem 5.7.1, we obtain

$$\beta = 1 - \left(\frac{1 - B(q^n)}{1 + B(q^n)}\right)^4. \tag{5.7.6}$$

Corollary 5.7.2. *Let $l = B(q)$, $m = B(q^3)$, $n = B(q^4)$, then*

(i) $l^4 - 4l^3m^3 + 6l^2m^2 - 4lm + m^4 = 0$

(ii) $l^4 + l^4n^4 + 4l^4n^3 + 6l^4n^2 + 4l^4n - 8n^3 - 8n = 0$

Proof: From [26, Ch. 19, Entry 5 (ii), pp. 230], we have

$$(\alpha\beta)^{\frac{1}{4}} + \{(1-\alpha)(1-\beta)\}^{\frac{1}{4}} = 1. \tag{5.7.7}$$

On using (5.7.6) with $n = 3$ and (5.7.3) in (5.7.7) and simplifying we complete the proof of Corollary 5.7.2.(i).

When β has degree 4 over α then we have from [26, Ch. 18, Eq. (24.22), pp. 215]

$$\sqrt{\beta} = \left(\frac{1 - (1-\alpha)^{\frac{1}{4}}}{1 + (1-\alpha)^{\frac{1}{4}}}\right)^2. \tag{5.7.8}$$

On using (5.7.6) with $n = 4$ and (5.7.3) in (5.7.8), we obtain

$$\sqrt{1-\left(\frac{1-n}{1+n}\right)^4} = \left\{\frac{1-\left(\frac{1-l}{1+l}\right)}{1+\left(\frac{1-l}{1+l}\right)}\right\}^2.$$

Squaring both sides of the above identity and simplifying we complete the proof of Corollary 5.7.2.(ii). □

5.8 EXPLICIT FORMULA FOR THE EVALUATION OF $A(q)$

Let $q_n = e^{-\pi\sqrt{n}}$ and α_n denote the corresponding values of α in (5.7.2). From Theorem 5.7.1 we have

$$B(e^{-\pi\sqrt{n}}) = \frac{1-(1-\alpha_n)^{\frac{1}{4}}}{1+(1-\alpha_n)^{\frac{1}{4}}}.$$

Hence

$$A(e^{-\pi\sqrt{n}}) = \frac{1}{2}e^{\pi\sqrt{n}}\frac{1-(1-\alpha_n)^{\frac{1}{4}}}{1+(1-\alpha_n)^{\frac{1}{4}}}. \tag{5.8.1}$$

From [26, Ch. 17, pp. 97], we have $\alpha_1 = \frac{1}{2}$, $\alpha_2 = (\sqrt{2}-1)^2$, $\alpha_4 = (\sqrt{2}-1)^4$.

Thus from (5.8.1), we have

$$\begin{aligned}
A(e^{-\pi}) &= \frac{1}{2}e^{\pi}\frac{\sqrt[4]{2}-1}{\sqrt[4]{2}+1}, \\
A(e^{-\pi\sqrt{2}}) &= \frac{1}{2}e^{\pi\sqrt{2}}\left\{\frac{1-(-2+2\sqrt{2})^{\frac{1}{4}}}{1+(-2+2\sqrt{2})^{\frac{1}{4}}}\right\}, \\
A(e^{-2\pi}) &= \frac{1}{2}e^{2\pi}\left\{\frac{1-(-16+12\sqrt{2})^{\frac{1}{4}}}{1+(-16+12\sqrt{2})^{\frac{1}{4}}}\right\}.
\end{aligned}$$

The Ramanujan-Weber class invariant is defined by

$$G_n := 2^{-1/4} q_n^{-1/24} (-q_n; q_n^2)_\infty,$$

and

$$g_n := 2^{-1/4} q_n^{-1/24} (q_n; q_n^2)_\infty, \tag{5.8.2}$$

where $q_n = e^{-\pi\sqrt{n}}$. Chan and Huang [44] has derived few explicit formula for evaluation of $S(e^{-\pi\sqrt{n}/2})$ in terms of Ramanujan Weber class. Similar works are also carried out by Adiga et. al. [5]. Analogous to these works we obtain explicit formula for the evaluation of $A(e^{-\pi\sqrt{n}})$.

Theorem 5.8.1. *For Ramanujan Weber class invariant as defined in* (5.8.2) *and let* $p = G_n^{12}$ *and* $p_1 = g_n^{12}$*, then*

$$A(e^{-\pi\sqrt{n}}) = \frac{1}{2} e^{\pi\sqrt{n}} \left[\frac{\{2p(p+\sqrt{p^2-1})\}^{1/4} - \{2p(p+\sqrt{p^2-1})-1\}^{1/4}}{\{2p(p+\sqrt{p^2-1})\}^{1/4} + \{2p(p+\sqrt{p^2-1})-1\}^{1/4}} \right], \tag{5.8.3}$$

$$A(e^{-\pi\sqrt{n}}) = \frac{1}{2} e^{\pi\sqrt{n}} \left[\frac{1 - \{-2p_1(p_1 - \sqrt{p_1^2+1})\}^{1/4}}{1 + \{-2p_1(p_1 - \sqrt{p_1^2+1})\}^{1/4}} \right]. \tag{5.8.4}$$

Proof: From [42, Eq. 4.7, pp. 85], we have

$$G_n = [4\alpha_n(1-\alpha_n)]^{-1/24}.$$

Hence,

$$\alpha_n = \frac{1}{(\sqrt{p(p+1)} + \sqrt{p(p-1)})^2}. \tag{5.8.5}$$

Using (5.8.5) in (5.8.1) we obtain (5.8.3).

Also from [42, Eq. 4.9, pp. 85], we have

$$2g_n^{12} = \frac{1}{\sqrt{\alpha_n}} - \sqrt{\alpha_n}\,.$$

Hence

$$\sqrt{\alpha_n} = \sqrt{(p_1^2+1)} - p_1\,. \tag{5.8.6}$$

Using (5.8.6) in (5.8.1) we obtain (5.8.4).

□

Example. *Let $n = 1$. Since $G_1 = 1$, from Theorem 5.8.1 we have*

$$A(e^{-\pi}) = \frac{1}{2}e^{\pi}\left(\frac{2^{1/4}-1}{2^{1/4}+1}\right).$$

Let $n = 2$. Since $g_2 = 1$, from Theorem 5.8.1 we have

$$A(e^{-\pi\sqrt{2}}) = \frac{1}{2}e^{\pi\sqrt{2}}\left\{\frac{1-(2\sqrt{2}-2)^{1/4}}{1+(2\sqrt{2}-2)^{1/4}}\right\}.$$

Let $n = 3$. Since $G_3^{12} = 2$, from Theorem 5.8.1 we have

$$A(e^{-\pi\sqrt{3}}) = \frac{1}{2}e^{\pi\sqrt{3}}\left\{\frac{(8+4\sqrt{3})^{1/4}-(7+4\sqrt{3})^{1/4}}{(8+4\sqrt{3})^{1/4}+(7+4\sqrt{3})^{1/4}}\right\}.$$

BIBLIOGRAPHY

Bibliography

[1] C. Adiga and N. Anitha, *A note on a continued fraction of Ramanujan*, Bull. Austral. Math. Soc., **70**(2004), 486-497.

[2] C. Adiga and N. Anitha, *On a reciprocity theorem of Ramanujan*, Tamsui Oxford Journal of Mathematical Sciences, **22(1)**(2006), 9-15.

[3] C. Adiga, B. C. Berndt, S. Bhargava and G. N. Watson, *Chapter 16 of Ramanujan's second notebook: theta-functions and q-series*, Mem. Amer. Math. Soc., No. **315**, **53**(1985), 1-85, Amer. Math. Soc., Providence, 1985.

[4] C. Adiga and P. S. Guruprasad, *On a three variable reciprocity theorem*, South East Asian Journal of Mathematics and Mathematical Sciences, **6(2)**(2008), 57–61.

[5] C. Adiga, T. Kim, M. S. Mahadeva Naika and H. S. Madhusudhan, *On Ramanujan's cubic continued fraction and explicit evaluations of theta functions*, Indian J. Pure and App. Math., **55(9)**(2004), 1047-1062.

[6] C. Adiga and D. D. Somashekara, *On some Rogers-Ramanujan type continued fraction identities*, Mathematical Balkanica, New Series, **12(1-2)**(1998), 37-45.

[7] W. A. Al-Salam and M. E. H. Ismail, *Orthogonal polynomials associated with the Rogers-Ramanujan continued fraction*, Pacific J. Math., **104**(1983), 269-283.

[8] G. E. Andrews, *On Ramanujan's summation of* ${}_1\psi_1(a, b; z)$, Proc. Amer. Math.

Soc., **22**(1969), 552-553.

[9] G. E. Andrews, *Ramanujan's Lost Notebook Part III*, New York, Springer Verlang, 2012.

[10] G. E. Andrews, *Ramanujan's 'Lost' Notebook V. Euler's partition identity*, Adv. Math., **61**(1986), 156-164.

[11] G. E. Andrews, *The Theory of Partitions*, Encyclopedia of mathematics and its applications, Vol. **2**, Addison-Wesley, Reading, Mass. 1979; reprinted by Cambridge University Press, Cambridge, 1985.

[12] G. E. Andrews, *q-Series: Their development and application in analysis, number theory, combinatorics, physics and computer algebra*, CBMS Regional conference series in Mathematics, No.**66**, Amer. Math. Soc., Providence, 1986.

[13] G. E. Andrews, *On a transformation of bilateral series with applications*, Proc. Amer. Math. Soc., **25**(1970), 554-558.

[14] G. E. Andrews, *A simple proof of Jacobi's triple product identity*, Proceedings of the American Mathematical Society, vol. **16**(1965), 333-334.

[15] G. E. Andrews, *Ramanujan's Lost Notebook 1: Partail θ-functions*, Adv. in Math., **41**(1981), 137-172.

[16] G. E. Andrews, *Ramanujan's 'Lost' Notebook III. The Rogers-Ramanujan continued fraction*, Adv. Math., **41**(1981), 186-208.

[17] G. E. Andrews and R. Askey, *A simple proof of Ramanujan's summation of the* ${}_1\psi_1$, Aeqa. Math., **18**(1978), 333-337.

[18] G. E. Andrews, R. Askey, R. Roy, *Special functions*, Encyclopedia of Mathematics and Its Applications, **71**, Cambridge University Press, Cambridge, 1999.

[19] G. E. Andrews and B. C. Berndt, *Ramanujan's Lost Notebook*, Part I, Springer-Verlag, to appear.

[20] R. Askey, *Ramanujan's* $_1\psi_1$ *and formal Laurent series*, Indian J. Math., **29**(1987), 101-105.

[21] R. Askey, *The q-gamma and q-beta functions*, Appl. Anal., **8**(1978), 125-141.

[22] A. O. L. Atkin and P. Swinnerton-Dyer, *Some properties of partitions*, Proc. London Math. Soc., **s3-4 (1)**(1954), 84-106

[23] W. N. Bailey, *Generalized hyper geometric series*, Cambridge Tracts in Math., Cambridge Univ. Press, Cambridge-New York, (1935), 65-72.

[24] W. N. Bailey, *On the basic bilateral hypergeometric series* $_2\psi_2$*,* Quart. J. Math., **1**(1950), 194-198.

[25] W. N. Bailey, *Series of hypergometric type which are infinite in both directions*, Quart. J. Math., **7**(1936), 105-115.

[26] B. C. Berndt, *Ramanujan's notebooks Part III*, Springer-Verlag, New York, 1991.

[27] B. C. Berndt, S.-S. Huang, J. Sohn, and S. H. Son, *Some theorems on the Rogers-Ramanujan continued fraction in Ramanujan's 'lost' notebook*, Trans. Amer. Math. Soc., **352**(2000), 2157-2177.

[28] B. C. Berndt and R. A. Rankin, *Ramanujan: Letters and Commentary*, Amer. Math. Soc., Providence, 1995.

[29] B. C. Berndt, S. H. Chan, B. P.Yeap and A. J. Yee, *A reciprocity theorem for certain q-series found in Ramanujan's lost notebook*, The Ramanujan Journal **13(1-3)**(2007), 27-37.

[30] N. A. Bhagirathi, *On certain continued fractions and q-series*, Math. Student, **56(1-4)**(1988), 97-104.

[31] S. Bhargava and C. Adiga, *A basic bilateral series summation formula and its applications*, Integr. Transf. Spec. F., **2**(1994), 165-184.

[32] S. Bhargava and C. Adiga, *On some continued fraction identities of Srinivasa Ramanujan*, Proc. Amer. Math. Soc., **92**(1984), 13-18.

[33] S. Bhargava and C. Adiga, *Two generalizations of Ramanujan's continued fraction identities*, Number Theory, K. Alladi (Ed.), Lecture Note in Math., No. **1122**, Springer-Verlag, Berlin (1985), 56-62.

[34] S. Bhargava and C. Adiga and D. D. Somashekara, *On some generalizations of Ramanujan's continued fraction identities*, Proc. Indian Acad. Sci., (Math. Sci.), **97**(1987), 31-43.

[35] S. Bhargava and C. Adiga and D. D. Somashekara, *On certain continued fractions related to* ${}_3\varphi_2$ *basic hypergeometric function*, J. Math. Phys. Sci., **21**(1987), 613-629.

[36] R. P. Boas, *Tannery's theorem*, Math. Mag., **38**(1965), 66.

[37] M. Boon, M. L. Glasser, J. Jak, I. J. Zucker, *Additive decompositions of θ functions of multiple arguments*, J. Phy., **A 15**(1982), 3439-3440.

[38] Y. Cai, S. Chen, Q-M Luo, *Some circular summation formulas for theta functions*, Boundary Value Problems, **2013(1)**(2013), 1.

[39] Y. Cai, Y. Q. Bi, Q-M Luo, *Some new circular summation formulas of theta functions*, Integral Transforms Spec. Funct., **25**(2014), 497-507.

[40] L. Carlitz, *Note on some continued fractions of the Rogers-Ramanujan type*, Duke Math. J., **32**(1965), 713-720.

[41] A. Cauchy, *Mémoire sur les fonctions dont plusieurs valeurs sont liées entre elles par une équation linéaire, et sur diverses transformations de produits composés d'un nombre nombre indéfini de facteurs*, C. R Acad. Sci. Paris, T. $XVII$, pp. 523, Oeuvres de Cauchy, 1^{re} série, T. $VIII$, Gauthier-Villars, Paris, (1893), 42-50.

[42] S. H. Chan, *Ramanujan's famous ${}_1\psi_1$ summation formula*, J. Appl. Theory, **132**(2005), 149-153.

[43] S. H. Chan and Z. G. Liu, *On a new circular summation of theta functions*, J. Number Theory., **130**(2010), 1190-1196.

[44] H. H. Chan and S. S. Huang, *On the Ramanujan Göllnitz Gordon Continued Fraction*, Ramanujan J., **1**(1997), 75-90.

[45] H. H. Chan, Z. G. Liu, S. T. Ng, *Circular summation of theta functions in Ramanujan's lost notebook*, J. Math. Anal. Appl., **316**(2006), 628-641.

[46] K. Chandrasekharan, *Elliptic Functions*, Springer-Verlag, Berlin Heideberg, 1985.

[47] K. S. Chua, *The root lattice* A_n^* *and Ramanujan's circular summation of theta functions*, Proc. Amer. Math. Soc., **130**(2002), 1-8.

[48] K. S. Chua, *Circular summation of the 13th powers of Ramanujan's theta function*, Ramanujan J., **5**(2001), 353-354.

[49] S. Corteel and J. lovejoy, *Frobenious partitions and the combinatorics of Ramanujan's* ${}_1\psi_1$ *summation*, J. Combin. Theory Ser. A, **97**(2002), 177-183.

[50] R. Y. Denis, *On certain q-series and continued fractions*, Math. Student, **44**(1984), 70-76.

[51] R. Y. Denis, *On basic hypergeometric functions and continued fractions*, Math. Student, **52**(1984), 129-136.

[52] R. Y. Denis, *On certain q-series and continued fraction identities*, Math. Student, **53**(1985), 243-248.

[53] R. Y. Denis, *On certain transformations of basic hypergeometric functions*, Bull. Cal. Math. Coc., **79**(1987), 134-138.

[54] J. Dougall, *On Vandermonde's theorem and some more general expansions*, Proc. Edin. Math. Soc., **25**(1907), 114-132.

[55] J. A. Ewell, *An easy proof of the triple-product identity*, Amer. Math. Monthly, **88**(1981), 270-272.

[56] N. J. Fine, *Basic Hypergeometric Series and Applications*, Amer. Math. Soc.,

Providence, 1988.

[57] D. Foata and G.-N. Han, *The triple, quintuple and septuple product identities revisited*, The Andrews festschrift: seventeen papers on Classical Number Theory and Combinatorics, D. Foata and G.-N. Han, (Eds.), Springer-Verlag, New York, (2001), 323-334.

[58] G. Gasper and M. Rahman, *Basic hypergeometric series*, 2nd edn. Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, **96**(2004).

[59] C. F. Gauss, *Hundert Theoreme über die neuen Transscendenten*, Werke, Bd.,**3**, Königlichen Gesell. Wiss., Göttingen, (1876), 461-459.

[60] B. Gordon, *Some continued fractions of the Rogers-Ramanujan type*, Duke Math. J., **32**(1965), 741-748.

[61] W. Hahn, *Beiträge zur Theorie der Heineschen Reihen*, Math. Nachr., **2**(1949), 340-379.

[62] W. Hahn, *Über Orthogonalpolynome, die q-differenzengleichungen genügen*, Math. Nachr. **2**(1949), 4-34.

[63] W. Hahn, *Über Polynome, die gleichzeitig zwei verschiedenen Orthogonalsystemen angehören*, Math. Nachr. **2**(1949), 263-278.

[64] G. H. Hardy, *Ramanujan*, Cambridge University Press, Cambridge, 1940; reprinted by Amer. Math. Soc., Providence, 1999, 3rd ed, Chelsea, New York, 1978.

[65] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, 5th ed., Clarendon Press, Oxford, 1983.

[66] G. H. Hardy and E. M. Wright, *Introduction to the Theory of Numbers*, Oxford Univ. Press, 1945.

[67] E. Heine, *Theorie der Kügelfunktionen*, Berlin, (1878), 97-125.

[68] E. Heine, *Untersuchungen über die reihe ...*, J. Reine Angew. Math., **34**(1847), 285-328.

[69] M. D. Hirschhorn, *A continued fraction of Ramanujan*, J. Austral. Math. Soc., Ser. A, **29**(1980), 80-86.

[70] M. E. H. Ismail, *A simple proof of Ramanujan's* ${}_1\psi_1$ *sum*, Proc. Amer. Math. Soc., **63**(1977), 185-186.

[71] F. H. Jackson, *A generalization of the function* $\Gamma(n)$ *and* x^n, Proc. Roy. Soc. London, **74**(1904), 64-72.

[72] F. H. Jackson, *A new transformation of Heinean series*, Quart. J. Pure and Appl. Math. **50**(1927), 377-384.

[73] M. Jackson, *On Lerch's transcendent and the basic bilateral hypergeometric series* ${}_2\psi_2$, J. London Math. Soc., **25**(1950), 189-196.

[74] C. G. J. Jacobi, *Fundamenta Nova Theoriae Functionum Ellipticarum*, Fratrum Borntrãger Regiomonti, 1829;Gesammelte Werke, Erster Band, G. Reimer, Berlin, (1881), 49-239; reprinted by Chelsea, New York, 1969.

[75] S. Y. Kang, *Generalizations of Ramanujan's reciprocity theorem and their*

applications, Journal of the London Mathematical Society, **75(1)**(2007), 18-34.

[76] S. Y. Kang, *Ramanujan's formulas for the explicit evaluations of the Rogers-Ramanujan continued fraction and theta-functions*, Acta. Arith., **90**(1999), 49-68.

[77] S. Y. Kang, *Some theorems on the Rogers-Ramanujan continued fraction and associated theta function identities in Ramanujan's lost notebook*, The Ramanujan J., **3**(1999), 91-111.

[78] R. P. Lewis, *A combinatorial proof of the triple product identity*, Amer. Math. Monthly, **91**(1984), 420-423.

[79] Z. G. Liu, *Some inverse relations and theta function identities*, Int. J. Number Theory, **8**(2012), 1977-2002.

[80] Z. G. Liu, *Some operator identities and q-series transformation formulas*, Discrete math., **265**(2003), 119-139.

[81] X. R. Ma, *Six-variable generalization of Ramanujan's receprocity theorem and it's variants*, J. Math. Anal. Appl., **353**(2009), 320-328.

[82] P. A. MacMahon, *Combinatory analysis*, Cambridge Univ. Press, 1915.

[83] K. Mimachi, *A proof of Ramanujan's identity by use of loop integral*, SIAM J. Math. Anal., **19**(1988), 1490-1493.

[84] D. Mumford, *Tata Lectures on Theta I*, Birkhäuser-Verlag Basel, 1983.

[85] T. Murayama, *On an identity of theta functions obtained from weight*

enumurators of linear codes, Proc. Japan Acad. Ser. A., **74**(1998), 98-100.

[86] K. Ono, *On the circular summation of the eleventh powers of Ramanujan's theta function*, J. Number Theory., **76**(1999), 62-65.

[87] V. Ramamani, *Some identities conjuctured by Srinivasa Ramanujan found in his lithographed notes connected with partition theory and elliptic modular functions-their proofs-inter connections with various other topics in the theory of numbers and some generalizations*, Doctoral thesis, Unversity of Mysore, 1970.

[88] K. G. Ramanathan, *On Ramanujan's continued fraction*, Acta. Arith. **43**(1984), 209-226.

[89] K. G. Ramanathan, *On the Rogers-Ramanujan continued fraction*, Proc. Indian Acad. Sci., (Math. Sci.), **93**(1984), 67-77.

[90] S. Ramanujan, *Collected Pupers*, Chelsea, New York, 1962.

[91] S. Ramanujan, *Notebooks (2 volumes)*, Tata Institute of Fundamental Research, Bombay, 1957.

[92] S. Ramanujan, *Proof of certain identities in combinatory analysis*, Proc. Cambridge Philos. Soc., **19**(1919), 214-216.

[93] S. Ramanujan, *The Lost Notebook and Other Unpublished Papers*, Narosa, New Delhi, 1988.

[94] S. S. Rangachari, *On a result of Ramanujan on theta functions*, J. Number Theory, **48**(1994), 364-372.

[95] L. J. Rogers, *Second memoir on the expansion of certain infinite products*, Proc. London Math. Soc., **25**(1894), 318-343.

[96] D. B. Sears, *On the transformation theory of basic hypergeometric functions,* Proc. London Math. Soc., **53(2)**(1951), 158-180.

[97] D. B. Sears, *Transformations of basic hypergeometric functions of any order*, Proc. London Math. Soc., **53(2)**(1951), 181-191.

[98] D. B. Sears, *Transformations of basic hypergeometric functions of special type*, Proc. London Math. Soc. **52(2)**(1951), 467-483.

[99] L. C. Shen, *On the additive formula of the theta functions and a collection of Lambert series pertaining to the modular equations of degree 5*, Trans. Amer. Math. Soc., **345**(1994), 323-345.

[100] L. C. Shen, *On the Product of Three Theta Functions*, The Ramanujan J., **3**(1999), 343-357.

[101] H. S. Shukla, *A note on the sums of certain bilateral hypergeometric series*, Math. Proc. Cambridge Philos. Soc. **55**(1959), 262-266.

[102] S. N. Singh, *On q-hypergeometric functions and continued fraction*, Math. Student, **56**(1988), 81-84.

[103] S. P. Singh, *On certain transformations involving q-series*, Private communication.

[104] L. J. Slater, *General transformations of bilateral series*, Quart. J. Math., **3**(1952), 73-80.

[105] L. J. Slater, *Generalized Hypergeometric Functions*, Cambridge University Press, Cambridge, 1966.

[106] D. D. Somashekara and D. Mamta, *On the three variable reciprocity theorem and its applications*, AJMAA, **9(1)**, Art. **13**(2012), 1-9.

[107] S. H. Son, *Circular summations of theta functions in Ramanujan's lost notebook*, Ramanujan J., **8**(2004), 235-272.

[108] S. H. Son, *Some integrals of theta functions in Ramanujan's 'lost' notebook*, CRM Proceedings and Lecture Notes, **19**(1999), 323-332.

[109] J. Thomae, *Beitrãge zur theorie der durch die Heinesche reihe* $\cdots$, J.Reine Angew.Math., **70**(1869), 258-281.

[110] K. Venkatachaliengar, *Development of Elliptic Functions According to Ramanujan*, Madurai Kamraj University, Madurai, 1988.

[111] A. Verma, R. Y. Denis and K. Srinivasa Rao, *New continued fractions involving basic hypergeometric* ${}_3\varphi_2$ *functions*, J. Math. Phys. Sci., **21**(1987), 585-592.

[112] G. N. Watson, *Theorems stated by Ramanujan (VII): Theorems on continued fractions*, J. London Math. Soc., **4**(1929), 39-48.

[113] A. J Yee, *Combinatorial proof of Ramanujan's* ${}_1\psi_1$ *summation and the q-Gauss summation*, J. Combin. Theory Ser. A., **105**(2004), 63-77.

[114] P. Xu, *An elementary proof of Ramanujan's circular summation formula and its generalization*, Ramanujan J., **27**(2012), 409-417.

[115] X. F. Zeng, *A generalized circular summation of theta function and its application*, J. Math. Anal. Appl., **356**(2009), 698-703.

[116] Y. Zhou, Q-M Luo, *Circular summation formulas for theta function $\theta_4(z|\tau)$*, Advances in Difference Equations, **2014(1)** (2014), 1.

[117] J. M. Zhu, *An alternate circular summation formula of theta functions and its applications*, Appl. Anal. Discrete Math., **6**(2012), 114-125.

[118] J. M. Zhu, *A note on a generalizaed circular summation formula of theta functions*, J. Number Theory., **132**(2012), 1164-1169.