A STUDY ON SOME TOPIC RELATED TO PARTITION THEORY, BASIC HYPERGEOMETRIC SERIES AND CONTINUED FRACTIONS

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IN

MATHEMATICS

by

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under the guidance of

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CERTIFICATE

This is to certify that the thesis, entitled **"A STUDY ON SOME TOPIC RELATED TO PARTITION THEORY, BASIC HYPERGEOMETRIC SERIES AND CONTINUED FRACTIONS"**, submitted by Mr **T. Kathiravan** for the award of the degree of Doctor of Philosophy in Mathematics, is the record of the research work done by the candidate during the period of his study (2013-2017) at Department of Mathematics, Ramanujan School of Mathematical Sciences, Pondicherry University, Puducherry, India, under my supervision and guidance and that no part thereof has been presented for any degree, diploma, associateship or fellowship earlier.

Place : Puducherry **S. N. Fathima**

Date : Supervisor

DECLARATION

I hereby declare that the work presented in this thesis is original and performed under the supervision and guidance of **Dr. S. N. Fathima**, Assistant Professor, Department of Mathematics, Ramanujan School of Mathematical Sciences, Pondicherry University, Puducherry. I further state that this work has not formed the basis for the award of any other degree of this university or any other universities.

Place: Puducherry

Date:

(T. Kathiravan)

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T. Kathiravan

LIST OF PUBLICATIONS

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Chapter 1

Introduction

1.1 Section 1

Srinivasa Ramanujan, acknowledged as the famous Indian mathematician was born on December 22, 1887. His bibliography is brilliantly penned by Robert Kanigel [62], in *The man who knew infinity* and by K. Srinivasa Rao [94], in *Srinivasa Ramanujan: a mathematical genius.*

During the year 1903-1914, Ramanujan recorded his mathematical discoveries in three notebooks, without providing proofs. The astounding number of results are related to Number theory, Hypergeometric functions, Modular functions and Analysis with significant contribution to the development of Partition theory, q-series and Continued fractions.

It was only in 1957, the Ramanujan's notebooks were made public when Tata Institute of Fundamental Research in Bombay published a photocopy edition. In 1976, when G. E. Andrews visited the Trinity College Library at Cambridge University, he unearthed about 140 handwritten pages of Ramanujan containing over 600 results, fall under the purview of mock theta functions, q-series, Continued fractions, Asymptotic expansions, Approximations and Class invariants. In 1988, Narosa Publishing House, New Delhi published a photocopy edition of the lost notebook along with few unpublished manuscripts of Ramanujan.

After the death of Ramanujan on April 20, 1920, G. H. Hardy urged that Ramanujan's notebooks be edited and published. Although in 1929, G. N. Watson and B. M. Wilson had undertaken the task of editing Ramanujan's notebooks, but the project never completed partly due to premature death of Wilson in 1935. Bruce C. Berndt of University of Illinois, USA, completed the task with the help of other mathematicians. As a result, we now have five edited volumes, Ramanujan's Notebooks Part *I-V* [21, 22, 23, 24, 25] which contain proofs of the theorems or references to the proof in the literature are provided. The five volumes contain 3254 results. Andrews and Berndt have published [9, 10, 11, 12] four of approximately five volumes devoted to the claims made by Ramanujan in the lost note book and other unpublished papers.

It is strongly believed by mathematicians several of the Ramanujan's results pertaining to theta function identities, modular equations, continued fractions remain to be elucidated by the methods known to Ramanujan.

The research work presented in this thesis for the most part is based on and motivated by the works of Ramanujan.

In what follows we employ the usual notations:

$$(a)_\infty := (a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n),$$

$$(a_1, a_2, \cdots, a_n; q)_\infty = (a_1;q)_\infty(a_2;q)_\infty \cdots (a_n;q)_\infty$$

and

$$(a)_n := (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k) = \frac{(a)_\infty}{(aq^n)_\infty}, \quad n : \text{any integer} \tag{1.1.1}$$

where a and q are complex number with $|q| < 1$. In particular, if n is a positive integer

$$(a)_{-n} = \frac{(-1)^n q^{n(n+1)/2}}{a^n(q/a)_n}, \quad a \neq 0. \tag{1.1.2}$$

we shall define ${}_rF_s$ generalized hypergeometric series by

$${}_rF_s\left[\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{matrix}; z\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \ldots (a_r)_n}{n!(b_1)_n (b_2)_n \ldots (b_s)_n} z^n,$$

where

$$(a)_n := a(a+1)(a+2)\cdots(a+n-1).$$

By the ratio test, the ${}_rF_s$ series converges absolutely for all z if $r \leq s$ and for $|z| < 1$ if $r = s+1$.

The basic hypergeometric series ${}_{s+1}\phi_s$ is defined by

$${}_{s+1}\phi_s\left[\begin{matrix} a_1, a_2, \ldots, a_{s+1} \\ b_1, b_2, \ldots, b_s \end{matrix}; q, z\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \ldots (a_{s+1})_n}{(q)_n (b_1)_n (b_2)_n \ldots (b_s)_n} z^n,$$

where $|z| < 1$ and $a_1, a_2, \cdots, a_{s+1},\ b_1, b_2, \cdots, b_s$ are arbitrary, except that of course $(b_j)_n \neq 0$, $1 \leq j \leq s$, $0 \leq n < \infty$ and $(a)_n$ is as in (1.1.1). For $0 < |q| < 1$, the series on the right hand side of ${}_{s+1}\phi_s$ converges absolutely for $|z| < 1$.

The basic bilateral hypergeometric series ${}_r\psi_r$ is defined by

$${}_s\psi_s\left[\begin{matrix} a_1, a_2, \ldots, a_s \\ b_1, b_2, \ldots, b_s \end{matrix}; q, z\right] = \sum_{n=-\infty}^{\infty} \frac{(a_1)_n (a_2)_n \ldots (a_s)_n}{(b_1)_n (b_2)_n \ldots (b_s)_n} z^n,$$

where $(a)_n$ and $(a)_{-n}$ are as defined in (1.1.1) and (1.1.2) and the denominator

factor are never zero. For $0 < |q| < 1$, the series converges absolutely in the annulus $\begin{bmatrix} b_1, \ldots, b_r \\ a_1, \ldots, a_r \end{bmatrix} < |z| < 1$.
We use the notation

$$\frac{a_1}{b_1 +}\,\frac{a_2}{b_2 +}\,\frac{a_3}{b_3 +}\cdots, \tag{1.1.3}$$

for the continued fraction

$$\cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cdots}}}.$$

We let A_n and B_n denote the n^{th} numerator and denominator respectively, for (1.1.3). Thus for $n \geq 1$,

$$\frac{a_1}{b_1 +}\,\frac{a_2}{b_2 +}\,\frac{a_3}{b_3 +}\cdots\frac{a_n}{+\, b_n} = \frac{A_n}{B_n}$$

where

$$\begin{aligned} A_n &= b_n A_{n-1} + a_n A_{n-2}, \\ B_n &= b_n B_{n-1} + a_n B_{n-2}, \\ A_{-1} &= 1 = B_0 \end{aligned}$$

and

$$A_0 = 0 = B_{-1}.$$

The set of natural numbers is denoted by N, the set of integers by Z, the set of real numbers by R and the set of complex numbers by C. We set $\widehat{R} = R \cup \{\infty\}$ and $\widehat{C} = C \cup \{\infty\}$.

If $p_N = 0$, we say the continued fraction (1.1.3) terminates, and we assign to it the value

$$f := \frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \frac{a_3}{b_3 +} \cdots + \frac{a_{N-1}}{b_{N-1}} = \frac{A_{N-1}}{B_{N-1}},$$

if $a_n \neq 0,\ 1 \leq n < N$. If $a_n \neq 0,\ 1 \leq n < \infty$, then the continued fraction (1.1.3) converges if $\lim_{n\to\infty} \left(\frac{A_n}{B_n}\right)$ exists in $\widehat{C}$. Its value is given by

$$f = \lim_{n\to\infty} \left(\frac{A_n}{B_n}\right),$$

and we write

$$f := \frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \frac{a_3}{b_3 +} \cdots.$$

If $\lim_{n\to\infty} \left(\frac{A_n}{B_n}\right)$ does not exist in $\widehat{C}$, (and $a_n \neq 0, 1 \leq n < \infty$), we say that (1.1.3) diverges.

1.2 Section 2

Leibniz observation about partition put in modern notation is, a partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$ of a non-negative integer n is a finite sequence of non-increasing positive integer parts λ_i such that $n = \sum_{i=1}^{k} \lambda_i$. The partition function $p(n)$ is the number of partitions of a non-negative integer n, with the convention that $p(0) = 1$. For example, we have $p(6) = 11$, as there are 11 paritions of 6, namely, (6), $(5,1)$, $(4,2)$, $(4,1,1)$, $(3,3)$, $(3,2,1)$, $(3,1,1,1)$, $(2,2,2)$, $(2,2,1,1)$, $(2,1,1,1,1)$ and $(1,1,1,1,1,1)$. The theory of partition is a subject that naturally fits into the theory of q-series and also it is highly combinatorial. Euler, Sylvester, MacMahon, Rogers, Hardy, Ramanujan and Rademacher have played a seminal role in the development of partitions.

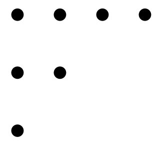
Euler gave the generating function for $p(n)$ using the q-series by

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)}.$$

Often generating functions leads us to relating one class of partition to another. For example, "The number of partitions of n in which the difference between any two parts is at least 2 equals the number of partitions of n into parts congruent to $\pm 1 \pmod 5$", and it is the combinatorial interpretation of the analytic identity due to Rogers and Ramanujan:

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty}, \ |q| < 1.$$

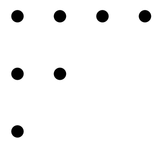
A partition is often represented with the help of a diagram called Ferrers-Young diagram. The Ferrers-Young diagram of the partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$ of n is formed by arranging n nodes in k rows so that the ith row has λ_i nodes. For example, the Ferrers-Young diagram of partition $\lambda = (4, 2, 1)$ of 7 is



The conjugate of a partition λ, denoted λ', is the partition whose Ferrers-Young diagram is the reflection along the main diagonal of the diagram of λ. Therefore, the conjugate of the partition $(4, 2, 1)$ is $(3, 2, 1, 1)$. A partition λ is self-conjugate if $\lambda = \lambda'$. For example, the partition (4,2,1,1) of 8 is self conjugate.

The nodes in the Ferrers-Young diagram of a partition are labeled by row and column coordinates as one would label the entries of a matrix. Let λ'_j denote the number of nodes in column j. The hook number $H(i, j)$ of the (i, j) node is defined as the number of nodes directly below and to the right of the node and including the node itself. That is, $H(i, j) = \lambda_i + \lambda'_j - j - i + 1$. A partition λ is said to be a t-core if and only if it has no hook numbers that are multiples of t.

Example. The Ferrers-Young diagram of the partition $\lambda = (4, 2, 1)$ of 7 is



The nodes $(1, 1)$, $(1, 2)$, $(1, 3)$, $(1, 4)$, $(2, 1)$, $(2, 2)$ and $(3, 1)$ have hook numbers 6, 4, 2, 1, 3, 1 and 1, respectively. Therefore, λ is a 5-core. Obviously, it is a t-core for $t \geq 7$.

In 1919, Ramanujan [86], [91, pp.210-213] gifted three simple congruences satisfied by $p(n)$, namely,

$$p(5n+4) \equiv 0 \pmod{5}, \tag{1.2.1}$$

$$p(7n+5) \equiv 0 \pmod{7}, \tag{1.2.2}$$

$$p(11n+6) \equiv 0 \pmod{11}. \tag{1.2.3}$$

He gave proofs of (1.2.1) and (1.2.2) in [86] and later in a short one page note [87], [88, p.230] announced that he had also found a proof of (1.2.3). In a posthumously published paper [88], [91, pp.232-238], Hardy extracted different proofs of (1.2.1)-(1.2.3) from an unpublished manuscript of Ramanujan [27], [90, pp.133-177].

Garvan, Kim and Stanton [47, 48] found that t-core are useful in establishing cranks, which are used to show a combinatorial proof of Ramanujan's famous congruences for the partition function. Garvan [46] also proved some "Ramanujan type" congruences for $a_p(n)$ for certain special small primes p. Hirschhorn and Sellers [56] proved multiplicative formulas for $a_4(n)$ and also conjectured similar multiplicative properties for $a_p(n)$ for other primes p.

The t-core conjecture has been the topic of a number of papers [43, 46, 51, 66, 67, 78, 79]. This conjecture asserted that every natural number has a t-core partition for every integer $t \geq 4$. Using the theory of modular forms and quadratic forms Granville and Ono [51, 78, 79] have proved the conjecture. Kiming [66] gave a simple proof for the conjecture. We also refer to [15, 16, 17, 19, 20, 55, 56, 59, 65, 81] for futher results and generalizations on t-core.

In chapter 2, we obtain infinite families of arithmetic identities involving 15-core and 23-core.

1.3 Section 3

For integer $\ell > 1$, a partition of n is called ℓ-regular if none of its parts is divisible by ℓ. If $b_\ell(n)$ denotes the number of ℓ-regular partitions of n, then the generating function for $b_\ell(n)$ satisfies

$$\sum_{n=0}^{\infty} b_\ell(n)q^n = \frac{f_\ell}{f_1}. \tag{1.3.1}$$

The arithmetic of ℓ-regular partition functions has received a great deal of attention. Gordon and Ono [50] proved that if p is prime and $p^{ord_p(\ell)} \geq \sqrt{\ell}$, then for any positive integers n such that $b_\ell(n) \equiv 0 \pmod{p^j}$ is one. Andrews, Hirschhorn and Sellers [13] established infinite families of congruences modulo 3 for $b_4(n)$, and analogous results were proven by Webb [96] for $b_{13}(n)$ and by Furcy and Penniston [45] for several other values of ℓ. And in [97] Xia found congruences for $b_4(n)$ modulo 8 (for more results on ℓ-regular partitions see [3, 31, 32, 35, 41, 42, 60, 63, 71, 72, 74, 80, 83, 84, 98, 99, 100]).

An ℓ-regular bipartitions of n is an ordered pair of ℓ-regular partitions (λ, μ) such that the sum of all of the parts equals n. Let $B_\ell(n)$ denote the number of ℓ-regular bipartitions of n. Then the generating function of $B_\ell(n)$ satisfies

$$\sum_{n=0}^{\infty} B_\ell(n)q^n = \frac{f_\ell^2}{f_1^2}. \tag{1.3.2}$$

In chapter 3, we establish some congruence modulo ℓ for ℓ-regular bipartitions, where $\ell \in \{5, 7, 13\}$.

1.4 Section 4

In [40], Corteel and Lovejoy introduced overpartitons. An overpartition of a positive integer n is a nonincreasing sequence of positive integers whose sum is n in which first occurrence of a distinct number may be overlined. Let $\overline{p}(n)$ denote the number of overpartitions of n. For example, the overpartitions of 3 are $(3), (\overline{3}), (2,1), (\overline{2},1), (2,\overline{1}), (\overline{2},\overline{1}), (1,1,1), (\overline{1},1,1)$.

The generating function for $\overline{p}(n)$, is given by

$$\sum_{n=0}^{\infty} \overline{p}(n)q^n = \frac{(-q;q)_\infty}{(q;q)_\infty} = \frac{f_2}{f_1^2}. \tag{1.4.1}$$

The function $\overline{p}(n)$ has been considered recently by number of mathematicians including Hirschhorn and Sellers [57, 58], Mahlburg [76] and Kim [64]. Overpartitions have been used in combinatorial proofs of many q-series identities and these partitions aries quite naturally in the study of hypergeometric series (see[38, 39, 40, 73, 82]). Overpartitions also arise in theoretical physics as jagged partitions in the solution of certain problems regarding seas of particles and fields (see[44]), where a jagged partition of n is an ordered sequence of nonnegative integers $(\lambda_m, \cdots, \lambda_1)$ that sum to n and satisfy the weakly decreasing conditions, $\lambda_j \geq \lambda_{j-1} - 1$ and $\lambda_j \geq \lambda_{j-2}$.

Recently, Andrews [8] introduced singular overpartitions. To introduce singular overpartitions, first he defined some properties of the entries in a Frobenius symbol for n, which is of the form

$$\begin{pmatrix} a_1 & a_2 & \cdot & \cdot & \cdot & a_r \\ b_1 & b_2 & \cdot & \cdot & \cdot & b_r \end{pmatrix}$$

where the rows are strictly decreasing sequences of non-negative integers and $\sum_{i=1}^{r}(a_i + b_i + 1) = n$. Andrews defined a column $\begin{matrix} a_j \\ b_j \end{matrix}$ in a Frobenius symbol as (k,i)-positive if $a_j - b_j \geq k - i - 1$, (k,i)-negative if $a_j - b_j \leq -i + 1$ and (k,i)-neutral if $-i+1 < a_j - b_j < k - i - 1$. He then divided the Frobenius symbol into (k,i)-parity blocks, where if two columns $\begin{matrix} a_n \\ b_n \end{matrix}$ and $\begin{matrix} a_j \\ b_j \end{matrix}$ are both (k,i)-positive or both (k,i)-negative, then they have the same (k,i)-parity. These blocks are the sets of contiguous columns maximally extended to the right:

$$\begin{matrix} a_n & a_{n+1} & \cdot & \cdot & \cdot & a_j \\ b_n & b_{n+1} & \cdot & \cdot & \cdot & b_j \end{matrix}$$

where all the entries have either the same (k,i)-parity or are (k,i)-neutral. The first non-neutral column in each parity block is called the anchor of the block.
A Frobenius symbol is said to be (k,i)-singular, if the following properties hold

1. there are no overlined entries, or
2. the one overlined entry on the top row occurs in the anchor of a (k,i)-positive block, or
3. the one overlined entry on the bottom row occurs in an anchor of a (k,i)-negative block, and
4. if there is one overlined entry in each row, then they occur in adjacent (k,i)-parity blocks.

Andrews denoted the number of such singular overpartitions of n as $\overline{Q}_{k,i}(n)$. He found that $\overline{Q}_{k,i}(n)$ is equal to $\overline{C}_{k,i}(n)$, the number of overpartitions of n in which no part is divisible by k and only parts $\equiv \pm i \pmod{k}$ may be overlined, i.e.,

$$\sum_{n=0}^{\infty} \overline{Q}_{k,i}(n)q^n = \sum_{n=0}^{\infty} \overline{C}_{k,i}(n)q^n = \frac{(q^k;q^k)_\infty(-q^i;q^k)_\infty(-q^{k-i};q^k)_\infty}{(q;q)_\infty}. \tag{1.4.2}$$

In chapter 4, we established several new congruences for $\overline{C}_{k,i}(n)$ for certain values of k and i by employing simple p-dissections of Ramanujan's theta functions.

Since our proofs mainly rely on various properties of Ramanujan's theta functions and dissections of certain q-products, we define a t-dissections and Ramanujan's general theta function and some of its special cases.

If $P(q)$ denotes a power series in q, than a t-dissection of $P(q)$ is given by

$$[P(q)]_{t-\text{dissection}} = \sum_{k=0}^{t-1} q^k P_k(q^t),$$

where P_k are power series in q^t.

Ramanujan's general theta function $f(a,b)$ is defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1.$$

Ramanujan's theta function $f(a,b)$ is equivalent of Jacobi's theta function [1, 23, 89]

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty$$

certain special cases of $f(a,b)$ are defined by

$$\varphi(q) := f(q,q) = \sum_{k=-\infty}^{\infty} q^{k^2} = \frac{(-q;q^2)_\infty (q^2;q^2)_\infty}{(q;q^2)_\infty (-q^2;q^2)_\infty}, \tag{1.4.3}$$

$$\psi(q) := f(q,q^3) = \frac{1}{2} f(1,q) = \sum_{n=0}^{\infty} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{1.4.4}$$

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty. \tag{1.4.5}$$

Following Ramanujan, we also define

$$\chi(q) := (-q;q^2)_\infty \tag{1.4.6}$$

one can easily show that

$$\varphi(q) = \frac{f_2^5}{f_1^2 f_4^2}, \quad \varphi(-q) = \frac{f_1^2}{f_2}, \quad \psi(q) = \frac{f_2^2}{f_1}, \quad \psi(-q) = \frac{f_1 f_4}{f_2},$$

$$\chi(q) = \frac{f_2^2}{f_1 f_4}, \quad \chi(-q) = \frac{f_1}{f_2} \text{ and } f(q) = \frac{f_2^3}{f_1 f_4}$$

where $f_n := f(-q^n)$.

1.5 Section 5

Prominent mathematicians like Jacobi, Gauss, Cauchy, Steiltjes and Ramanujan have contributed significantly to the theory of continued fractions.

Continued fractions are important in several branches of mathematics. Finite simple continued fractions are useful to solve linear Diophantine equation $ax + by = c$ whereas infinite continued fractions have been used in computer algorithm for computing rational approximation of real numbers. Also these infinite continued fractions play an important role to find solutions of Pell's equation $x^2 - dy^2 = N$.

Ramanujan's contribution to the field of continued fraction is magnificent. This notebooks contains nearly 200 results related to continued fraction. Chapter 12 of his second notebook [89] is entirely devoted to continued fractions. Several of his interesting continued fractions can be found in chapter 16 of his second notebook and in the 'lost' notebook of Ramanujan. Ramanujan's most crowing achievements in the theory of continued fraction is the Rogers-Ramanujan continued fraction identity,

$$\begin{aligned} R(q) &:= \frac{q^{1/5}}{1} \, \frac{q}{1+} \, \frac{q^2}{1+} \, \frac{}{1+\cdots} \\ &= q^{1/5} \frac{(q;q^5)_\infty (q^4;q^5)_\infty}{(q^2;q^5)_\infty (q^3;q^5)_\infty} \quad |q| < 1. \end{aligned} \tag{1.5.1}$$

The first proof of (1.5.1) was given by Rogers [93]. Ramanujan [89] rediscovered and proved the continued fraction (1.5.1), therefore the continued fraction (1.5.1) enjoy the name Rogers-Ramanujan's continued fraction.

On page 46 of his 'lost' notebook [90], Ramanujan gave the following integral representation for $R(q)$ is

$$R(q) = \frac{\sqrt{5}-1}{2} exp\left(\frac{-1}{5}\right) \int_q^1 \frac{(1-t)^5(1-t^2)^5\cdots}{(1-t^5)(1-t^{10})\cdots} \frac{dt}{t}.$$

Several generalization and ramification of the continued fraction $R(q)$ have been recorded by Ramanujan in his 'lost' notebook. Also, in his letters [91] to Hardy, Ramanujan communicated the values of $R(e^{-2\pi})$, $R(-e^{-\pi})$ and $R(e^{-2\pi/\sqrt{5}})$. Many mathematicians like B. C. Berndt and H. H. Chan [26] and K. G. Ramanathan [85] have extensively studied the values of $R(q)$. Generalizations and related works of $R(q)$ may be found in papers by Al-Salam and Ismail [7], B. Gordon [49], M. D. Hirschhorn [52], S. Bhargava and C. Adiga [28], S. Bhargava, C. Adiga and D. D. Somashekara [29] and many others.

On page 366 of his 'lost' notebook, Ramanujan investigated the continued fraction

$$G(q) := \frac{q^{1/3}}{1}_{+} \frac{q+q^2}{1}_{+} \frac{q^2+q^4}{1}_{+} \frac{q^3+q^6}{1}_{+\cdots}, \quad |q| < 1,$$

which is known as Ramanujan's cubic continued fraction. H. H. Chan [33] has established several modular equations relating $G(q)$ with $G(-q)$, $G(q^2)$ and $G(q^3)$.

Chan and Sen-Shan Huang [34] studied the Ramanujan-Göllnitz-Gordon continued fraction

$$H(q) := \frac{q^{1/2}}{1+q}_{+} \frac{q^2}{1+q^3}_{+} \frac{q^4}{1+q^5}_{+} \frac{q^6}{1+q^7}_{+\cdots}, \quad |q| < 1.$$

Recently C. Adiga and T. Kim [2] established an integral representation of a q-continued fraction of Ramanujan and obtained its explicit evaluations, also they derived its relation with $H(q)$.

Motivated by these works in chapter 5 we derive several identities involving the Ramanujan's continued fraction $M(q)$ given

$$\begin{aligned} M(q) &:= \frac{q^{1/2}}{1-q\,+}\;\frac{q(1-q)}{1+q^2}\;+\frac{q(1-q^3)^2}{(1-q)(1+q^4)}\;+\frac{q(1-q^5)^2}{(1-q)(1+q^6)}\;+\cdots, \quad |q|<1 \\ &= q^{1/2}\,\frac{(q^4;q^4)_\infty^2}{(q^2;q^4)_\infty^2}. \end{aligned} \tag{1.5.2}$$

Chapter 2

Congruences for 15-core and 23-core partition

2.1 Introduction

If $a_t(n)$ denotes the number of partitions of n that are t-core, then the generating function for $a_t(n)$ is given by [47, Equation (2.1)], [77, Proposition (3.3)]

$$\sum_{n=0}^{\infty} a_t(n)q^n = \frac{(q^t;q^t)_\infty^t}{(q;q)_\infty}. \tag{2.1.1}$$

The study of t-cotes for t prime first arose in connection with Nakayama's conjecture [61, 92]. Using the theory of modular forms, Granville and One [78] proved that

$$a_3(n) = d_{1,3}(3n+1) - d_{2,3}(3n+1), \tag{2.1.2}$$

where $d_{r,3}(n)$ is the number of divisors of n congruent to $r \pmod 3$. Baruah and Berndt [15] used a modular equation of Ramanujan to prove that

$$a_3(4n+1) = a_3(n), \quad \text{for all} \quad n \geq 0. \tag{2.1.3}$$

In [55, 56], Hirschhorn and Sellers used some elementary generating function manipulations to find certain congruences and the following infinite families of arithmetic relations involving 4-cores: for $k \geq 1$,

$$3^k a_4(3n) = a_4\left(3^{2k+1}n + \frac{5 \times 3^{2k} - 5}{8}\right), \tag{2.1.4}$$

$$(2 \times 3^k - 1)a_4(3n+1) = a_4\left(3^{2k+1}n + \frac{13 \times 3^{2k} - 5}{8}\right), \tag{2.1.5}$$

$$\left(\frac{3^{k+1}-1}{2}\right) a_4(9n+2) = a_4\left(3^{2k+2}n + \frac{7 \times 3^{2k+1} - 5}{8}\right), \tag{2.1.6}$$

$$\left(\frac{3^{k+1}-1}{2}\right) a_4(9n+8) = a_4\left(3^{2k+2}n + \frac{23 \times 3^{2k+1} - 5}{8}\right). \tag{2.1.7}$$

In the next section, we obtain our main results.

2.2 Main Theorems

In order to prove our main results, we collect a few lemmas.

By the binomial theorem, for any positive integer k,

$$f_1^{2^k} \equiv f_2^{2^{k-1}} \pmod{2^k}. \tag{2.2.1}$$

Lemma 2.2.1. (Cui and Gu [41, Theorem 2.2]) If $p \geq 5$ is a prime and

$$\frac{\pm p-1}{6} := \begin{cases} \dfrac{p-1}{6}, & \text{if } p \equiv 1 \pmod{6}, \\ \dfrac{-p-1}{6}, & \text{if } p \equiv -1 \pmod{6}, \end{cases}$$

then

$$(q;q)_\infty = \sum_{\substack{k=-\frac{p-1}{2} \\ k \neq \frac{\pm p-1}{6}}}^{\frac{p-1}{2}} (-1)^k q^{\frac{3k^2+k}{2}} f\left(-q^{\frac{3p^2+(6k+1)p}{2}}, -q^{\frac{3p^2-(6k+1)p}{2}}\right)$$
$$+ (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} (q^{p^2};q^{p^2})_\infty. \tag{2.2.2}$$

Furthermore, if $\frac{-(p-1)}{2} \leq k \leq \frac{(p-1)}{2}$, $k \neq \frac{(\pm p-1)}{6}$, then $\frac{3k^2+k}{2} \not\equiv \frac{p^2-1}{24} \pmod{p}$.

Lemma 2.2.2. (Ahmed and Baruah [6, Eqn. (3.5)])

$$\frac{1}{(q;q)_\infty(q^{15};q^{15})_\infty} = \frac{1}{(q^2;q^2)_\infty^2(q^{30};q^{30})_\infty^2} \left(\psi(q^6)\psi(q^{10}) + qf(q^{90},q^{150})f(q^2,q^{14})\right.$$
$$\left. + q^{15} f(q^{30},q^{210})f(q^6,q^{10})\right). \tag{2.2.3}$$

Theorem 2.2.1. For any non-negative integer k, we have

$$\sum_{n=0}^{\infty} a_{23}\left(8\cdot 23^{2k+1}n+23^{2k+1}-22\right)q^n \equiv f_1f_2+q^2f_1f_4f_{46} \pmod{2}. \tag{2.2.4}$$

Proof. From (2.1.1), we have

$$\sum_{n=0}^{\infty} a_{23}(n)q^n \equiv \frac{f_{184}^3}{f_1f_{23}} \pmod{2}. \tag{2.2.5}$$

Now from [101, Lemma 2.1.], we have

$$\frac{1}{f_1f_{23}} \equiv \sum_{n=0}^{\infty} p_{1^1 23^1}(2n)q^{2n}+q+q^3f_2f_{46} \pmod{2} \tag{2.2.6}$$

where $p_{1^1 23^1}(n)$ is defined by

$$\sum_{n=0}^{\infty} p_{1^1 23^1}(n)q^n = \frac{1}{f_1f_{23}}.$$

From (2.2.5) and (2.2.6), we obtain

$$\sum_{n=0}^{\infty} a_{23}(n)q^n \equiv f_{184}^3\left(\sum_{n=0}^{\infty} p_{1^1 23^1}(2n)q^{2n}+q+q^3f_2f_{46}\right) \pmod{2}. \tag{2.2.7}$$

Extracting the terms involving q^{2n+1} from both sides of (2.2.7), dividing both sides by q and then replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} a_{23}(2n+1)q^n \equiv f_{92}^3+q\frac{f_2f_{46}^7}{f_1f_{23}} \pmod{2}. \tag{2.2.8}$$

Now, substituting (2.2.6) in (2.2.8) and extracting the terms involving q^{2n} from both sides of the resulting congruence and then replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} a_{23}(4n+1)q^n \equiv f_{46}^3 + q\frac{f_2 f_{184}}{f_1 f_{23}} + q^2 f_2 f_{184} \pmod{2}. \tag{2.2.9}$$

Again substituting (2.2.6) in (2.2.9) and extracting the terms involving q^{2n} from both sides of the resulting congruence and then replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} a_{23}(8n+1)q^n \equiv f_{23}^3 + q^2 f_2 f_{23}^5 \pmod{2}. \tag{2.2.10}$$

Taking $p = 23$ in (2.2.2), and q replacing by q^2, we get

$$f_2 = \left(\sum_{\substack{k=-11 \\ k\neq -4}}^{11} (-1)^k q^{3k^2+k} f\left(-q^{46(35+3k)}, -q^{46(35-3k)}\right) + q^{44} f_{1058} \right) \tag{2.2.11}$$

Note that for $-11 \leq k \leq 11$ and $k \neq -4$,

$$3k^2 + k \not\equiv 44 \pmod{23}.$$

Employing (2.2.11) in (2.2.10) and extracting the terms involving q^{23n} from both sides of the resulting congruence and then replacing q^{23} by q, we obtain

$$\sum_{n=0}^{\infty} a_{23}(184n+1)q^n \equiv f_1 f_2 + q^2 f_1 f_4 f_{46} \pmod{2}. \tag{2.2.12}$$

Which is the $k = 0$ case of (2.2.4). Now suppose (2.2.4) holds for some $k \geq 0$. Again taking $p = 23$ in (2.2.2), we obtain

$$f_1 = \left(\sum_{\substack{k=-11 \\ k \neq -4}}^{11} (-1)^k q^{\frac{3k^2+k}{2}} f\left(-q^{23(35+3k)}, -q^{23(35-3k)}\right) + q^{22} f_{529} \right) \tag{2.2.13}$$

Note that for $-11 \leq k \leq 11$ and $k \neq -4$,

$$\frac{3k^2+k}{2} \not\equiv 22 \pmod{23}.$$

If we replace q by q^4 in (2.2.13), we get

$$f_4 = \left(\sum_{\substack{k=-11 \\ k \neq -4}}^{11} (-1)^k q^{2\cdot(3k^2+k)} f\left(-q^{92(35+3k)}, -q^{92(35-3k)}\right) + q^{88} f_{2116} \right) \tag{2.2.14}$$

Note that for $-11 \leq k \leq 11$ and $k \neq -4$,

$$2 \cdot (3k^2+k) \not\equiv 88 \pmod{23}.$$

Employing (2.2.11), (2.2.13) and (2.2.14) in (2.2.12) and extracting the terms involving q^{23n+20} from both sides of the resulting congruence, dividing both sides by q^{20} and then replacing q^{23} by q, we obtain

$$\sum_{n=0}^{\infty} a_{23}\left(8 \cdot 23^{2k+2} n + 7 \cdot 23^{2k+2} - 22\right) q^n \equiv q^2 f_{23}^3 + q^4 f_2 f_{23}^5 \pmod{2}. \tag{2.2.15}$$

Employing (2.2.11) in (2.2.15) and extracting the terms involving q^{23n+2} from both sides of the resulting congruence, dividing both sides by q^2 and then replacing q^{23} by q, we obtain

$$\sum_{n=0}^{\infty} a_{23}\left(8\cdot 23^{2k+3}n+23^{2k+3}-22\right)q^n \equiv f_1f_2+q^2f_1f_4f_{46} \pmod{2}. \qquad (2.2.16)$$

This completes the proof by induction of (2.2.4). $\square$

Theorem 2.2.2. If $\ell \in \{5,7,10,11,14,15,17,19,20,21,22\}$, then for all $n \geq 0$,

$$a_{23}\left(8(23n+\ell)+1\right) \equiv 0 \pmod{2} \qquad (2.2.17)$$

and if $m \in \{1,7,9,12,13,16,17,19,21,22\}$, then for all $n \geq 0$,

$$a_{23}\left(8\cdot 23^{2k+2}(23n+m)+7\cdot 23^{2k+2}-22\right) \equiv 0 \pmod{2}. \qquad (2.2.18)$$

Proof. Employing (2.2.11) in (2.2.10) and then equating the coefficients of $q^{23n+\ell}$ from both sides we obtain (2.2.17). And also employing (2.2.11) in (2.2.15) and then equating the coefficients of q^{23n+m} from both sides we obtain (2.2.18).

$\square$

Theorem 2.2.3. For any non-negative integer k, we have

$$\sum_{n=0}^{\infty} a_{15}\left(8\cdot 5^{2k+1}n+\frac{70\cdot 5^{2k}-28}{3}\right)q^n \equiv f_5f_3^3 \pmod{2}. \qquad (2.2.19)$$

Proof. Again from (2.1.1), we have

$$\sum_{n=0}^{\infty} a_{15}(n)q^n \equiv \frac{f_{240}}{f_1f_{15}} \pmod{2}. \qquad (2.2.20)$$

Substituting (2.2.3) in (2.2.20) and extracting the terms involving q^{2n} from both sides of the resulting congruence and then replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} a_{15}(2n)q^n \equiv \frac{f_{120}f_{12}f_{20}}{f_2f_{30}f_3f_5} \pmod 2. \tag{2.2.21}$$

Now, from [18, Eq. (4.11)], we have

$$\sum_{n=0}^{\infty} p_{3^15^1}(2n+1)q^n = q\frac{f_2^2f_{30}^2}{f_3^2f_5^2f_1f_{15}}, \tag{2.2.22}$$

where $p_{3^15^1}(n)$ is defined by

$$\sum_{n=0}^{\infty} p_{3^15^1}(n)q^n = \frac{1}{f_3f_5}.$$

Extracting the terms involving q^{2n+1} from both sides of the congruence, dividing both sides by q, replacing q^2 by q and then employing (2.2.22), we obtain

$$\sum_{n=0}^{\infty} a_{15}(4n+2)q^n \equiv qf_2f_{30}^3 \pmod 2. \tag{2.2.23}$$

From (2.2.23), we have

$$\sum_{n=0}^{\infty} a_{15}(8n+6)q^n \equiv f_1f_{15}^3 \pmod 2. \tag{2.2.24}$$

Ramanujan [91] stated the following identity without proof:

$$f_1 = f_{25}\left(R^{-1} - q - q^2R\right), \tag{2.2.25}$$

where

$$R = \frac{(q^5, q^{20}; q^{25})_\infty}{(q^{10}, q^{15}; q^{25})_\infty}.$$

Substituting (2.2.25) in (2.2.24) and extracting the terms involving q^{5n+1} from both sides of the resulting congruence, dividing both sides by q and then replacing q^5 by q, we obtain

$$\sum_{n=0}^{\infty} a_{15}(40n+14)q^n \equiv f_5 f_3^3 \pmod 2. \tag{2.2.26}$$

Which is the $k = 0$ case of (2.2.19). Now suppose (2.2.19) holds for some $k \geq 0$. Next, take power three on both side in (2.2.25) and replacing q by q^3, we obtain

$$f_3^3 = f_{75}^3 \left(S^{-3} - 3q^3 S^{-2} + 5q^9 - 3q^{15} S^2 - q^{18} S^3\right) \tag{2.2.27}$$

where

$$S = \frac{(q^{15}, q^{60}; q^{75})_\infty}{(q^{30}, q^{45}; q^{75})_\infty}.$$

Substituting (2.2.27) in (2.2.26) and extracting the terms involving q^{5n+4} from both sides of the resulting congruence, dividing both sides by q^4 and then replacing q^5 by q, we obtain

$$\sum_{n=0}^{\infty} a_{15}\left(8 \cdot 5^{2k+2} n + \frac{110 \cdot 5^{2k+1} - 28}{3}\right) q^n \equiv q f_1 f_{15}^3 \pmod 2. \tag{2.2.28}$$

Substituting (2.2.25) in (2.2.28) and extracting the terms involving q^{5n+2} from both sides of the resulting congruence, dividing both sides by q^2 and then replacing q^5 by q, we obtain

$$\sum_{n=0}^{\infty} a_{15}\left(8\cdot 5^{2k+3}n+\frac{70\cdot 5^{2k+2}-28}{3}\right)q^n \equiv f_5 f_3^3 \pmod 2. \tag{2.2.29}$$

This completes the proof by induction of (2.2.19). □

Theorem 2.2.4. For all $n \geq 0$,

$$a_{15}(8n+2) \equiv 0 \pmod 2 \tag{2.2.30}$$

$$a_{15}(8(5n+s)+6) \equiv 0 \pmod 2,\ s\in\{3,4\} \tag{2.2.31}$$

$$a_{15}\left(8\cdot 5^{2k+1}(5n+s)+\frac{70\cdot 5^{2k}-28}{3}\right) \equiv 0 \pmod 2,\ s\in\{1,2\} \tag{2.2.32}$$

Proof. The result (2.2.30) follows from (2.2.23). Employing (2.2.25) in (2.2.24) and then equating the coefficients of q^{5n+s} from both sides we obtain (2.2.31). Employing (2.2.27) in (2.2.19), we obtain (2.2.32).

□

Chapter 3

Congruences for ℓ-Regular bipartition modulo ℓ

3.1 Introduction

We stated in the introductory chapter that the generating function of $B_\ell(n)$ satisfies

$$\sum_{n=0}^{\infty} B_\ell(n)q^n = \frac{f_\ell^2}{f_1^2}. \tag{3.1.1}$$

For example,

$$\sum_{n=0}^{\infty} B_4(n)q^n = \frac{f_4^2}{f_1^2}. \tag{3.1.2}$$

Recently Lin [68] studied the arithmetic properties of the function $ped_{-2}(n)$ whose generating function is identical to that of $B_4(n)$, and in [69] and [70] established infinite families of congruences modulo 3 for $B_7(n)$ and $B_{13}(n)$. For example,

$$B_7\left(3^\alpha n + \frac{5 \cdot 3^{\alpha-1}}{2}\right) \equiv 0 \pmod{3} \tag{3.1.3}$$

and

$$B_{13}(3^\alpha n + 2 \cdot 3^{\alpha-1} - 1) \equiv 0 \pmod{3}. \tag{3.1.4}$$

for all $\alpha \geq 2$ and $n \geq 0$.

In next section we obtain our main result.

3.2 Main Theorems

In this chapter we establish some congruences modulo ℓ for ℓ-regular bipartitions, where $\ell \in \{5, 7, 13\}$.

By the binomial theorem, it is easy to see that for any prime number ℓ,

$$f_\ell \equiv f_1^\ell \pmod{\ell}.$$

Lemma 3.2.1. (Hirschhorn and Sellers [60])

$$\frac{f_5}{f_1} = \frac{f_8 f_{20}^2}{f_2^2 f_{40}} + q\frac{f_4^3 f_{10} f_{40}}{f_2^3 f_8 f_{20}}. \tag{3.2.1}$$

Theorem 3.2.1. For all $\alpha \geq 1$ and $n \geq 0$, we have

$$B_5\left(4^\alpha n + \frac{4^\alpha - 1}{3}\right) \equiv 2^\alpha B_5(n) \pmod{5} \tag{3.2.2}$$

and

$$B_5\left(4^\alpha n + \frac{5 \times 4^\alpha - 2}{6}\right) \equiv 0 \pmod{5}. \tag{3.2.3}$$

Proof. If we square both sides of (3.2.1), extract the terms involving odd powers of q, then divide by q and replace q by $q^{\frac{1}{2}}$, we find that

$$\sum_{n=0}^{\infty} B_5(2n+1)q^n = 2\frac{f_2^3 f_5 f_{10}}{f_1^5}. \tag{3.2.4}$$

This yields

$$\sum_{n=0}^{\infty} B_5(2n+1)q^n \equiv 2f_2^3 f_{10} \pmod{5}. \tag{3.2.5}$$

It follows that, extract the terms involving even powers of q and replace q by $q^{\frac{1}{2}}$, we find that

$$\sum_{n=0}^{\infty} B_5(4n+1)q^n \equiv 2f_1^3 f_5 \equiv 2\frac{f_5^2}{f_1^2} \equiv 2\sum_{n=0}^{\infty} B_5(n)q^n \pmod{5}. \tag{3.2.6}$$

If from (3.2.5), we extract the terms involving odd powers of q, then divide by q and replace q by $q^{\frac{1}{2}}$, we find that

$$\sum_{n=0}^{\infty} B_5(4n+3)q^n \equiv 0 \pmod{5}. \tag{3.2.7}$$

and thus

$$B_5(4n+1) \equiv 2B_5(n) \pmod{5} \tag{3.2.8}$$

and

$$B_5(4n+3) \equiv 0 \pmod{5}. \tag{3.2.9}$$

for all $n \geq 0$. Iteratively replacing n by $4n+1$ in (3.2.8) yields (3.2.2), while replacing n by $4n+3$ in (3.2.2) and utilizing (3.2.9) yields (3.2.3). □

Theorem 3.2.2. For all $n \geq 0$,

$$B_7(25n+12) \equiv 5B_7(5n+2) + 4B_7(n) \pmod{7}. \tag{3.2.10}$$

Proof. Ramanujan [91] stated the following identity without proof:

$$\frac{f_1}{f_{25}} = R^{-1} - q - q^2 R, \tag{3.2.11}$$

where

$$R = \frac{(q^5, q^{20}; q^{25})_\infty}{(q^{10}, q^{15}; q^{25})_\infty}. \tag{3.2.12}$$

Using quintuple product identity (see [37]), Watson [95] gave a proof for (3.2.11). Later Hirschhorn [53] generalized (3.2.11) and established the identity

$$\frac{f_5^6}{f_{25}^6} = R^{-5} - 11q^5 - q^{10}R^5. \tag{3.2.13}$$

From (3.1.1), we have

$$\begin{aligned}\sum_{n=0}^{\infty} B_7(n) q^n &= \frac{f_7^2}{f_1^2} \\ &\equiv f_1^{12} \pmod{7}. \end{aligned} \tag{3.2.14}$$

Substituting (3.2.11) into (3.2.14), we have

$$
\begin{aligned}
\sum_{n=0}^{\infty} B_7(n)q^n &\equiv f_{25}^{12}(R^{-1} - q - q^2R)^{12} \pmod 7 \\
&\equiv \frac{f_{25}^{12}}{R^{12}}(1 + 2qR + 5q^2R^2 + 3q^3R^3 + 6q^4R^4 + 3q^5R^5 + q^6R^6 + 2q^7R^7 \\
&\quad + 4q^9R^9 + 3q^{10}R^{10} + q^{11}R^{11} + 4q^{12}R^{12} + 6q^{13}R^{13} + 3q^{14}R^{14} \\
&\quad + 3q^{15}R^{15} + 5q^{17}R^{17} + q^{18}R^{18} + 4q^{19}R^{19} + 6q^{20}R^{20} + 4q^{21}R^{21} \\
&\quad + 5q^{22}R^{22} + 5q^{23}R^{23} + q^{24}R^{24}) \pmod 7. \qquad (3.2.15)
\end{aligned}
$$

If from (3.2.15), we extract those terms in which the power of q is congruent to 2 modulo 5 and then divide by q^2, we find that

$$
\begin{aligned}
\sum_{n=0}^{\infty} B_7(5n+2)q^{5n} &\equiv f_{25}^{12}(5R^{-10} + 2q^5R^{-5} + 4q^{10} + 5q^{15}R^5 + 5q^{20}R^{10}) \pmod 7 \\
&\equiv f_{25}^{12}\left(5(R^{-5} - 11q^5 - q^{10}R^5)^2 + 4q^{10}\right) \pmod 7 \\
&\equiv f_{25}^{-}\left(5\left(\frac{f_5^6}{f_{25}^6}\right)^2 + 4q^{10}\right) \pmod 7 \\
&\equiv 5f_5^{12} + 4q^{10}f_{25}^{12} \pmod 7. \qquad (3.2.16)
\end{aligned}
$$

On replacing q by $q^{\frac{1}{5}}$, we find

$$
\begin{aligned}
\sum_{n=0}^{\infty} B_7(5n+2)q^{n} &\equiv 5f_1^{12} + 4q^2f_5^{12} \pmod 7 \\
&\equiv 5\frac{f_7^2}{f_1^2} + 4q^2\frac{f_{35}^2}{f_5^2} \pmod 7 \\
&\equiv 5\sum_{n=0}^{\infty} B_7(n)q^n + 4\sum_{n=0}^{\infty} B_7(n)q^{5n+2} \pmod 7. \qquad (3.2.17)
\end{aligned}
$$

Theorem 3.2.2 follows from (3.2.17).

□

Corollary 3.2.1. For $\alpha \geq 0$ and for all $n \geq 0$,

$$B_7\left(5^{8\alpha}n + \frac{5^{8\alpha}-1}{2}\right) \equiv 3^{\alpha} B_7(n) \pmod{7}, \tag{3.2.18}$$
$$B_7\left(5^{8\alpha+1}n + \frac{5^{8\alpha+1}-1}{2}\right) \equiv 3^{\alpha} B_7(5n+2) \pmod{7}, \tag{3.2.19}$$
$$B_7\left(5^{8\alpha+2}n + \frac{5^{8\alpha+2}-1}{2}\right) \equiv 3^{\alpha} \left(5B_7(5n+2) + 4B_7(n)\right) \pmod{7}, \tag{3.2.20}$$
$$B_7\left(5^{8\alpha+3}n + \frac{5^{8\alpha+3}-1}{2}\right) \equiv 3^{\alpha} \left(B_7(5n+2) + 6B_7(n)\right) \pmod{7}, \tag{3.2.21}$$
$$B_7\left(5^{8\alpha+4}n + \frac{5^{8\alpha+4}-1}{2}\right) \equiv 3^{\alpha} \left(4B_7(5n+2) + 4B_7(n)\right) \pmod{7}, \tag{3.2.22}$$
$$B_7\left(5^{8\alpha+5}n + \frac{5^{8\alpha+5}-1}{2}\right) \equiv 3^{\alpha} \left(3B_7(5n+2) + 2B_7(n)\right) \pmod{7}, \tag{3.2.23}$$
$$B_7\left(5^{8\alpha+6}n + \frac{5^{8\alpha+6}-1}{2}\right) \equiv 3^{\alpha} \left(3B_7(5n+2) + 5B_7(n)\right) \pmod{7}, \tag{3.2.24}$$
$$B_7\left(5^{8\alpha+7}n + \frac{5^{8\alpha+7}-1}{2}\right) \equiv 3^{\alpha} \left(6B_7(5n+2) + 5B_7(n)\right) \pmod{7}. \tag{3.2.25}$$

Theorem 3.2.3. For all $n \geq 0$,

$$B_7(9n+4) \equiv 2B_7(3n+1) + 2B_7(n) \pmod{7}. \tag{3.2.26}$$

Proof. Entry 1(iv) on page 345 of [23] is Ramanujan's cubic continued fraction

$$f_1^3 = f_9^3(u^{-1} - 3q + 4q^3u^2), \tag{3.2.27}$$

where

$$u = \frac{f_3 f_{18}^3}{f_6 f_9^3}.$$

Again from (3.1.1), we have

$$\begin{aligned}\sum_{n=0}^{\infty} B_7(n)q^n &= \frac{f_7^2}{f_1^2} \\ &\equiv f_1^{12} \pmod 7 \qquad (3.2.28)\end{aligned}$$

Substitution (3.2.27) into (3.2.28), we have

$$\begin{aligned}\sum_{n=0}^{\infty} B_7(n)q^n &\equiv f_9^{12}(u^{-1} - 3q + 4q^3u^2)^4 \pmod 7 \\ &\equiv \frac{f_9^{12}}{u^4}(1 + 2qu + 5q^2u^2 + 6q^3u^3 + 5q^5u^5 + 5q^7u^7 \\ &\quad + 3q^8u^8 + 4q^9u^9 + 2q^{10}u^{10} + 4q^{12}u^{12}) \pmod 7. \qquad (3.2.29)\end{aligned}$$

If from (3.2.29), we extract those terms in which the power of q is congruent to 1 modulo 3 and then divide by q and replace q by $q^{\frac{1}{3}}$, we find that

$$\begin{aligned}\sum_{n=0}^{\infty} B_7(3n+1)q^n &\equiv f_3^{12}(2v^{-3} + 5q^2v^3 + 2q^3v^6) \pmod 7 \\ &\equiv 2f_3^{12}\left((v^{-1} + 4qv^2)^3 - 5q\right) \pmod 7 \\ &\equiv 2f_3^{12}\left(\frac{f_1^{12}}{f_3^{12}} + 22q\right) \pmod 7. \qquad (3.2.30)\end{aligned}$$

Here we have used Entry 1 on the page 345 in [23], namely,

$$\frac{f_1^{12}}{f_3^{12}} + 27q = (v^{-1} + 4qv^2)^3, \qquad (3.2.31)$$

where

$$v := \frac{f_1 f_6^3}{f_2 f_3^3}.$$

Using (3.2.30), we find

$$\begin{aligned}\sum_{n=0}^{\infty} B_7(3n+1)q^n &\equiv 2f_1^{12} + 2qf_3^{12} \pmod 7\\ &\equiv 2\frac{f_7^2}{f_1^2} + 2q\frac{f_{21}^2}{f_3^2} \pmod 7\\ &\equiv 2\sum_{n=0}^{\infty} B_7(n)q^n + 2\sum_{n=0}^{\infty} B_7(n)q^{3n+1} \pmod 7. \qquad (3.2.32)\end{aligned}$$

Theorem 3.2.3 follows from (3.2.32). □

Theorem 3.2.4. For all $n \geq 0$,

$$B_{13}(25n+24) \equiv 7B_{13}(5n+4) + 5B_{13}(n) \pmod{13}. \qquad (3.2.33)$$

Proof. Again from (3.1.1), we have

$$\begin{aligned}\sum_{n=0}^{\infty} B_{13}(n)q^n &= \frac{f_{13}^2}{f_1^2}\\ &\equiv f_1^{24} \equiv f_{25}^{24}(R^{-1} - q - q^2R)^{24} \pmod{13}\\ &\equiv \frac{f_{25}^{24}}{R^{24}}(1 + 2qR + 5q^2R^2 + 10q^3R^3 + 7q^4R^4 + 12q^5R^5 + 6q^6R^6\\ &\quad + q^8R^8 + 4q^9R^9 + 3q^{10}R^{10} + 8q^{11}R^{11} + 10q^{12}R^{12} + 3q^{13}R^{13}\\ &\quad + 10q^{14}R^{14} + 8q^{15}R^{15} + 10q^{16}R^{16} + 5q^{17}R^{17} + 7q^{18}R^{18}\\ &\quad + 4q^{19}R^{19} + 8q^{20}R^{20} + q^{21}R^{21} + 8q^{22}R^{22} + 10q^{23}R^{23}\end{aligned}$$

$$+ 5q^{24}R^{24} + 3q^{25}R^{25} + 8q^{26}R^{26} + 12q^{27}R^{27} + 8q^{28}R^{28}$$
$$+ 9q^{29}R^{29} + 7q^{30}R^{30} + 8q^{31}R^{31} + 10q^{32}R^{32} + 5q^{33}R^{33}$$
$$+ 10q^{34}R^{34} + 10q^{35}R^{35} + 10q^{36}R^{36} + 5q^{37}R^{37} + 3q^{38}R^{38}$$
$$+ 9q^{39}R^{39} + q^{40}R^{40} + 6q^{42}R^{42} + q^{43}R^{43} + 7q^{44}R^{44}$$
$$+ 3q^{45}R^{45} + 5q^{46}R^{46} + 11q^{47}R^{47} + q^{48}R^{48}) \pmod{13}. \tag{3.2.34}$$

If from (3.2.34), we extract those terms in which the power of q is congruent to 4 modulo 5 and then divide by q^4, we find that

$$\begin{aligned}\sum_{n=0}^{\infty} B_{13}(5n+4)q^{5n} &\equiv f_{25}^{24}(7R^{-20} + 4q^5R^{-15} + 10q^{10}R^{-10} + 4q^{15}R^{-5} + 5q^{20} \\ &\qquad + 9q^{25}R^5 + 10q^{30}R^{10} + 9q^{35}R^{15} + 7q^{40}R^{20}) \pmod{13} \\ &\equiv f_{25}^{24}\left(7(R^{-5} - 11q^5 - q^{10}R^5)^4 + 5q^{20}\right) \pmod{13} \\ &\equiv f_{25}^{24}\left(7\left(\frac{f_5^6}{f_{25}^6}\right)^4 + 5q^{20}\right) \pmod{13} \\ &\equiv 7f_5^{24} + 5q^{20}f_{25}^{24} \pmod{13}. \end{aligned} \tag{3.2.35}$$

On replacing q by $q^{\frac{1}{5}}$, we find

$$\begin{aligned}\sum_{n=0}^{\infty} B_{13}(5n+4)q^{n} &\equiv 7f_1^{24} + 5q^4f_5^{24} \pmod{13} \\ &\equiv 7\frac{f_{13}^2}{f_1^2} + 5q^4\frac{f_{65}^2}{f_5^2} \pmod{13} \\ &\equiv 7\sum_{n=0}^{\infty} B_{13}(n)q^n + 5\sum_{n=0}^{\infty} B_{13}(n)q^{5n+4} \pmod{13}. \end{aligned} \tag{3.2.36}$$

Theorem 3.2.4 follows from (3.2.36). □

Theorem 3.2.5. For all $n \geq 0$,

$$B_{13}(9n+8) \equiv 5B_{13}(3n+2) + 4B_{13}(n) \pmod{13}. \tag{3.2.37}$$

Proof. Again from (3.1.1), we have

$$\begin{aligned}\sum_{n=0}^{\infty} B_{13}(n)q^n &= \frac{f_{13}^2}{f_1^2} \\ &\equiv f_1^{24} \equiv f_9^{24}(u^{-1} - 3q + 4q^3u^2)^8 \pmod{13} \\ &\equiv \frac{f_9^{24}}{u^8}(1 + 2qu + 5q^2u^2 + 2q^3u^3 + 6q^4u^4 + 6q^5u^5 + 6q^6u^6 + 4q^7u^7 \\ &\quad + 10q^8u^8 + 9q^9u^9 + 5q^{11}u^{11} + 11q^{13}u^{13} + 9q^{14}u^{14} + 3q^{15}u^{15} \\ &\quad + 9q^{16}u^{16} + q^{17}u^{17} + q^{18}u^{18} + q^{19}u^{19} + 5q^{20}u^{20} + 6q^{21}u^{21} \\ &\quad + 8q^{22}u^{22} + 3q^{24}u^{24}) \pmod{13}. \end{aligned} \tag{3.2.38}$$

If from (3.2.38), we extract those terms in which the power of q is congruent to 2 modulo 3 and then divide by q^2 and replace q by $q^{\frac{1}{3}}$, we find that

$$\begin{aligned}\sum_{n=0}^{\infty} B_{13}(3n+2)q^n &\equiv f_3^{24}(5v^{-6} + 6qv^{-3} + 10q^2 + 5q^3v^3 \\ &\quad + 9q^4v^6 + q^5v^9 + 5q^6v^{12}) \pmod{13} \\ &\equiv f_3^{24}(5(v^{-1} + 4qv^2)^6 - 10q(v^{-1} + 4qv^2)^3 + 9q^2) \pmod{13}. \end{aligned} \tag{3.2.39}$$

Using (3.2.31), we find

$$
\begin{aligned}
\sum_{n=0}^{\infty} B_{13}(3n+2)q^n &\equiv f_3^{24}\left(5\frac{f_1^{24}}{f_3^{24}}+4q^2\right) \pmod{13}\\
&\equiv 5f_1^{24}+4q^2f_3^{24} \pmod{13}\\
&\equiv 5\frac{f_{13}^2}{f_1^2}+4q^2\frac{f_{39}^2}{f_3^2} \pmod{13}\\
&\equiv 5\sum_{n=0}^{\infty} B_{13}(n)q^n+4\sum_{n=0}^{\infty} B_{13}(n)q^{3n+2} \pmod{13}. \qquad (3.2.40)
\end{aligned}
$$

Theorem 3.2.5 follows from (3.2.40).

□

Remark 1. Families of congruences analogous to those in Corollary 3.2.1 can be derived from (3.2.26), (3.2.33) and (3.2.37).

Chapter 4

Congruences for Andrews' singular overpartitions

4.1 Introduction

We stated in the introductory chapter that Andrews [8] introduced singular overpartition denoted by $\overline{C}_{\delta,i}(n)$, which count the number of overpartitions of n in which no part is divisible by δ and only parts $\equiv \pm i \pmod{\delta}$ may be overlined. The generating function for $\overline{C}_{\delta,i}(n)$, is given by, $\delta \geq 3$ and $1 \leq i \leq \left\lfloor \frac{\delta}{2} \right\rfloor$,

$$\sum_{n=0}^{\infty} \overline{C}_{\delta,i}(n) q^n = \frac{(q^\delta; q^\delta)_\infty (-q^i; q^\delta)_\infty (-q^{\delta-i}; q^\delta)_\infty}{(q;q)_\infty}. \tag{4.1.1}$$

.

In his paper [8], G. E. Andrews also proved that for $n \geq 0$,

$$\overline{C}_{3,1}(9n+3) \equiv \overline{C}_{3,1}(9n+6) \equiv 0 \pmod{3}.$$

Chan et al. [36] generalized and found infinite families of congruences modulo 3 for $\overline{C}_{3,1}(n)$, $\overline{C}_{6,1}(n)$, $\overline{C}_{6,2}(n)$ and modulo 2 for $\overline{C}_{4,1}(n)$. For example, they proved that for $n, k \geq 0$,

$$\overline{C}_{3,1}(2^k(6n+5)) \equiv 0 \pmod{8}.$$

Recently, Ahmed and Baruah [5] using simple p-dissections of Ramanujan's theta functions have proved several congruences for $\overline{C}_{3,1}(n)$, $\overline{C}_{8,2}(n)$, $\overline{C}_{12,2}(n)$, $\overline{C}_{12,4}(n)$, $\overline{C}_{24,8}(n)$ and $\overline{C}_{48,16}(n)$. Subsequently, Naika and Gireesh [75] prove congruence modulo 6, 12, 16, 18 and 24 for $\overline{C}_{3,1}$ and infinite families of congruence modulo 12, 18, 48, and 72 for $\overline{C}_{3,1}(n)$. In the next section, we obtain our new congruences for $\overline{C}_{3,1}(n)$, $\overline{C}_{12,3}(n)$, $\overline{C}_{44,11}(n)$, $\overline{C}_{60,15}(n)$, $\overline{C}_{75,25}(n)$ and $\overline{C}_{92,23}(n)$.

4.2 Main Theorems

In order to prove our main results, we collect a few lemmas.

Lemma 4.2.1. (Hirschhorn and Sellers [57]) The following 3-dissection holds

$$\frac{f_2}{f_1^2} = \frac{f_6^4 f_9^6}{f_3^8 f_{18}^3} + 2q\frac{f_6^3 f_9^3}{f_3^7} + 4q^2\frac{f_6^2 f_{18}^3}{f_3^6}. \tag{4.2.1}$$

Lemma 4.2.2. (Baruah and Ojah [18, Theorem 4.3]) The following 2-dissection holds

$$\frac{1}{f_1 f_3} = \frac{f_8^2 f_{12}^5}{f_2^2 f_4 f_6^4 f_{24}^2} + q\frac{f_4^5 f_{24}^2}{f_2^4 f_6^2 f_8^2 f_{12}}. \tag{4.2.2}$$

Multiplying both sides of (4.2.2) by f_1^2 and replacing q by q^{11}, we find

$$\frac{f_{11}}{f_{33}} \equiv \frac{f_{22}^5}{f_{132}} + q^{11}\frac{f_{132}}{f_{22}} \pmod 2. \tag{4.2.3}$$

Lemma 4.2.3. (Hirschhorn, Garvan and Borwein [54]) The following 2-dissection holds

$$\frac{f_3^3}{f_1} = \frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q\frac{f_{12}^3}{f_4}. \tag{4.2.4}$$

Lemma 4.2.4. (Ahmed and Baruah [4, Lemma 2.3]) If $p \geq 3$ is prime, then

$$(q;q)_\infty^3 = \sum_{\substack{k=0 \\ k\neq \frac{p-1}{2}}}^{p-1} (-1)^k q^{\frac{k(k+1)}{2}} \sum_{n=0}^{\infty} (-1)^n (2pn+2k+1) q^{pn\cdot\frac{pn+2k+1}{2}}$$
$$+ p(-1)^{\frac{p-1}{2}} q^{\frac{p^2-1}{8}} (q^{p^2};q^{p^2})_\infty^3. \tag{4.2.5}$$

Furthermore, if $k \neq \frac{p-1}{2}, 0 \leq k \leq p-1$, then $\frac{k^2+k}{2} \not\equiv \frac{p^2-1}{8} \pmod{p}$.

Lemma 4.2.5. (Hirschhorn [53]) We have,

$$\frac{1}{f_1} = \frac{f_{25}^5}{f_5^6}\left(\frac{1}{R^4(q^5)} + \frac{q}{R^3(q^5)} + \frac{2q^2}{R^2(q^5)} + \frac{3q^3}{R(q^5)} + 5q^4 - 3q^5R(q^5) \right.$$
$$\left. + 2q^6R^2(q^5) - q^7R^3(q^5) + q^8R^4(q^5)\right), \tag{4.2.6}$$

where $R(q)$ is the Rogers-Ramanujan continued fraction defined, for $|\, q \,| < 1$, by

$$R(q) := \frac{q^{1/5}}{1}\underset{+}{}\frac{q}{1}\underset{+}{}\frac{q^2}{1}\underset{+}{}\cdots.$$

Lemma 4.2.6. (Baruah and Ahmed [14, Eqn. (2.4)])

$$\frac{1}{(q;q)_\infty(q^{11};q^{11})_\infty} \equiv \frac{1}{(q^2;q^2)_\infty^2(q^{22};q^{22})_\infty^2}\left(\psi(q^{12}) + q^6\frac{\psi(-q^{66})\chi(q^{22})}{\chi(-q^4)}\right.$$
$$\left. + q\frac{\psi(-q^6)\chi(q^2)}{\chi(-q^{44})} + q^{15}\psi(q^{132})\right) \pmod{2}. \tag{4.2.7}$$

Lemma 4.2.7. (Berndt [23, Entry 31, p. 48])

Let $U_n = a^{n(n+1)/2}b^{n(n-1)/2}$ and $V_n = a^{n(n-1)/2}b^{n(n+1)/2}$ for an integer n. Then

$$f(U_1, V_1) = \sum_{r=0}^{n-1} U_r f\left(\frac{U_{n+r}}{U_r}, \frac{V_{n-r}}{U_r}\right). \tag{4.2.8}$$

Theorem 4.2.1. For all $n \geq 0$,

$$\overline{C}_{3,1}(12n+11) \equiv 0 \pmod{144}. \tag{4.2.9}$$

Proof. From [75, Eq. 3.19], we have

$$\sum_{n=0}^{\infty} \overline{C}_{3,1}(4n+3)q^n = 6\frac{f_2^3 f_6^3}{f_1^6}. \tag{4.2.10}$$

Substituting (4.2.1) in (4.2.10), we have

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{C}_{3,1}(4n+3)q^n &= 6f_6^3\left(\frac{f_6^4 f_9^6}{f_3^8 f_{18}^3} + 2q\frac{f_6^3 f_9^3}{f_3^7} + 4q^2\frac{f_6^2 f_{18}^3}{f_3^6}\right)^3 \\ &= 6\frac{f_6^{15} f_9^{18}}{f_3^{24} f_{18}^9} + 36q\frac{f_6^{14} f_9^{15}}{f_3^{23} f_{18}^6} + 144q^2\frac{f_6^{13} f_9^{12}}{f_3^{22} f_{18}^3} + 336q^3\frac{f_6^{12} f_9^9}{f_3^{21}} \\ &+ 576q^4\frac{f_6^{11} f_9^6 f_{18}^3}{f_3^{20}} + 576q^5\frac{f_6^{10} f_9^3 f_{18}^6}{f_3^{19}} + 384q^6\frac{f_6^9 f_{18}^9}{f_3^{18}}. \end{aligned} \tag{4.2.11}$$

It follows that

$$\sum_{n=0}^{\infty} \overline{C}_{3,1}(12n+11)q^n = 144\frac{f_2^{13} f_3^{12}}{f_1^{22} f_6^3} + 576q\frac{f^{10} f_3^3 f_6^6}{f_1^{19}}. \tag{4.2.12}$$

Theorem 4.2.1 follows from (4.2.12).

□

Theorem 4.2.2. If p is prime with $p \equiv 5 \pmod 6$ and $\alpha \geq 0$, then

$$\sum_{n=0}^{\infty} \overline{C}_{3,1}\left(24p^{2\alpha}n + 7p^{2\alpha}\right) q^n \equiv 36p^{\alpha}(-1)^{\alpha\cdot\frac{p-2}{3}}(q;q)_\infty^3(q^4;q^4)_\infty \pmod{128}. \tag{4.2.13}$$

Proof. It follows from (4.2.11) that

$$\sum_{n=0}^{\infty} \overline{C}_{3,1}(12n+7)q^n = 36\frac{f_2^{14} f_3^{15}}{f_1^{23} f_6^6} + 576q\frac{f_2^{11} f_3^6 f_6^3}{f_1^{20}}. \tag{4.2.14}$$

Using (2.2.1) in (4.2.14), we have

$$\sum_{n=0}^{\infty} \overline{C}_{3,1}(12n+7)q^n \equiv 36\frac{f_2^3 f_{12}}{f_1 f_3} + 64q f_2 f_{12}^3 \pmod{128}. \tag{4.2.15}$$

Substituting (4.2.4) into (4.2.15), we have

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{C}_{3,1}(12n+7)q^n &\equiv 36 f_2^3 f_{12}\left(\frac{f_8}{f_{12}} + q\frac{f_{24}}{f_4}\right) + 64q f_2 f_{12}^3 \pmod{128}, \\ &\equiv 36 f_2^3 f_8 + 100 q f_2 f_{12}^3 \pmod{128}. \end{aligned} \tag{4.2.16}$$

From (4.2.16), we have

$$\sum_{n=0}^{\infty} \overline{C}_{3,1}(24n+7)q^n \equiv 36 f_1^3 f_4 \pmod{128}, \tag{4.2.17}$$

which is the $\alpha = 0$ case of (4.2.13). Now suppose that (4.2.13) holds for some $\alpha \geq 0$. Substituting (2.2.2) and (4.2.5) in (4.2.13), we have

$$\begin{aligned}&\sum_{n=0}^{\infty} \overline{C}_{3,1}\left(24p^{2\alpha}n + 7p^{2\alpha}\right)q^n \\ &\equiv 36p^{\alpha}(-1)^{\alpha\left(\frac{\pm p-1}{6}+\frac{p-1}{2}\right)}\left[\sum_{\substack{k=0 \\ k\neq \frac{p-1}{2}}}^{p-1} (-1)^k q^{\frac{k(k+1)}{2}} \sum_{n=0}^{\infty}(-1)^n(2pn+2k+1)q^{pn\cdot\frac{pn+2k+1}{2}}\right. \\ &\left. + p(-1)^{\frac{p-1}{2}} q^{\frac{p^2-1}{8}} (q^{p^2};q^{p^2})_{\infty}^3 \right]\end{aligned}$$

$$\times \left[\sum_{\substack{k=-\frac{p-1}{2} \\ k\neq \frac{\pm p-1}{6}}}^{\frac{p-1}{2}} (-1)^k q^{4\frac{3k^2+k}{2}} f\left(-q^{4\frac{3p^2+(6k+1)p}{2}}, -q^{4\frac{3p^2-(6k+1)p}{2}}\right) \right.$$

$$\left. + (-1)^{\frac{\pm p-1}{6}} q^{4\frac{p^2-1}{24}} (q^{4p^2}; q^{4p^2})_\infty \right] \pmod{128}. \tag{4.2.18}$$

For a prime $p \geq 5$, $0 \leq k \leq p-1$ and $\frac{-(p-1)}{2} \leq m \leq \frac{(p-1)}{2}$, now consider the congruence

$$\frac{k^2+k}{2} + 4 \cdot \frac{3m^2+m}{2} \equiv \frac{7p^2-7}{24} \pmod{p}, \tag{4.2.19}$$

which is equivalent to

$$3(2k+1)^2 + (12m+2)^2 \equiv 0 \pmod{p}.$$

Since $\left(\frac{-3}{p}\right) = -1$ as $p \equiv 5 \pmod{6}$ the solution (4.2.19) is $k = \frac{p-1}{2}$ and $m = \frac{p-1}{6}$. Therefore, extracting the terms involving $q^{pn+\frac{7p^2-7}{24}}$ from both sides of (4.2.18) and then replacing q^p by q, we find that

$$\sum_{n=0}^{\infty} \overline{C}_{3,1}\left(24p^{2\alpha+1}n + 7p^{2\alpha+2}\right) q^n \equiv 36p^{\alpha+1}(-1)^{(\alpha+1)\cdot\frac{p-2}{3}} (q^p; q^p)_\infty^3 (q^{4p}; q^{4p})_\infty \pmod{128}. \tag{4.2.20}$$

Extracting the terms containing q^{pn} from both sides of identity (4.2.20) and then replacing q^p by q, we find that

$$\sum_{n=0}^{\infty}\overline{C}_{3,1}\left(24p^{2\alpha+2}n+7p^{2\alpha+2}\right)q^n \equiv 36p^{\alpha+1}(-1)^{(\alpha+1)\cdot\frac{p-2}{3}}(q;q)_\infty^3(q^4;q^4)_\infty \pmod{128} \tag{4.2.21}$$

This completes the proof by induction of (4.2.13). □

Theorem 4.2.3. If p is prime $p \geq 5$, such that $\left(\dfrac{-3}{p}\right) = -1$, than for any nonnegative integer α and n,

$$\overline{C}_{3,1}\left(24p^{2\alpha+1}(pn+j)+7p^{2\alpha+2}\right) \equiv 0 \pmod{128}.$$

Proof. Employing (2.2.2) and (4.2.5) and then comparing the coefficients of q^{pn+j}, $1 \leq j \leq p-1$, on both side of (4.2.20), we deduce Theorem 4.2.3. □

Theorem 4.2.4. If p is prime with $p \equiv 5$ or $7 \pmod 8$ and $\alpha \geq 0$, then

$$\sum_{n=0}^{\infty}\overline{C}_{3,1}\left(24p^{2\alpha}n+19p^{2\alpha}\right)q^n \equiv 100p^{\alpha}(-1)^{\alpha\cdot\frac{p-2}{3}}(q^6;q^6)_\infty^3(q;q)_\infty \pmod{128}. \tag{4.2.22}$$

Proof. From (4.2.11), we have

$$\sum_{n=0}^{\infty}\overline{C}_{3,1}(24n+19)q^n \equiv 100f_6^3f_1 \pmod{128} \tag{4.2.23}$$

which is the $\alpha = 0$ case of (4.2.22). Now suppose that (4.2.22) holds for some $\alpha \geq 0$. Substituting (2.2.2) and (4.2.5) in (4.2.22), we have

$$\sum_{n=0}^{\infty}\overline{C}_{3,1}\left(24p^{2\alpha}n+19p^{2\alpha}\right)q^n$$

$$\equiv 100p^{\alpha}(-1)^{\alpha(\frac{\pm p-1}{6}+\frac{p-1}{2})}\left[\sum_{\substack{k=0\\k\neq\frac{p-1}{2}}}^{p-1}(-1)^k q^{6\frac{k(k+1)}{2}}\sum_{n=0}^{\infty}(-1)^n(2pn+2k+1)q^{6pn\cdot\frac{pn+2k+1}{2}}\right.$$

$$\left.+p(-1)^{\frac{p-1}{2}}q^{6\frac{p^2-1}{8}}(q^{6p^2};q^{6p^2})_\infty^3\right]$$

$$\times\left[\sum_{\substack{k=-\frac{p-1}{2}\\k\neq\frac{\pm p-1}{6}}}^{\frac{p-1}{2}}(-1)^k q^{\frac{3k^2+k}{2}}f\left(-q^{\frac{3p^2+(6k+1)p}{2}},-q^{\frac{3p^2-(6k+1)p}{2}}\right)\right.$$

$$\left.+(-1)^{\frac{\pm p-1}{6}}q^{\frac{p^2-1}{24}}(q^{p^2};q^{p^2})_\infty\right] \pmod{128}. \tag{4.2.24}$$

For a prime $p\geq 5$, $0\leq k\leq p-1$ and $\frac{-(p-1)}{2}\leq m\leq\frac{(p-1)}{2}$, now consider the congruence

$$6\cdot\frac{k^2+k}{2}+\frac{3m^2+m}{2}\equiv\frac{19p^2-19}{24} \pmod{p}, \tag{4.2.25}$$

which is equivalent to

$$2(6k+3)^2+(6m+1)^2\equiv 0 \pmod{p}.$$

Since $\left(\frac{-2}{p}\right)=-1$ as $p\equiv 5$ or $7 \pmod 8$ the solution to (4.2.25) is $k=\frac{p-1}{2}$ and $m=\frac{p-1}{6}$. Therefore, extracting the terms involving $q^{pn+\frac{19p^2-19}{24}}$ from both sides of (4.2.24) and then replacing q^p by q, we find that

$$\sum_{n=0}^{\infty} \overline{C}_{3,1}\left(24p^{2\alpha+1}n + 19p^{2\alpha+2}\right) q^n \equiv 100p^{\alpha+1}(-1)^{(\alpha+1)\cdot\frac{p-2}{3}}(q^{6p};q^{6p})_\infty^3 (q^p;q^p)_\infty \pmod{128}. \tag{4.2.26}$$

Extracting the terms containing q^{pn} from both sides of the above and then replacing q^p by q, we find that

$$\sum_{n=0}^{\infty} \overline{C}_{3,1}\left(24p^{2\alpha+2}n + 19p^{2\alpha+2}\right) q^n \equiv 100p^{\alpha+1}(-1)^{(\alpha+1)\cdot\frac{p-2}{3}}(q^{6};q^{6})_\infty^3 (q;q)_\infty \pmod{128}, \tag{4.2.27}$$

This completes the proof by induction of (4.2.22). □

Theorem 4.2.5. If p is prime $p \geq 5$, such that $\left(\frac{-2}{p}\right) = -1$, than for any nonnegative integer α and n,

$$\overline{C}_{3,1}(24p^{2\alpha+1}\left(pn+j\right) + 19p^{2\alpha+2}\right) \equiv 0 \pmod{128}.$$

Proof. Employing (2.2.2) and (4.2.5) and then comparing the coefficients of q^{pn+j}, $1 \leq j \leq p-1$, from both side of (4.2.26), we deduce Theorem 4.2.5.

□

Theorem 4.2.6. For $k \geq 0$, we have

$$\overline{C}_{12,3}\left(4^k n + \frac{4^k-1}{3}\right) \equiv \overline{C}_{12,3}(n) \pmod{2}, \tag{4.2.28}$$

$$\overline{C}_{12,3}\left(4^{k+1} n + \frac{10\cdot 4^k-1}{3}\right) \equiv 0 \pmod{2}, \tag{4.2.29}$$

$$\overline{C}_{12,3}\left(4^{k+1} n + \frac{4^k(6m+1)-1}{3}\right) \equiv 0 \pmod{2}, \quad 1 \leq m \leq 7. \tag{4.2.30}$$

Proof. From (4.1.1), we have

$$\sum_{n=0}^{\infty} \overline{C}_{12,3}(n)q^n \equiv \frac{f_3^3}{f_1} \pmod{2}. \tag{4.2.31}$$

Using (4.2.4) in (4.2.31), we found

$$\sum_{n=0}^{\infty} \overline{C}_{12,3}(2n+1)q^n \equiv \frac{f_6^3}{f_2} \pmod{2}. \tag{4.2.32}$$

It follows that

$$\overline{C}_{12,3}(4n+1) \equiv \overline{C}_{12,3}(n) \pmod{2} \tag{4.2.33}$$

and

$$\overline{C}_{12,3}(4n+3) \equiv 0 \pmod{2}. \tag{4.2.34}$$

The results (4.2.28) and (4.2.29) follow by induction, using (4.2.33) and (4.2.34) respectively. Again from (4.2.31), we have

$$\overline{C}_{12,3}(2n) \equiv f_8 \pmod{2}. \tag{4.2.35}$$

It follow that

$$\overline{C}_{12,3}(16n) \equiv f_1 \pmod{2} \tag{4.2.36}$$

and

$$\overline{C}_{12,3}(16n+2m) \equiv 0 \pmod{2}, \tag{4.2.37}$$

V for $1 \le m \le 7$, using (4.2.28) in (4.2.37), we have the result (4.2.30).

□

Theorem 4.2.7. For all $n \ge 0$,

$$\overline{C}_{44,11}(16n+2) \equiv 0 \pmod{2}, \tag{4.2.38}$$

$$\overline{C}_{44,11}(16n+14) \equiv 0 \pmod{2}, \tag{4.2.39}$$

$$\overline{C}_{44,11}(16n+10) \equiv 0 \pmod{2}, \tag{4.2.40}$$

$$\overline{C}_{44,11}(176n+16m+6) \equiv 0 \pmod{2}, \quad 1 \le m \le 10. \tag{4.2.41}$$

Proof. Again from (4.1.1), we have

$$\sum_{n=0}^{\infty} \overline{C}_{44,11}(n)q^n \equiv \frac{f_{22}^2}{f_1 f_{11}} \pmod{2}. \tag{4.2.42}$$

Substituting (4.2.7) in (4.2.42) and extracting the terms involving q^{2n} from both sides of the congruence and then replacing q^2 by q, we have

$$\sum_{n=0}^{\infty} \overline{C}_{44,11}(2n)q^n \equiv \frac{1}{f_2}\left(\psi(q^6) + q^3 \frac{f_{66}^2 f_4 f_{11}}{f_{22} f_2 f_{33}}\right) \pmod{2}. \tag{4.2.43}$$

Using (4.2.3) in (4.2.43) and extracting the terms involving q^{2n+1} from both sides of the congruence, dividing both sides by q and then replacing q^2 by q, we have

$$\sum_{n=0}^{\infty} \overline{C}_{44,11}(4n+2)q^n \equiv q\frac{f_{33}^2 f_{11}^5}{f_{11}f_{66}}, \equiv qf_{44} \pmod{2}. \tag{4.2.44}$$

Extracting the terms involving q^{4n+1} from both sides of the congruence, dividing both sides by q and then replacing q^4 by q, we have

$$\sum_{n=0}^{\infty} \overline{C}_{44,11}(16n+6)q^n \equiv f_{11} \pmod{2} \tag{4.2.45}$$

The results (4.2.38)-(4.2.40), follow from (4.2.44). The result (4.2.41) follows from (4.2.45). □

Theorem 4.2.8. For any non-negative integer k, we have

$$\sum_{n=0}^{\infty} \overline{C}_{60,15}\left(20 \cdot 5^{2k}n + \frac{19 \cdot 5^{2k+1} - 11}{6}\right) q^n \equiv f_3^3 f_{10} \pmod{2}. \tag{4.2.46}$$

Proof. From (4.1.1), we have

$$\sum_{n=0}^{\infty} \overline{C}_{60,15}(n)q^n \equiv \frac{f_{30}^2}{f_1 f_{15}} \pmod{2}. \tag{4.2.47}$$

Substituting (2.2.3) in (4.2.47) and extracting the terms involving q^{2n} from both sides of the resulting congruence and then replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} \overline{C}_{60,15}(2n)q^n \equiv \frac{f_6^2 f_{10}^2}{f_2 f_3 f_5} \pmod{2}. \tag{4.2.48}$$

Extracting the terms involving q^{2n+1} from both sides of the congruence, dividing both sides by q, replacing q^2 by q and then employing (2.2.22), we obtain

$$\sum_{n=0}^{\infty} \overline{C}_{60,15}(4n+2)q^n \equiv q f_2 f_{15}^3 \pmod{2}. \tag{4.2.49}$$

Now replacing q by q^2 in (2.2.25), we get

$$f_2 = f_{50}\left(M^{-1} - q^2 - q^4 M\right), \tag{4.2.50}$$

where

$$M = \frac{(q^{10}, q^{40}; q^{50})_\infty}{(q^{20}, q^{30}; q^{50})_\infty}.$$

Substituting (4.2.50) in (4.2.49) and extracting the terms involving q^{5n+3} from both sides of the resulting congruence, dividing both sides by q^3 and then replacing q^5 by q, we obtain

$$\sum_{n=0}^{\infty} \overline{C}_{60,15}(20n+14)q^n \equiv f_3^3 f_{10} \pmod{2}. \tag{4.2.51}$$

Which is the $k = 0$ case of (4.2.46). Now suppose (4.2.46) holds for some $k \geq 0$. Substituting (2.2.27) in (4.2.51) and extracting the terms involving q^{5n+4} from both sides of the resulting congruence, dividing both sides by q^4 and then replacing q^5 by q, we obtain

$$\sum_{n=0}^{\infty} \overline{C}_{60,15}\left(20 \cdot 5^{2k+1} n + \frac{23 \cdot 5^{2k+2} - 11}{6}\right) q^n \equiv q f_2 f_{15}^3 \pmod{2}. \tag{4.2.52}$$

Substituting (4.2.50) in (4.2.52) and extracting the terms involving q^{5n+3} from both sides of the resulting congruence, dividing both sides by q^3 and then replacing q^5 by q, we obtain

$$\sum_{n=0}^{\infty} \overline{C}_{60,15}\left(20 \cdot 5^{2k+2}n + \frac{19 \cdot 5^{2k+3} - 11}{6}\right) q^n \equiv f_3^3 f_{10} \pmod{2}. \tag{4.2.53}$$

This completes the proof by induction of (4.2.46). □

Theorem 4.2.9. For all $n \geq 0$,

$$\overline{C}_{60,15}\left(20 \cdot 5^{2k}(5n + s) + \frac{19 \cdot 5^{2k+1} - 11}{6}\right) q^n \equiv 0 \pmod{2}, \quad s \in \{1, 2\}. \tag{4.2.54}$$

Proof. Employing (2.2.27) in (4.2.46), we obtain (4.2.54).

□

Theorem 4.2.10. For all $n \geq 0$,

$$\overline{C}_{75,25}(10n + 9) \equiv 0 \pmod{2}, \tag{4.2.55}$$

$$\overline{C}_{75,25}(80n + 20m + 14) \equiv 0 \pmod{2}, \quad 1 \leq m \leq 3. \tag{4.2.56}$$

Proof. Again from (4.1.1), we have

$$\sum_{n=0}^{\infty} \overline{C}_{75,25}(n) q^n \equiv \frac{f_{25}}{f_1} \pmod{2}. \tag{4.2.57}$$

Substituting (4.2.6) in (4.2.57), extracting the terms involving q^{5n+4} from both sides of the congruence, dividing both sides by q^4 and then replacing q^5 by q, we have

$$\sum_{n=0}^{\infty} \overline{C}_{75,25}(5n+4)q^n \equiv \frac{f_5^6}{f_1^6} \equiv \frac{f_{10}^3}{f_2^3} \pmod 2. \tag{4.2.58}$$

The result (4.2.55) follow from (4.2.58). Also from (4.2.58), we have

$$\sum_{n=0}^{\infty} \overline{C}_{75,25}(10n+4)q^n \equiv \frac{f_5^3}{f_1^3} \equiv \frac{f_{10}f_5}{f_2 f_1} \pmod 2. \tag{4.2.59}$$

Substituting (3.2.1) in (4.2.59), extracting the terms involving q^{2n+1} from both sides of the congruence, dividing both sides by q and then replacing q^2 by q, we have

$$\sum_{n=0}^{\infty} \overline{C}_{75,25}(20n+14)q^n \equiv \frac{f_4 f_{40}}{f_8} \pmod 2. \tag{4.2.60}$$

The result (4.2.56) follows from (4.2.60). □

Theorem 4.2.11. For any non-negative integer k, we have

$$\sum_{n=0}^{\infty} \overline{C}_{92,23}\left(4\cdot 23^{2k}n + \frac{23^{2k+1}-17}{6}\right)q^n \equiv f_{23} + qf_1 f_{46} + q^2 f_2 f_{23}^3 \pmod 2. \tag{4.2.61}$$

Proof. From (4.1.1), we have

$$\sum_{n=0}^{\infty} \overline{C}_{92,23}(n)q^n \equiv \frac{f_{46}^2}{f_1 f_{23}} \pmod 2. \tag{4.2.62}$$

From (2.2.6) and (4.2.62), we have

$$\sum_{n=0}^{\infty}\overline{C}_{92,23}(n)q^n \equiv f_{46}^2\left(\sum_{n=0}^{\infty}p_{1^1 23^1}(2n)q^{2n}+q+q^3f_2f_{46}\right) \pmod{2}. \tag{4.2.63}$$

Extracting the terms involving q^{2n+1} from both sides of (4.2.63), dividing both sides by q and then replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty}\overline{C}_{92,23}(2n+1)q^n \equiv f_{46}+q\frac{f_2f_{46}^2}{f_1f_{23}} \pmod{2}. \tag{4.2.64}$$

Now substituting (2.2.6) in (4.2.64) and extracting extracting the terms involving q^{2n} from both sides of the resulting congruence and then replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty}\overline{C}_{92,23}\left(4n+1\right)q^n \equiv f_{23}+qf_1f_{46}+q^2f_2f_{23}^3 \pmod{2}. \tag{4.2.65}$$

Which is the $k=0$ case of (4.2.61). Now suppose (4.2.61) holds for some $k\geq 0$. Employing (2.2.13) and (2.2.11) in (4.2.65) and extracting the terms involving q^{23n} from both sides of the resulting congruence and then replacing q^{23} by q, we obtain

$$\sum_{n=0}^{\infty}\overline{C}_{92,23}\left(4\cdot 23^{2k+1}n+\frac{23^{2k+1}-17}{6}\right)q^n \equiv f_1+qf_2f_{23}+q^2f_1f_2f_{46} \pmod{2}. \tag{4.2.66}$$

Employing (2.2.13) and (2.2.11) in (4.2.66) and extracting the terms involving q^{23n} from both sides of the resulting congruence, dividing both sides by q^{22} and then replacing q^{23} by q, we obtain

$$\sum_{n=0}^{\infty} \overline{C}_{92,23}\left(4\cdot 23^{2k+2}n+\frac{23^{2k+3}-17}{6}\right)q^n \equiv f_{23}+qf_1f_{46}+q^2f_2f_{23}^3 \pmod{2}. \tag{4.2.67}$$

This completes the proof by induction of (4.2.61). □

Theorem 4.2.12. If $\ell \in \{5,7,10,11,14,15,17,19,20,21,22\}$ then for all $n \geq 0$,

$$\overline{C}_{92,23}\left(4\cdot 23^{2k}(23n+\ell)+\frac{23^{2k+1}-17}{6}\right) \equiv 0 \pmod{2}. \tag{4.2.68}$$

Proof. Employing (2.2.13), (2.2.11) in (4.2.61) and then equating the coefficients of $q^{23n+\ell}$ from both sides we obtain Theorem 4.2.12.

□

Theorem 4.2.13. For any non-negative integer k, we have

$$\sum_{n=0}^{\infty} \overline{C}_{92,23}\left(2\cdot 23^{2k}n+\frac{7\cdot 23^{2k+1}-73}{88}\right)q^n \equiv f_{23}^2+qf_1f_{23}^3 \pmod{2}. \tag{4.2.69}$$

Proof. Again from (4.1.1), we have

$$\sum_{n=0}^{\infty} \overline{C}_{92,23}(n)q^n \equiv \frac{f_{46}^2}{f_1f_{23}} \pmod{2}. \tag{4.2.70}$$

Now, from [18, Eq. (1.9)], we have

$$\sum_{n=0}^{\infty} p_{[1^1 23^1]}(2n+1)q^n = \frac{f_2f_{46}}{f_1^2f_{23}^2}+q\frac{f_2^2f_{46}^2}{f_1^3f_{23}^3}, \tag{4.2.71}$$

where $p_{[1^1 23^1]}(n)$ is defined by

$$\sum_{n=0}^{\infty} p_{[1^1 23^1]}(n)q^n := \frac{1}{f_1 f_{23}}. \tag{4.2.72}$$

Extracting the terms involving q^{2n+1} from both sides of (4.2.70), replacing q^2 by q and then employing (4.2.71), we have

$$\sum_{n=0}^{\infty} \overline{C}_{92,23}(2n+1)q^n \equiv f_{23}^2 + q f_1 f_{23}^3 \pmod{2}, \tag{4.2.73}$$

which is the $k = 0$ case of (4.2.69). Now suppose (4.2.69) holds for some $k \geq 0$. Setting $U_1 = a = -q$, $V_1 = b = -q^2$ and $n = 23$ in (4.2.8) and using the identity $f(a, b) = af(a^{-1}, a^2 b)$, we find the following 23-dissection of $f(-q, -q^2) = f_1$.

$$\begin{aligned} f_1 &= f(-q^{782}, -q^{805}) - qf(-q^{851}, -q^{736}) - q^2 f(-q^{713}, -q^{874}) - q^5 f(-q^{920}, -q^{667}) \\ &+ q^7 f(-q^{644}, -q^{943}) - q^{12} f(-q^{989}. - q^{598}) - q^{15} f(-q^{575}, -q^{1012}) \\ &+ q^{22} f(-q^{1058}, -q^{529}) + q^{26} f(-q^{506}, -q^{1081}) - q^{35} f(-q^{1127}, -q^{460}) \\ &- q^{40} f(-q^{437}, -q^{1150}) + q^{51} f(-q^{1196}, -q^{391}) + q^{57} f(-q^{368}, -q^{1219}) \\ &- q^{70} f(-q^{1265}, -q^{322}) - q^{77} f(-q^{299}, -q^{1288}) + q^{92} f(-q^{1334}, -q^{253}) \\ &+ q^{100} f(-q^{230}, -q^{1357}) - q^{117} f(-q^{1403}, -q^{184}) - q^{126} f(-q^{161}, -q^{1426}) \\ &+ q^{145} f(-q^{1472}, -q^{115}) + q^{155} f(-q^{92}, -q^{1495}) - q^{176} f(-q^{1541}, -q^{46}) \\ &- q^{187} f(-q^{23}, -q^{1564}). \end{aligned} \tag{4.2.74}$$

Employing (4.2.74) in (4.2.69) extracting the terms involving q^{23n} from both sides of the resulting congruence, replacing q^{23} by q, we have

$$\sum_{n=0}^{\infty}\overline{C}_{92,23}\left(2\cdot 23^{2k}n+\frac{7\cdot 23^{2k+1}-73}{88}\right)q^n \equiv f_1^2+qf_1^3f_{23}$$
$$\equiv f_2+qf_1f_2f_{23} \pmod 2 \quad (4.2.75)$$

Next, squaring (4.2.74), we have

$$\begin{aligned}
f_2 \equiv\ & f^2(-q^{782},-q^{805})+q^2f^2(-q^{851},-q^{736})+q^4f^2(-q^{713},-q^{874})+q^{10}f^2(-q^{920},-q^{667})\\
+\ & q^{14}f^2(-q^{644},-q^{943})+q^{24}f^2(-q^{989}.-q^{598})+q^{30}f^2(-q^{575},-q^{1012})\\
+\ & q^{44}f^2(-q^{1058},-q^{529})+q^{52}f^2(-q^{506},-q^{1081})+q^{70}f^2(-q^{1127},-q^{460})\\
+\ & q^{80}f^2(-q^{437},-q^{1150})+q^{102}f^2(-q^{1196},-q^{391})+q^{114}f^2(-q^{368},-q^{1219})\\
+\ & q^{140}f^2(-q^{1265},-q^{322})+q^{154}f^2(-q^{299},-q^{1288})+q^{184}f^2(-q^{1334},-q^{253})\\
+\ & q^{200}f^2(-q^{230},-q^{1357})+q^{234}f^2(-q^{1403},-q^{184})+q^{252}f^2(-q^{161},-q^{1426})\\
+\ & q^{290}f^2(-q^{1472},-q^{115})+q^{310}f^2(-q^{92},-q^{1495})+q^{352}f^2(-q^{1541},-q^{46})\\
+\ & q^{374}f^2(-q^{23},-q^{1564}) \pmod 2. \qquad (4.2.76)
\end{aligned}$$

Employing (4.2.74) and (4.2.76) in (4.2.75), extracting the terms involving q^{23n+21} from both sides of the congruence, dividing both sides by q^{21} and then replacing q^{23} by q, we have

$$\sum_{n=0}^{\infty}\overline{C}_{92,23}\left(2\cdot 23^{2k+1}n+\frac{7\cdot 23^{2k+2}-73}{88}\right)q^n \equiv f_{23}^2+qf_1f_{23}^3, \qquad (4.2.77)$$

This completes the proof by induction of (4.2.69). □

Theorem 4.2.14. If $m \in \{5,7,10,11,14,15,17,19,20,21,22\}$, then for all $n \geq 0$,

$$\overline{C}_{92,23}\left(2\cdot 23^{2k}(23n+m)+\frac{7\cdot 23^{2k+1}-73}{88}\right)\equiv 0 \pmod 2. \tag{4.2.78}$$

Proof. Employing (4.2.74) in (4.2.69) and then equating the coefficients of q^{23n+m} from both sides we deduce Theorem 4.2.14. □

In the next theorem we have some interesting congruences for $\overline{C}_{12,3}(n)$, $\overline{C}_{44,11}(n)$ and $b_2(n)$ modulo 2. From (4.2.45), we have

$$\sum_{n=0}^{\infty}\overline{C}_{44,11}(176n+6)q^n \equiv f_1 \pmod 2 \tag{4.2.79}$$

Theorem 4.2.15. For any prime $p \geq 5$, $\alpha \geq 1$, and $n \geq 0$,

$$\overline{C}_{12,3}\left(16p^{2\alpha}n+\frac{2(24i+p)p^{2\alpha-1}-2}{3}\right)\equiv 0 \pmod 2, \tag{4.2.80}$$

For $i = 1, 2, \cdots, p-1$. For any prime $p \geq 5$, $\alpha \geq 0$, and $n \geq 0$,

$$\overline{C}_{12,3}\left(16p^{2\alpha+1}n+\frac{2(24i+1)p^{2\alpha}-2}{3}\right)\equiv 0 \pmod 2, \tag{4.2.81}$$

where j is an integer with $0 \leq j \leq p-1$ such that $\left(\frac{24j+1}{p}=-1\right)$.

Proof. We note for 2-regular partitions modulo 2

$$\sum_{n=0}^{\infty} b_2(n)q^n \equiv f_1 \pmod 2.$$

In [41] Cui and Gu have proved several interesting results, for example, for any prime $p \geq 5$, $\alpha \geq 1$, and $n \geq 0$,

$$b_2\left(p^{2\alpha}n + \frac{(24i+p)p^{2\alpha-1}-1}{24}\right) \equiv 0 \pmod 2. \tag{4.2.82}$$

And for any prime $p \geq 5$, $\alpha \geq 0$, and $n \geq 0$,

$$b_2\left(p^{2\alpha+1}n + \frac{(24j+1)p^{2\alpha}-1}{24}\right) \equiv 0 \pmod 2, \tag{4.2.83}$$

where j is an integer with $0 \leq j \leq p-1$ such that $\left(\frac{24j+1}{p} = -1\right)$. Theorem 4.2.15 follows from (4.2.36) and the results (4.2.80) and (4.2.81). $\square$

Remark: Similar results can be obtained for (4.2.79).

Chapter 5

An Interesting q-Continued Fractions of Ramanujan

5.1 Introduction

The celebrated Rogers-Ramanujan continued fraction is define by

$$R(q) := \frac{q^{1/5}}{1}\,\frac{q}{+1+}\,\frac{q^2}{1+}\,\frac{q^3}{1+\cdots}, \quad |q| < 1. \tag{5.1.1}$$

In his first two letters to Hardy [91], Ramanujan communicated several theorems about $R(q)$ and $S(q) := -R(-q)$. In these two letters, Ramanujan claimed that

$$R(e^{-2\pi}) = \sqrt{\frac{5+\sqrt{5}}{2}} - \frac{\sqrt{5}+1}{2},$$

and

$$S(e^{-\pi}) = \sqrt{\frac{5-\sqrt{5}}{2}} - \frac{\sqrt{5}-1}{2}.$$

On page 365 of his 'lost' notebook, Ramanujan wrote five modular equations relating $R(q)$ with $R(-q)$, $R(q^2)$, $R(q^3)$, $R(q^4)$ and $R(q^5)$. Motivated by these works, in this chapter, we study the Ramanujan continued fraction

$$M(q) := \frac{q^{1/2}}{1-q\,+}\;\frac{q(1-q)}{1+q^2}\;+\frac{q(1-q^3)^2}{(1-q)(1+q^4)}\;+\frac{q(1-q^5)^2}{(1-q)(1+q^6)}\;+\cdots, \quad |q|<1$$

$$= q^{1/2}\frac{(q^4;q^4)_\infty^2}{(q^2;q^4)_\infty^2}. \tag{5.1.2}$$

In Chapter 16 Entry 12 of [23], Ramanujan has recorded the following continued fraction

$$\frac{(a^2q^3;q^4)_\infty(b^2q^3;q^4)_\infty}{(a^2q;q^4)_\infty(b^2q;q^4)_\infty} = \frac{1}{1-ab\,+}\;\frac{(a-bq)(b-aq)}{(1-ab)(1+q^2)\;+}\;\frac{(a-bq^3)(b-aq^3)}{(1-ab)(1+q^4)}\;+\cdots, \qquad |ab|<1, |q|<1. \tag{5.1.3}$$

In fact setting $a = q^{1/2}$ and $b = q^{1/2}$ in (5.1.3), we obtain (5.1.2).

In Section 5.2 we obtain an interesting q-identity related to $M(q)$ using Ramanujan's ${}_1\psi_1$ summation formula [23, Ch. 16, Entry 17]

$$\sum_{n=-\infty}^{\infty}\frac{(a)_n}{(b)_n}z^n = \frac{(az)_\infty(q/az)_\infty(q)_\infty(b/a)_\infty}{(z)_\infty(b/az)_\infty(b)_\infty(q/a)_\infty}, \qquad |b/a|<|z|<1, \tag{5.1.4}$$

and Andrew's identity [9, p. 57]

$$\sum_{n=0}^{\infty}\frac{q^{kn}}{1-q^{ln+k}} = \sum_{n=0}^{\infty}q^{ln^2+2kn}\frac{1+q^{ln+k}}{1-q^{ln+k}}. \tag{5.1.5}$$

In Section 5.3 we obtain several relation of $M(q)$ with theta function $\varphi(q)$, $\psi(q)$ and $\chi(q)$. In Section 5.4 we obtain an integral representation of $M(q)$. In Section 5.5 we derive a formula that help us to obtain relation among $M(q^{1/2})$,

$M(q)$, $M(q^2)$ and $M(q^4)$. We establish explicit formulas for the evaluation of $\dfrac{M(e^{-\pi\sqrt{n}})}{M(e^{-\pi\sqrt{n}/2})}$ in Section 5.6.

5.2 q-Identity related to $M(q)$

Theorem 5.2.1.

$$M(q)=\sum_{n=0}^{\infty}q^{n(8n+4)+1/2}\,\frac{1+q^{8n+2}}{1-q^{8n+2}}-\sum_{n=0}^{\infty}q^{(n+1)(8n+4)+1/2}\,\frac{1+q^{8n+6}}{1-q^{8n+6}} \tag{5.2.1}$$

Proof. Changing q to q^2, then setting $a=q^2$, $b=q^{10}$ and $z=q^2$ in ${}_1\psi_1$ summation formula (5.1.4) we obtain

$$\frac{(q^4;q^4)^2_\infty}{(q^2;q^4)^2_\infty}=\sum_{n=0}^{\infty}\frac{q^{2n}}{1-q^{8n+2}}-\sum_{n=0}^{\infty}\frac{q^{6n+4}}{1-q^{8n+6}} \tag{5.2.2}$$

employing Andrews identity (5.1.5) with $k=2, l=8$ and $k=6, l=8$ in both the summations in right side of the identity (5.2.2) respectively and finally multiplying both sides of the resulting identity with $q^{1/2}$ and using product represtation of $M(q)$ (5.1.2), we complete the proof of Theorem 5.2.1. □

5.3 Some Identities involving $M(q)$

We obtain relation of $M(q)$ in terms of theta function $\varphi(q)$, $\psi(q)$ and $\chi(q)$.

Theorem 5.3.1.

$$\begin{aligned}
M(q) &= q^{1/2}\,\frac{\psi^4(q)}{\varphi^2(q)}\,, && (5.3.1)\\
8M(q^2) &= \varphi^2(q)-\varphi^2(-q)\,, && (5.3.2)\\
16M^2(q) &= \varphi^4(q)-\varphi^4(-q)\,, && (5.3.3)\\
\frac{M^2(q)}{M(q^2)} &= \varphi^2(q^2)\,, && (5.3.4)\\
4M(q^2) &= \varphi^2(q)-\varphi^2(q^2)\,, && (5.3.5)\\
\frac{M^{-1}(q)+M(q)}{M^{-1}(q)-M(q)} &= \frac{1+q\psi^4(q^2)}{1-q\psi^4(q^2)}\,, && (5.3.6)\\
8M(q^2) &= \frac{\chi^2(q)}{\chi^2(-q)}\,\phi^2(-q^2)-\phi^2(-q)\,. && (5.3.7)
\end{aligned}$$

Proof. Using [23, Ch. 16, Entry 22(ii)] in (5.1.2), we obtain

$$M(q) = q^{1/2}\psi^2(q^2)\,. \tag{5.3.8}$$

Employing [23, Ch. 16, Entry 25(iv)] in (5.3.8), we obtain (5.3.1).
From (5.3.1), we have

$$M(q^2) = q\frac{\psi^4(q^2)}{\varphi^2(q^2)}\,. \tag{5.3.9}$$

Employing [23, Ch. 16, Entry 25(vii)] and [23, Ch. 16, Entry 25(vi)] in (5.3.9), we obtain (5.3.2). Identity (5.3.3) immediately follows from (5.3.8) and [23, Ch. 16, Entry 25(vii)].
Again from (5.3.1), we have

$$\frac{M^2(q)}{M(q^2)} = \frac{\psi^8(q)\varphi^2(q^2)}{\psi^4(q^2)\varphi^4(q)}, \tag{5.3.10}$$

employing [23, Ch. 16, Entry 25(iv)], in the identity (5.3.10) we obtain (5.3.4).
From (5.3.2) and (5.3.3), we have

$$64M^2(q^2) + 16M^2(q) = 16\varphi^2(q)M(q^2), \tag{5.3.11}$$

dividing the above identity (5.3.11) throughout by $16M(q^2)$ and using (5.3.4) we obtain (5.3.5).
From (5.3.1), we deduce that

$$M^{-1}(q) + M(q) = \frac{\varphi^4(q) + q\psi^8(q\psi)}{q^{1/2}\varphi^2(q)\psi^4(q)}, \tag{5.3.12}$$

and

$$M^{-1}(q) - M(q) = \frac{\varphi^4(q) - q\psi^8(q\psi)}{q^{1/2}\varphi^2(q)\psi^4(q)}. \tag{5.3.13}$$

On dividing (5.3.12) by (5.3.13) and using [23, Ch. 16, Entry 25(iv)] in the resulting identity, we complete the proof of (5.3.6).

From (1.4.3) and (1.4.6), we have

$$\varphi(-q) + \frac{\chi(q)}{\chi(-q)}\varphi(-q^2) = \frac{(q;q)_\infty}{(-q;q)_\infty}\left[1 + \frac{f(q,q)}{f(-q,-q)}\right],$$

employing [23, Ch. 16, Entry 30(ii)] in right hand side of above identity we obtain

$$\varphi(-q) + \frac{\chi(q)}{\chi(-q)}\varphi(-q^2) = \frac{2(q^8;q^8)^5_\infty(q^{32};q^{32})^2_\infty}{(q^4;q^4)^2_\infty(q^{16};q^{16})^2_\infty(q^{64};q^{64})^4_\infty}\,\frac{M(q^{16})}{q^8}. \tag{5.3.14}$$

Again from (1.4.3) and (1.4.6), we have

$$\varphi(-q) - \frac{\chi(q)}{\chi(-q)}\varphi(-q^2) = \frac{(q;q)_\infty}{(-q;q)_\infty}\left[1 - \frac{f(q,q)}{f(-q,-q)}\right],$$

employing [23, Ch. 16, Entry 30(ii)] and [23, Ch. 16, Entry 18(ii)] in right hand side of above identity we obtain

$$\varphi(-q) - \frac{\chi(q)}{\chi(-q)}\varphi(-q^2) = \frac{-4q(-q^8;q^8)_\infty(q^{64};q^{64})_\infty^3}{(-q^{16};q^{32})_\infty}\frac{q^8}{M(q^{16})}\,. \tag{5.3.15}$$

Multiplying (5.3.14) and (5.3.15) we complete the proof of (5.3.7). □

Theorem 5.3.2. Let $u = M(q)$, $v = M(-q)$ and $w = (q^2)$, then

$$u^2 - v^2 = 8w^2$$

Proof. On substituting (5.3.4) in (5.3.5), we obtain

$$\varphi^2(q) = \frac{4M^2(q^2) + M^2(q)}{M(q^2)}. \tag{5.3.16}$$

Changing q to $-q$ in (5.3.16), we have

$$\varphi^2(-q) = \frac{4M^2(q^2) + M^2(-q)}{M(q^2)}. \tag{5.3.17}$$

Subtracting (5.3.17) from (5.3.16) and using identity (5.3.2), we complete the proof of Theorem 5.3.2. □

5.4 Integral Representation of $M(q)$

Theorem 5.4.1. For $0 < |q| < 1$,

$$M(q) = \exp\int\left(\frac{1}{2q} + \frac{4}{q}\left[\frac{\varphi^4(-q)-1}{8} + \frac{q\varphi'(q)}{2\varphi(q)}\right]\right)dq\,, \tag{5.4.1}$$

where $\varphi(q)$ and $\psi(q)$ are as defined in (1.4.3) and (1.4.4).

Proof. Taking log on both sides of (5.3.1), we have

$$\log M(q) = \frac{1}{2}\log q + 4\log\psi(q) - 2\log\varphi(q). \tag{5.4.2}$$

Employing [23, Ch. 16, Entry 23(ii)] and [23, Ch. 16, Entry 23(i)] on right hand side of (5.4.2), we obtain

$$\log M(q) = \frac{1}{2}\log q + 4\sum_{n=1}^{\infty}\frac{q^{2n}}{2n(1+q^{2n})}. \tag{5.4.3}$$

Differentiating (5.4.3) and simplifying, we have

$$\frac{d}{dq}\log M(q) = \frac{1}{2q} + \frac{4}{q}\left[\sum_{n=1}^{\infty}\frac{(-1)^n q^n}{(1+q^n)^2} + \sum_{n=1}^{\infty}\frac{q^{2n-1}}{(1+q^{2n-1})^2}\right]. \tag{5.4.4}$$

Using Jacobi's identity [23, Ch. 16, 33.5, p. 54)] and [23, Ch. 16, Entry 23(i)] and integrating both sides and finally taking exponentiating both sides of identity (5.4.4), we complete the proof of Theorem 5.4.1. □

5.5 Modular Equation of Degree n and Relation Between $M(q)$ and $M(q^n)$

In the terminology of hypergeometric function, a modular equation of degree n is a relation between α and β that is induced by

$$n\frac{{}_2F_1(1/2,1/2;1;1-\alpha)}{{}_2F_1(1/2,1/2;1;\alpha)} = \frac{{}_2F_1(1/2,1/2;1;1-\beta)}{{}_2F_1(1/2,1/2;1;\beta)},$$

where

$${}_2F_1(a,b;c;x) = \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k k!}x^k,$$

and

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}.$$

Let $Z_1(r) =_2 F_1(1/r, r-1/r; 1; \alpha)$ and $Z_n(r) =_2 F_1(1/r, r-1/r; 1; \beta)$, where n is the degree of the modular equation. The multiplier $m(r)$ is defined by the equation

$$m(r) = \frac{Z_1(r)}{Z_n(r)}.$$

Theorem 5.5.1. If

$$q = \exp\left(-\pi \frac{{}_2F_1(1/2, 1/2; 1; 1-\alpha)}{{}_2F_1(1/2, 1/2; 1; \alpha)}\right), \tag{5.5.1}$$

then

$$\alpha = 16\frac{M^4(q)}{M^4(q^{1/2})} \tag{5.5.2}$$

Proof. From (5.1.2) and (1.4.3), we have

$$\begin{aligned} M(q)\varphi^2(q) &= q^{1/2}\frac{(q^4;q^4)_\infty^2}{(q^2;q^4)_\infty^2}\frac{(-q;q^2)_\infty^4(q^4;q^4)_\infty^2}{(-q^2;q^2)_\infty^4(q^2;q^4)_\infty^2} \\ &= M^2(q^{1/2}). \end{aligned} \tag{5.5.3}$$

Substitution (5.5.3) in (5.3.3), we obtain

$$16M^2(q) = \frac{M^4(q^{1/2})}{M^2(q)}\left[1 - \frac{\varphi^4(-q)}{\varphi^4(q)}\right]. \tag{5.5.4}$$

From a know identity [23, Ch. 16, p.100, Entry 5] and (5.5.1) it is implied that

$$\alpha = 1 - \frac{\varphi^4(-q)}{\varphi^4(q)}. \tag{5.5.5}$$

Using (5.5.5) in (5.5.4), we complete the proof of (5.5.2).

Let α and β related by (5.5.1). If β has degree n over α then from Theorem 5.5.1, we obtain

$$\beta = 16\frac{M^4(q^n)}{M^4(q^{n/2})} \tag{5.5.6}$$

□

Corollary 5.5.1. Let $u = M(q^{1/2})$, $v = M(q)$, $w = M(q^2)$ and $x = M(q^4)$, then

$$16x^4v^2 + 32x^3wv^2 - 4x^3wu^4 + 24x^2w^2v^2 + 8xw^3v^2 - xw^3u^4 + w^4v^2 = 0. \tag{5.5.7}$$

Proof. From [23, Ch. 16, p.216, Entry 24(v)], we have

$$\sqrt{1-\alpha} = \left(\frac{1-\beta^{1/4}}{1+\beta^{1/4}}\right)^2. \tag{5.5.8}$$

On using (5.5.6) with $n = 4$ and (5.5.2) in (5.5.8), we obtain

$$\sqrt{\frac{u^4 - 16v^4}{u^4}} = \left(\frac{w-2x}{w+2x}\right)^2. \tag{5.5.9}$$

Squaring both side of (5.5.9) and then simplifying, we obtain (5.5.7). □

5.6 Evaluations of $M(q)$

As an application of Theorem 5.5.1, we establish few explicit evaluation of $M(q)$. Let $q_n = e^{-\pi\sqrt{n}}$ and let α_n denote the corresponding value of α in (5.5.1). Then by Theorem 5.5.1, we have

$$\frac{M(e^{-\pi\sqrt{n}})}{M(e^{-\pi\sqrt{n}/2})} = \frac{1}{2}\alpha_n^{1/4}. \tag{5.6.1}$$

From [23, p. 97, Ch. 17], we have $\alpha_1 = \frac{1}{2}$, $\alpha_2 = \left(\sqrt{2}-1\right)^2$ and $\alpha_4 = \left(\sqrt{2}-1\right)^4$. Thus from (5.6.1), it immediately follows

$$\frac{M(e^{-\pi})}{M(e^{-\pi/2})} = \left(\frac{1}{2}\right)^{5/4}, \tag{5.6.2}$$

$$\frac{M(e^{-\sqrt{2}\pi})}{M(e^{-\pi/\sqrt{2}})} = \frac{1}{2}\sqrt{\sqrt{2}-1}, \tag{5.6.3}$$

$$\frac{M(e^{-2\pi})}{M(e^{-\pi})} = \frac{\sqrt{2}-1}{2}. \tag{5.6.4}$$

Ramanujan has recorded several modular equation in his notebook [89, p. 204-237] and [89, p. 156-160] which are very useful in the computation of class invariants and the values of theta function. Ramanujan has also recorded values of theta function $\varphi(q)$ and $\psi(q)$ in his notebook. For example

$$\varphi(e^{-\pi}) = \frac{\pi^{1/4}}{\Gamma(3/4)}, \tag{5.6.5}$$

$$\psi(e^{-\pi}) = 2^{-5/8}e^{\pi/8}\frac{\pi^{1/4}}{\Gamma(3/4)}, \tag{5.6.6}$$

$$\frac{\varphi(e^{-\pi})}{\varphi(e^{-3\pi})} = \sqrt[4]{6\sqrt{3}-9}. \tag{5.6.7}$$

From (5.3.8) and (5.6.6), we have

$$M(e^{-\pi/2}) = 2^{-5/4}\frac{\sqrt{\pi}}{\Gamma^2(3/4)}, \tag{5.6.8}$$

Using (5.6.8) and (5.6.2), we obtain

$$M(e^{-\pi}) = \frac{\sqrt{\pi}}{\Gamma^2(3/4)}. \tag{5.6.9}$$

Setting (5.6.9) in (5.6.4), we obtain

$$M(e^{-2\pi}) = \frac{\sqrt{2}-1}{2}\frac{\sqrt{\pi}}{\Gamma^2(3/2)} \tag{5.6.10}$$

J.M. Borwein and P.B. Borwein [30] are the first to observe that class invariant could be used to evaluated certain of $\varphi(e^{-n\pi})$. The Ramanujan Weber class invariants are defined by

$$G_n := 2^{-1/4}q_n^{-1/24}(-q_n; q_n^2)_\infty$$

and

$$g_n := 2^{-1/4}q_n^{-1/24}(-q_n; q_n^2)_\infty, \tag{5.6.11}$$

where $q_n = e^{-\pi\sqrt{n}}$. Chan and Huang has derived few explicit formulas for evaluating $K(e^{-\pi\sqrt{n}/2})$ in the terms of Ramanujan Weber class. Similar works are done by Adiga et.,al. Analogoues to these works we obtain explicit formulas to evaluate $\dfrac{M(e^{-\pi\sqrt{n}})}{M(e^{-\pi\sqrt{n}/2})}$.

Theorem 5.6.1. For Ramanujan Weber class invariant defined as in (5.6.11), let $p = G_n^{12}$ and $p_1 = g_n^{12}$, then

$$\frac{M(e^{-\pi\sqrt{n}})}{M(e^{-\pi\sqrt{n}/2})} = \frac{1}{2}\frac{1}{\sqrt{\sqrt{p(p+1)}+\sqrt{p(p-1)}}}, \tag{5.6.12}$$

$$\frac{M(e^{-\pi\sqrt{n}})}{M(e^{-\pi\sqrt{n}/2})} = \frac{1}{2}\sqrt{\sqrt{p_1^2+1}-p_1}. \tag{5.6.13}$$

Proof. From [34], we have

$$g_n = [4\alpha_n(1-\alpha_n)]^{-1/24}.$$

Hence

$$\alpha_n = \frac{1}{(\sqrt{p(p+1)}+\sqrt{p(p-1)})^2} \tag{5.6.14}$$

Using (5.6.14) in (5.6.1), we obtain (5.6.12).

Also from [34], we have

$$2g_n^{12} = \frac{1}{\sqrt{\alpha_n}} - \sqrt{\alpha_n}.$$

Hence

$$\sqrt{\alpha_n} = \sqrt{(p_1^2+1)} - p_1. \tag{5.6.15}$$

Using (5.6.15) in (5.6.1), we complete the proof of (5.6.13). □

Example: Let $n = 1$, Since $G_1 = 1$, from Theorem 5.6.1 we have

$$\frac{M(e^{-\pi})}{M(e^{-\pi/2})} = \left(\frac{1}{2}\right)^{5/4}.$$

Let $n = 2$. Since $g_2 = 1$, from Theorem 5.6.1 we have

$$\frac{M(e^{-\sqrt{2}\pi})}{M(e^{-\pi/\sqrt{2}})} = \frac{1}{2}\sqrt{\sqrt{2} - 1}.$$

Remark: Using [89, p.229] it is easily verified that $M(q)$ and $K(q)$ are related by the equation

$$M(q^2)K(q) + K(q)M(q) - M(q^2) = 0.$$

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