SOME NEW LIFE DISTRIBUTIONS: SURVIVAL PROPERTIES AND APPLICATIONS

Thesis submitted in partial fulfillment of the requirements for the award of the degree of

DOCTOR OF PHILOSOPHY

in

STATISTICS

by

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CERTIFICATE

This is to certify that the research work done in this thesis entitled "**SOME NEW LIFE DISTRIBUTIONS: SURVIVAL PROPERTIES AND APPLICATIONS**" submitted to Pondicherry University in partial fulfillment of the requirements for the award of the degree of *Doctor of Philosophy* in Statistics by *Mr. Subhradev Sen* is a bonafide record of thesis embodies the results of his investigation during the period (2014-2017) he worked as Ph.D. research scholar (part-time:external) under my supervision and no part of this thesis has been submitted for award of any degree / diploma / associateship / fellowship or any similar title in earlier.

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Declaration of the Candidate

I hereby declare that the work incorporated in this thesis entitled "**SOME NEW LIFE DISTRIBUTIONS: SURVIVAL PROPERTIES AND APPLICATIONS**" submitted to Pondicherry University for partial fulfillment of the requirement for the award of degree of *Doctor of Philosophy* in *Statistics*, is a record of original research work carried out by me under the supervision of **Dr. Navin Chandra**, Assistant Professor, Department of Statistics, Ramanujan School of Mathematical Sciences, Pondicherry University, Puducherry, India, and that part of this thesis has not been submitted to any Degree/Diploma/Associateship/Fellowship/ on any similar title before this University or elsewhere.

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To my Mother . . . (Late) Smt. Rita Sen

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Ph.D Thesis entitled

"SOME NEW LIFE DISTRIBUTIONS: SURVIVAL PROPERTIES AND APPLICATIONS"

by **Subhradev Sen**

Abstract

In a wide variety of scientific and technological fields, such as public health, actuarial science, biomedical studies, demography, and industrial reliability, modeling and analysis of lifetimes is an important aspect of statistical work. For modeling of such survival data, variety of probability distributions have been proposed in the literature based on failure rate types. Besides a long list of probability distributions developed in last two decades or so, exponential, gamma, Weibull and lognormal are still considered as important probability distributions for modeling real world phenomena, especially in modeling lifetime (or time-to-event) of any system or unit.

In one way or other, none of the distributions, so far developed and studied, provides universal adequacy (in the light of the adequacy of fit or goodness of fit criteria) in modeling data coming from diverse field of applications and the usefulness of statistical distributions remains a need of time altogether. Hence, newer probability distributions are been developed, their theory and applications are widely studied. The interest, as well as need, in developing more flexible statistical distributions than the existing ones remains a strong focus of this research work.

The entire research work presented in this dissertation can broadly be classified into three important and interlinked segments.

A new non-negative continuous probability distribution, named as the *xgamma distribution*, is been introduced and studied initially. The distribution is synthesized as a special finite mixture of exponential distribution with parameter θ and gamma distribution with shape parameter 3 and scale parameter θ (i.e., $gamma(3, \theta)$), with mixing proportions $\theta/(1+\theta)$ and $1/(1+\theta)$, respectively, and hence the name *xgamma* proposed.

The one parameter xgamma distribution, thus obtained, is unimodal and the shape is regulated by different values of the parameter. The xgamma distribution is initially DFR and then IFR and is found to have added flexibility over the constant hazard rate of exponential distribution. Moreover, mean residual life (MRL) function is decreasing, i.e., DMRL, for entire range. The xgamma random variables are ordered with strongest likelihood ratio ordering and thereby other stochastic orderings and xgamma random variable is found to be stochastically larger than those of Lindley and exponential for similar parametric set up. It is seen that the moment estimator of the parameter in xgamma distribution is positively biased. Comprehensive sample generation algorithms are proposed for complete and censored situations and Monte-Carlo simulation studies ensure that estimates of the parameter behave satisfactorily for larger samples. It is recommended to use Bayesian estimate for parameter provided a prior information is available; otherwise the method of maximum likelihood would be a better choice under progressively type-II right censored situation. Real lifetime data analyses confirm that xgamma lifetime model provides better fit as compared to exponential, gamma, Weibull, log-normal and Lindley distributions and the distribution has potential of a competent life distribution for modeling time-to-event data sets.

At the second segment of the work, main concentration is given in introducing and studying two different versions of xgamma distribution, namely, truncated (lower, upper and double) xgamma distributions and weighted xgamma distribution (including length biased version as special case).
The truncated versions are been introduced owing the fact that designed lifetimes of equipment are usually finite and hence, the truncated versions of xgamma distribution are been proposed. Different properties of the upper truncated xgamma distribution are been studied in details and is found thath the upper truncated xgamma distribution is unimodal. Moreover, the distribution is sometimes IFR

and sometimes DFR depending on the particular range of the concerned random variable. Maximum likelihood method of estimation provides satisfactory result in estimating unknown parameters in the proposed truncated version. Real data illustration shows that upper truncated version of xgamma distribution can be better alternative in modeling lifetime data sets compared to the some other popular lifetime models.

The weighted versions of popular life distributions available in literature mainly have been utilized on the areas like cell kinetics, early detection of diseases, encountered data analysis, equilibrium population analysis subject to harvesting and predation, etc. The aim in proposing and studying weighted version is to find an application in lifetime data and expectation is been fulfilled when the length biased xgamma distribution, studied as a special case of weighted xgamma version, provides satisfactory fit to a lifetime data set and shows superiority over exponential, gamma, Weibull and recently introduced length biased weighted exponential models. It is observed that the length biased xgamma is a special case of weighted xgamma distribution, is a special finite mixture of $gamma(2, \theta)$ and $gamma(4, \theta)$, is unimodal; and it holds IFR and DMRL property. The length biased xgamma random variable possesses strong hazard rate, mean residual life and stochastic ordering for certain restriction on parameter. Moreover, method of maximum likelihood estimation works nicely without having much sacrifice on the procedure and simulation study confirms the fact for different sample sizes.

Adding extra parameters to an existing probability distribution is a popular technique for generalization and the resultant family of distributions, thus obtained, can provide additional flexibility in distributional and/or survival properties and in modeling real life data sets. Hence, at the last segment of this research, two extensions (or generalizations) of xgamma distribution are been proposed and studied for modeling survival (or time-to event) data sets. The extensions (or generalizations) are named as the *quasi xgamma* (QXG) distribution and the *two-parameter xgamma* (TPXG) distribution. Both the proposed distributions are special finite mixtures of $exp(\theta)$ and $gamma(3, \theta)$ distributions with different mixing proportions.

Both the distributions proposed provide additional flexibility over xgamma distribution in view of their distributional and survival properties and possess strong likelihood ratio ordering. Moreover, TPXG random variables are stochastically smaller than those of QXG in likelihood ratio and other orderings. Classical methods (method of moments and method of maximum likelihood) of estimation are been suggested for parameter estimation in complete sample cases and simulation studies confirm the behavior for larger sample sizes. Real data analyses revealed that both the proposed distributions are quite competent in modeling time-to-event data sets.

Publications

Papers in Refereed Journals

1. Subhradev Sen, Sudhansu S. Maiti and N. Chandra. **The xgamma Distribution: Statistical properties and Application**, *Journal of Modern Applied Statistical Methods*, **15(1)**, 774-788 (May, 2016).

2. Subhradev Sen and N. Chandra. **The quasi xgamma distribution with application in bladder cancer data**, *Journal of Data Science*, **15(1)**, 61-76 (January, 2017).

3. Subhradev Sen, N. Chandra and Sudhansu S. Maiti. **The weighted xgamma distribution: Properties and Application**, *Journal of Reliability and Statistical Studies*, **10(1)**, 43-58 (June, 2017).

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Abbreviations

AI	**A**geing **I**ntensity
AIC	**A**kaike **I**nformation **C**riteria
BIC	**B**ayesian **I**nformation **C**riteria
cAIC	**c**onsistent **A**kaike **I**nformation **C**riteria
cdf	**c**umulative **d**istribution **f**unction
CI	**C**onfidence **I**nterval
DFR	**D**ecreasing **F**ailure **R**ate
ER	**E**xpected **R**isk
EV	**E**xpected **V**alue
FR	**F**ailure **R**ate
HR	**H**azard **R**ate
i.i.d	**i**ndependently and **i**dentically **d**istributed
IFR	**I**ncreasing **F**ailure **R**ate
LR	**L**ikelihood **R**atio
mgf	**m**oment **g**enerating **f**unction
ML	**M**aximum **L**ikelihood
MLE	**M**aximum **L**ikelihood **E**stimator
MLEs	**M**aximum **L**ikelihood **E**stimators/**E**stimates
MRL	**M**ean **R**esidual **L**ife
MSE	**M**ean **S**quare **E**rror
MSEs	**M**ean **S**quare **E**rrors
MTTF	**M**ean **T**ime **T**o **F**ailure

PC	**P**rogressive **C**ensoring
pdf	**p**robability **d**ensity **f**unction
sf	**s**urvival **f**unction
ST	**St**ochastic
SELF	**S**quared **E**rror **L**oss **F**unction
Std. Error	Standard Error

Symbols

X, Y, U, V, W	random variables
$\sim$	distributed as
X_i	i^{th} random sample on X
$\bar{X}$	sample mean
$X_{1:n}$	smallest order statistic
$X_{n:n}$	largest order statistic
n	sample size
$\boldsymbol{x}$	set of observations of size n
$\tilde{x}$	progressive type-II censored sample
μ	mean
ln	natural logarithm function
σ^2	variance
μ'_r	r^{th} moment about zero, $r = 1, 2, \ldots$.
μ_r	r^{th} moment about mean, $r = 1, 2, \ldots$.
$\phi_X(t)$	characteristic function
$M_X(t)$	moment generating function
$K_X(t)$	cumulant generating function
$H_R(\gamma)$	Rényi entropy measure
$H(f)$	Shannon entropy measure
$S_q(X)$	Tsallis entropy measure
$f(x)$	probability density function

$F(x)$	cumulative distribution function
$S(x)$	survival function
$h(x)$	hazard rate (or failure rate) function
$m(x)$	mean residual life function
$r(x)$	reversed hazard rate function
Θ	parameter space
$\Gamma(\cdot)$	gamma function
$\gamma(,)$	lower incomplete function
$\Gamma(,)$	upper incomplete function
γ	coefficient of variation
$\sqrt{\beta_1}$	coefficient of skewness
β_2	coefficient of kurtosis
θ_0	initial solution for θ
$\Re$	set real numbers
R	R software
$\underline{R}$	progressive censoring scheme

Chapter 1

Introduction

With the advances in science and technology, a wealth of information has gifted the statistician and data modeler to think in a broader way in the process of gathering knowledge. For making inference about the population of interest, statisticians gather and analyze these information keeping the responsibility of accurate inference. In recent years, it has been observed that many well-known probability distributions used to model data sets do not offer enough flexibility to provide an adequate fit. It is, therefore, the need of time that guides the statisticians to model the real life scenario by introducing newer probability distributions that are more suitable and flexible.

The present chapter of this thesis dissertation deals with general descriptions on the notion of life distributions in section 1.1, basic distributional (section 1.2) and survival properties (section 1.3) of probability distributions, lifetime data and censoring (section 1.4), methods of estimating unknown parameters in a distribution (section 1.5), review of literature (section 1.6) related to genesis of the thesis (section 1.7).

1.1 Notion of life distributions

The probability distributions that are effectively used for modeling lifetime data sets are termed as "lifetime probability distributions" or simply "life distributions". Usually, life is defined by a non-negative random variable in statistical literature owing the fact that life can not be negative. Although "lifetime" literally means *the duration of a thing's existence or usefulness*, in statistical point of view the term "lifetime" (or simply "life") has a broader meaning.
By notation, a non-negative random variable, X, which denotes time to occurrence of an intended event (or expected event), can be termed as a "lifetime random variable".

The probability distribution of X is usually continuous, however, sometime lives are viewed as discrete where clock time is not the best scale to describe lifetime, more details can be seen in Xekalaki (1983), Adams and Watson (1989), Bain (1991) and Shaked et al. (1995). Thus, life distributions, thought suitably, come under the purview of parametric lifetime models or parametric life distributions for modeling time-to-event data sets. In general, lifetimes of biological organisms and of human made devices are the focus of survival and reliability analysis, respectively.

However, non-negative random variables, depending on continuous set up or discrete, arise in a wide variety of applications. These can be waiting times for delays in traffic, intervals between floods or earthquakes, or required time for a task learning. In magnitudes related to physical objects also non-negative random variables arise, for example, measurements on atmospheric characteristics, lengths of cracks, diameters or heights of trees, speed of wind, strengths of materials, stream flows, rainfall, tire wear, or composition in different types of chemicals. The another area where non-negative random variables arise is economics. Some examples may be, income, size of firms, prices and losses encountered in actuarial science.

1.2 Basic measures of probability distributions

The following set of basic distributional properties and measures will be used throughout the thesis. Most of these basic properties can be found in Hogg et al. (2005).

Let X be a non-negative continuous random variable having probability density function (pdf), $f(x)$, and cumulative distribution function (cdf), $F(x) = Pr(X \leq x)$. For very obvious reason, the range of the random variable is taken as $(0, \infty)$ here.

The following sub-sections presenting the notions of various statistical measures that characterize any probability distribution. These measures are also called sometimes as distributional properties.

1.2.1 Non-central and central moments

The symbol, μ'_r, refers to the r^{th} order non-central moment (or moments about the origin) of a continuous random variable X having a distribution function $F(x)$. For $r \geq 1$. The r^{th} non-central moment is given by

$$\mu'_r = E(X^r) = \int_0^\infty x^r f(x) dx, \tag{1.1}$$

where μ'_1, the first moment about zero, is called the mean and it is a measure of central tendency denoted by μ.

On the other hand, the symbol, μ_r, refers to the r^{th} order central moment (or the r^{th} moment about the mean) of a non-negative continuous random variable X having a distribution function $F(x)$. For $r \geq 1$. the r^{th} order central moment is given by $\mu_r = E[(X - \mu)^r]$ such that

$$E[(X - \mu)^r] = \int_0^\infty (x - \mu)^r f(x) dx. \tag{1.2}$$

The following recurrence relation between central and non-central moments are well known.

$$\mu_j = E[(X-\mu)^j] = \sum_{r=0}^{j} \binom{j}{r} \mu'_r (-\mu)^{j-r}, \tag{1.3}$$

where μ_j denotes j^{th} order central moment.

1.2.2 Characteristic and generating functions

The characteristic function of a probability distribution completely specifies the distribution and it is denoted by $\phi_X(t)$ for $t \in \Re$, and is defined as

$$\phi_X(t) = E[e^{itX}] = \int_0^\infty e^{itx} f(x)dx, \tag{1.4}$$

where $i = \sqrt{-1}$.

The moment generating function is denoted by $M_X(t)$ for $t \in \Re$, and is defined as

$$M_X(t) = E[e^{tX}] = \int_0^\infty e^{tx} f(x)dx. \tag{1.5}$$

The cumulant generating function is denoted by $K_X(t)$, for $t \in \Re$, and can be obtained by taking natural logarithm of $M_X(t)$. Hence, it is defined as

$$K_X(t) = \ln[M_X(t)]. \tag{1.6}$$

1.2.3 Order statistics

Let us denote $X_1, X_2, \ldots, X_n$ as a random sample of size n drawn from an arbitrary pdf $f(x)$ and cdf $F(x)$. Then, $X_{1:n} = Min\{X_1, X_2, \ldots, X_n\}$ denotes the smallest order statistics (or the first order statistic) and $X_{n:n} = Max\{X_1, X_2, \ldots, X_n\}$ denotes the largest order statistic (or the n^{th} order statistic). In general, $X_{j:n}$

denotes the j^{th} order statistic.

The pdf of $X_{j:n}$ is given by

$$f_{X_{j:n}}(x) = \frac{n!}{(j-1)!(n-j)!}[F(x)]^{j-1}[1-F(x)]^{n-j}f(x), 0 < x < \infty, \tag{1.7}$$

for $j = 1, 2, \ldots, n$.

In particular, the pdf of $X_{1:n}$ is given by

$$f_{X_{1:n}}(x) = n[1-F(x)]^{n-1}f(x), 0 < x < \infty, \tag{1.8}$$

and, that of $X_{n:n}$ is given by

$$f_{X_{n:n}}(x) = n[F(x)]^{n-1}f(x), 0 < x < \infty. \tag{1.9}$$

The extreme order statistics, $X_{1:n}$ and $X_{n:n}$ represent the life of series and parallel systems and important applications of them can be found in system reliability analysis.

For a complete theory and methods on order statistics one could refer to Balakrishnan and Rao (1998) and Arnold et al. (2008).

1.2.4 Measures of entropy

The idea of information is too extensive to be captured completely by a single definition. However, for any probability distribution, a quantity known as the entropy, that has several properties that accept as true with the intuitive notion of what a measure of information ought to be. *Mutual entropy* is measure defined as an extended version of this notion and represents amount of information one random variable contains about the other. Hence, the self-information of a random variable is then explained by entropy. A more general quantity, termed as *relative entropy*, which contains mutual entropy as special case, describes a distance measure between two probability distributions, see for more details Cover and Thomas (2012).

The concept of entropy was introduced in thermodynamics to dispense an assertion of the second law of thermodynamics, see Bein-Naim (2008) for complete insight of statistical thermodynamic based on information. A connection between thermodynamic entropy and the logarithm of the number of micro-states in a macro state of the system is provided by statistical mechanics, see Jaynes (1957), Wilson (1970) and Chandler (1987) for more details.

Therefore, for a random variable X, entropy is viewed as a measure of variation or uncertainty. A popular measure of entropy is Rényi entropy, see Rényi (1961). If a non-negative random variable X has the pdf $f(x)$, then the Rényi entropy is defined as

$$H_R(\gamma) = \frac{1}{1-\gamma} \ln \left[\int_0^\infty f^\gamma(x) dx \right] \text{ for } \gamma > 0 (\neq 1). \tag{1.10}$$

Shannon measure of entropy (Shannon, 1948), which is a special case of Rényi entropy, is defined as

$$H(f) = E[-\ln f(x)] = -\int_0^\infty \ln f(x) f(x) dx. \tag{1.11}$$

In physics, Tsallis entropy or q-entropy (see Tsallis, 1988) is a generalization of the standard Boltzmann–Gibbs entropy. It is defined as

$$S_q(X) = \frac{1}{q-1} \ln \left[1 - \int_0^\infty f^q(x) dx \right] \text{ for } q > 0\, (\neq 1). \tag{1.12}$$

1.3 Survival properties

Every life distribution possesses certain basic properties related to its survival or reliability characteristics through some functions. Various alternative functions

are in common use that can be thought as survival properties related to the corresponding density function or distribution function to describe the distribution of a non-negative random variable mathematically. Some basic of these functions include survival function, hazard rate or failure rate, reversed hazard rate and mean residual life. If these functions exist, any one of them can be obtained from the any other, at least theoretically. Although none of these functions is uniformly best, there are some beneficial reasons for interest of studying all of these functions. Moreover, it might be easier to estimate some of these functions than others.
The following basic survival properties (or survival characteristics) along with their respective definitions and notations have been utilized throughout the thesis.

1.3.1 Survival function

The survival function (or the reliability function) is defined as the probability of performing, without failure, a specific function under given conditions for a specified period of time. The term survival function is used in an extensive range of applications, whereas, reliability function is common in engineering, see Finkelstein (2008) for more.
Mathematically, survival function is the complementary probability statement of the distribution function and is denoted by $S(x)$ throughout the thesis. However, it is also denoted by $\bar{F}(x)$ or $R(x)$ in reliability context. So,

$$S(x) = 1 - F(x) = \Pr(X > x) = \int_x^\infty f(z)dz. \tag{1.13}$$

1.3.2 Hazard rate or failure rate function

Hazard rate or failure rate, also called *force of mortality* or *the mortality rate* or *intensity rate* or *instantaneous force of mortality* in actuarial and demographic literature, is a function of time or a constant. The interpretation of this function plays a pivotal role in survival analysis, reliability analysis and other fields, see Lee and Wang (2003), Colett (2015) and references therein.

For an absolutely continuous non-negative random variable, X, with pdf $f(x)$ and cdf $F(x)$, hazard rate function (or failure rate function) can be defined as the limiting value of the ratio of the conditional probability of failure in a small interval $(x, x + \Delta x]$ to Δx. Therefore, hazard rate function (or failure rate function), denoted by $h(x)$, is given by

$$h(x) = \lim_{\Delta x \to 0} \frac{\Pr(x < X < x + \Delta x | X > x)}{\Delta x}, \tag{1.14}$$

$$= \lim_{\Delta x \to 0} \frac{F(x + \Delta x) - F(x)}{\Delta x} \times \frac{1}{S(x)} = \frac{f(x)}{S(x)}. \tag{1.15}$$

When Δx is sufficiently small, $h(x)\Delta x$ is popularly interpreted as an approximate conditional probability of failure in $(x, x + \Delta x]$ and $f(x)\Delta x$ is defined as the corresponding approximate unconditional probability of failure in $(x, x+\Delta x]$. the survival function and hazard rate function maintain an useful identity,

$$S(x) = exp\left[-\int_0^x h(z)dz\right] = exp[-H(x)], \tag{1.16}$$

where $H(x) = \int_0^x h(z)dz$ is called *cumulative hazard rate.*

A life distribution is very well characterized by hazard rate or failure rate function and, in turn, it led to the concepts of a class of life distributions corresponding to the notion of adverse aging, see Barlow and Proschan (1975). The cdf $F(x)$ is called increasing (decreasing) failure rate distribution or IFR (DFR) if $h(x)$ increases (decreases) in x.

1.3.3 Mean residual life function

An another way to describe a distribution is the mean residual life function or MRL function. The study of the failure rate function is difficult without considering other measures. The MRL function is probably foremost among these. These functions complement each other nicely, see Finkelstein (2008).

Let $X_{(x)}$ denotes residual or remaining life at age "x", i.e., $X_{(x)} = (X - x|X > x)$. The MRL function, denoted by $m(x)$, is defined as

$$m(x) = E[X - x|X > x] = \frac{1}{S(x)} \int_x^\infty S(z)dz. \tag{1.17}$$

The survival function can be represented in term of MRL function, given by Cox (1962), Muth (1977) and Gupta (1979), as

$$S(x) = \frac{\mu}{m(x)} exp\left[-\int_0^x \frac{dz}{m(z)}\right]. \tag{1.18}$$

The MRL function has a tremendous range of applications, such as, in studying burn in, setting rates of benefits for life insurance, the duration of wars and strikes or of jobs. Guess and Proschan (1988), Hall and Wellner (1981), Kupka and Loo (1989), and Ghai and Mi (1999) have provided an excellent review of the theory and applications of the MRL function.

1.3.4 Reversed hazard rate function

The reversed hazard rate was introduced by von Mises (1936), however, Keilson and Sumita (1982) were among the first to define reversed hazard rate function and called it the "dual failure function". The name reversed hazard rate was first used by Lagakos et al. (1988). It extends the concept of hazard rate to a reverse time direction. This function is denoted by $r(x)$ and is defined as

$$r(x) = \lim_{\Delta x \to 0} \frac{\Pr(x - \Delta x < X < x|X \leq x)}{\Delta x}, \tag{1.19}$$

$$= \lim_{\Delta x \to 0} \frac{F(x) - F(x - \Delta x)}{\Delta x} \times \frac{1}{F(x)} = \frac{f(x)}{F(x)}. \tag{1.20}$$

Hence, the reversed hazard rate function can be interpreted as the instantaneous conditional probability that the life has survived the instant $(x - \Delta x)$, given that it fails before time x. Properties and other related characterizations of reversed hazard rate functions can be found in Chandra and Roy (2001), Nanda and Shaked (2001) and Shaked and Shanthikumar (1994).

1.3.5 Stochastic orderings

For a non-negative continuous random variable, X, stochastic ordering is an important tool for judging the comparative behavior.

Definition 1.1. A random variable X is said to be smaller than a random variable Y in the

(i) stochastic order $(X \leq_{ST} Y)$ if $F_X(x) \geq F_Y(x)$ for all x.

(ii) hazard rate order $(X \leq_{HR} Y)$ if $h_X(x) \geq h_Y(x)$ for all x.

(iii) mean residual life order $(X \leq_{MRL} Y)$ if $m_X(x) \leq m_Y(x)$ for all x.

(iv) likelihood ratio order $(X \leq_{LR} Y)$ if $\frac{f_X(x)}{f_Y(x)}$ decreases in x.

The following implications (see Shaked and Shanthikumar, 1994) are well known.

$$X \leq_{LR} Y \Rightarrow X \leq_{HR} Y \Rightarrow X \leq_{MRL} Y \text{ and } X \leq_{HR} Y \Rightarrow X \leq_{ST} Y. \tag{1.21}$$

In the next section, we describe time-to-event (or lifetime) data and notions of censoring that are very frequently encountered in survival and reliability analyses.

1.4 Time-to-event data and censoring

Lifetime or time-to-event data is a term utilized for describing data that measure time to occurrence of some event. The event could be different depending on the variable of interest. It could be death, occurrence or outset of some disease, relapse or recurrence of a disease, equipment breakdown etc., see Kalbfleisch and Prentice (2002) and Lawless (2011) for more examples on time-to-event data appearing in the fields of survival and reliability analysis.

Time-to-event data present themselves in various ways that create distinctive problems in analyzing and inferring such data. Often, one peculiar feature observed in time-to-event data, known as censoring, which, in a broader description, is realized when some lifetimes are known to have occurred within certain time intervals and the remaining of the lifetimes are only known exactly. There are various categories or schemes of censoring. Censoring schemes can be broadly classified as conventional censoring, such as, right censoring, left censoring, interval censoring, and progressive censoring.

1.4.1 Conventional censoring

1.4.1.1 Right censoring

Right censoring is the most common form of censoring with lifetime data in both engineering and medical applications. In right censoring, just lower limits on lifetime are accessible for a few individuals. Right censoring arises only in certain situations because some individuals survive at the time of termination of the study. In other instances, individual might move away from the study area for reasons that are directly unrelated with the study and hence, contacts for those individuals are lost eventually. In some other situations, individuals might be withdrawn or might decide to withdraw from the study on account of improving and/or worsening prognosis. Two types of right censoring are constructed into the experimental design for reducing the time taken for completion of the study.

Type-I censoring: Sometimes experiments are continued upto a fixed time period in such a fashion that lifetime of an individual will be known to the researcher exactly only if it is lesser than a predetermined time. In such situations, the data are termed as Type-I or time censored. For example, suppose in a life testing experiment n items are simultaneously put into operation and the study is terminated at a predetermined or predefined time point t_0. Suppose that r items gets failed by this time and the remaining $n - r$ items become operative till t_0. Then, it is said that $n - r$ items are censored and the data consist of lifetimes of r failed items and for the remaining $n - r$ items the censoring time is t_0. Type-I censoring occurs frequently in medical research when a decision is taken to terminate a study at a particular fixed date on which all individual's lifetime will not will be known.

Type-II censoring: The term Type-II censoring refers to the situation where n individuals start on study at the same time and the study terminates as soon as the k (a pre-specified number between 1 to n) lifetimes have been observed. Thus, only the smallest k lifetimes in a random sample of n are observed in such situation. This type of censoring is also sometimes known as order censoring or failure censoring.

1.4.1.2 Left censoring

In life testing applications, left censoring occurs when a unit has failed at the time of its first inspection and we realize only that the unit has failed before the inspection time. In other situations, left censored observations might arise when the exact value of a response has not been observed, rather we know only an upper limit on that response. As an example, consider a measuring instrument that possess a lack of sensitivity needed to measure observations below a known threshold level. When the measurement is used, if the indication comes below the instrument threshold, we know only that the measurement taken by it is less than the threshold.

A data set may contain both left and right censored observations and in that case lifetimes are termed as *doubly censored*. As an example, suppose a psychiatrist has collected data for determining the age at which children might have learned to execute a particular task. In such situation, lifetime must be the time the child has taken to learn to perform the task from date of birth. Hence, those children who already knew how to perform the task at the start of the study, were left censored and those who could not learn the task even at the end of the study were eventually right censored observations.

1.4.1.3 Interval censoring

Interval censoring, an another type of censoring, occurs when the lifetime is only known to occur within a time interval. Such pattern might be observed in a clinical trial where study participants (patients) are followed up periodically and an intended event time of a participant is only known to fall in an interval. For more details on conventional censoring schemes, one can see Klein and Moeschberger (2005).

1.4.2 Progressive censoring

There are many situations in life testing or reliability experimentation in which units are lost or removed from experiment before failure takes place and we loss information about certain lifetimes. The loss might occur unintentionally, or it might be intentional as per the design of the study. Unintentional loss might happen, for example, in the case of accidental breakage of an experimental unit or if an individual under study drops out or if the experimentation itself must cease on account of some unanticipated circumstances such as, unavailability of testing facilities and depletion of funds.

Sometimes, the removal of experimental units from experimentation is pre-designed and intentional and is planned to be done so in order to free up testing facilities to save cost and time or for other experimentation purposes. In some situations, intentional removal of items or termination of the experiment might be due to ethical considerations when there are live units on test.

If an experimenter intends to remove live experimental units at different time points other than the final termination point of the experiment then the conventional schemes described above will not be of use. None of the conventional censoring schemes allows for the units to be lost or removed from the test at different time points other than the final termination point. This allowance shall be desirable, as happens in the situation of studies of wear, in which the actual aging process requires items to be disassembled at the maximum extent at different stages in the experiment.

Intermediate removal might also be desirable is cases when a trade-off between reduced time of experimentation and the observation of at least some extreme lifetimes is pursued, or when some of the surviving items under experiment that are removed early on (such as, items under test are arduous to acquire or highly expensive) can be used for some other tests. Sometimes, the loss of items at points other than the final termination point might also be unavoidable, as in the case of loss of contact with individuals under study or accidental breakage of experimental units. These reasons and motivations lead practitioners and theoreticians directly into the area called *progressive censoring* or *progressively censoring*. There are different types of progressive censoring schemes described in literature. For example, progressively type-II right censoring is more generalized than conventional type-II censoring and so on. For more detailed insight on progressive censoring schemes, one could refer to Balakrishnan and Aggarwala (2000), Balakrishnan and Cramer (2014).

In the next section we describe some important methods of estimating unknown parameters involved in any probability distribution or any life distribution.

1.5 Parameter estimation

The important segment of modeling time-to-event data sets is to estimate the unknown parameter of the assumed probability distribution, at least assumed to be approximately appropriate in modeling, through the light of real life observations. Important purpose of estimating model parameters is to understand the complete form of the probability distribution by estimated parameter values and to evaluate its goodness of fit, that is, how well it fits the observed data. Goodness of fit is assessed by finding parameter values of a model that best fits the data-a procedure called *parameter estimation*, see Lehmann and Casella (2006).
The methods of estimating unknown parameter(s) of a life distribution can be broadly classified into two segments, namely, classical methods and Bayesian methods. In this thesis, the following methods of estimation are been adopted.

1.5.1 Method of moments

The method of moments estimation is a classical parameter estimation technique, introduced by *Karl Pearson* in 1894, based on the assumption that the sample moments (about the origin) are good estimators of the population moments (about the origin) because of the unbiasedness and consistency criteria of estimators, see Casella and Berger (2002) for more details. The method works as follows.
Let $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ denote the vector of sample observations. It is well known that the population moments are functions of unknown parameters, say θ, of the probability distribution. If there are k parameters in the population probability distribution, i.e., θ is a $k \times 1$ vector, then first k raw moments for the population probability distribution, viz. μ'_r, $r = 1, 2, \ldots, k$, are obtained and they are equated to the corresponding sample raw moments, viz., m'_r, $r = 1, 2, \ldots, k$,

which gives the following k equations

$$E[X^r] = \mu'_r = m'_r = \sum_{i=1}^{n} \frac{X_i^r}{n} \; r = 1, 2, \ldots, k. \tag{1.22}$$

These k equations are called *moment equations* and the solutions obtained by solving these equations for θ are called moment estimators of the k parameters. This method can be used only if the moments of the distribution exist.

1.5.2 Method of maximum likelihood

This is an important and widely used classical parameter estimation method, introduced by *R. A. Fisher* in 1912.

Let $X_1, X_2, \ldots, X_n$ be an independent and identically distributed (i.i.d) sample from a pdf $f(x;\theta), \theta \in \Theta$ (parameter space). Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ be a particular realization on $X_1, X_2, \ldots, X_n$. For given $\boldsymbol{x}$, the function, $L(\theta|\boldsymbol{x})$, called the likelihood function of the sample, is considered as a function of the parameter θ. By notation,

$$L(\theta|\boldsymbol{x}) = \prod_{i=1}^{n} f(x_i;\theta).$$

It should be noted that there is a conceptual distinction between the joint pdf of sample data and the likelihood function of the sample, the domain of the former is sample space whereas the domain of the latter is the parameter space Θ, see Lehmann and Casella (2006) for more insights.

Definition 1.2. The maximum likelihood estimator (MLE) of the parameter θ is defined as the value $\hat{\theta} \in \Theta$, such that

$$L(\hat{\theta}|\boldsymbol{x}) = \sup_{\theta \in \Theta} L(\theta|\boldsymbol{x}). \tag{1.23}$$

Generally, maximization techniques based on differential calculus are used to obtain MLEs. For simplicity of calculation, MLEs are obtained by maximizing $l(\theta) = \ln L(\theta|\boldsymbol{x})$. Optimal properties of MLEs, viz., asymptotic and non-asymptotic, are useful to understand the quality of the estimators. One important property of MLEs is the consistency of the estimators, see Lehmann and Casella (2006), Rajagopalan and Dhanavanthan (2012).

1.5.3 Bayesian method

This is an alternative approach in statistical inference, called Bayesian approach, which views probability of an event as a measure of degree of one's personal belief in the occurrence of the event. In Bayesian approach, before drawing the sample, the information is collected about the parameter involved in the population distribution-called *prior information* about the population.
If θ is the unknown parameter of the population, then in Bayesian approach θ is treated as a random variable. The prior distribution of θ, say, $\pi(\theta)$, is a probability distribution defined on the parameter space Θ based on prior information that has been closely scrutinized and processed. Owing that the joint pdf of sample, $f(\boldsymbol{x};\theta)$ is actually the conditional distribution of the observations given the parameter, the conditional pdf of θ given the sample observations $\boldsymbol{x}$, denoted by $\pi(\theta|\boldsymbol{x})$ is obtained by applying Bayes' theorem, see Berger (2013). So,

$$\pi(\theta|\boldsymbol{x}) = \frac{\pi(\theta) f(\boldsymbol{x};\theta)}{\int \pi(\theta) f(\boldsymbol{x};\theta) d\theta}, \theta \in \Theta, \tag{1.24}$$

where $\pi(\theta|\boldsymbol{x})$ is called posterior distribution of θ.

Definition 1.3. The Bayes estimator of θ is the value $\hat{\theta} \in \Theta$, such that

$$\pi(\hat{\theta}|\boldsymbol{x}) \geq \pi(\theta|\boldsymbol{x}) \; \forall \; \theta \in \Theta. \tag{1.25}$$

However, under the decision theory approach, Bayes estimator of θ is obtained by minimizing Bayes Risk considering suitable loss function, see Zacks (1971) and Ghosh et al. (2007).

Now, in the next section, we represent a comprehensive review of literature on exponential and gamma life distributions and finite mixture of probability distributions along with application areas and recent developments that build the base of the main research work presented in the thesis.

1.6 Review of literature

Among many useful life distributions only few viz., exponential, gamma, Weibull, Rayleigh and lognormal, have been considered as (standard) life distributions in statistical literature. To describe only important (standard) life distributions and keeping the genesis of this thesis in mind, two relevant (standard) life distributions, namely, exponential and gamma have been reviewed and been briefly discussed respectively in sub-sections 1.6.1 and 1.6.2 below. In sub-section 1.6.3, a thorough review of finite mixtures of distributions, considered as main technique and base of the present research work, has been accomplished along with fields of applications and recent developments on the topic.

1.6.1 Exponential distribution

Among a lengthy list of life distributions, exponential distribution is the most important one parameter family of life distribution. The importance is partly because of the fact that many of the most commonly used families of life distributions are actually two or three parameter extensions of the exponential distributions, and hence, it is standard among other families of life distributions. Moreover, the exponential distribution, with its constant hazard rate property, form a base for evaluating other families of life distributions. The exponential distribution is quite simple to describe and is uncommonly tractable in statistical analyses since it has only one parameter.

If X is a non-negative continuous random variable following an exponential distribution with parameter $\theta(>0)$ or mean $1/\theta$, then the pdf of X is given by

$$f(x) = \theta e^{-\theta x}, x > 0. \tag{1.26}$$

let us denote it by $X \sim exp(\theta)$. The corresponding cdf is given by

$$F(x) = 1 - e^{-\theta x}, x > 0. \tag{1.27}$$

It is easy to verify that the residual life distribution at $t(>0)$ is independent of t for exponential distribution. In fact, this characterizes exponential distributions. As a result, an another characterizing property asserts that the mean residual life of exponential life is independent of the age its age. Singpurwalla (2003) has made extensive use of the fact that for any life distribution G with the cumulative hazard rate function H,

$$G(x) = e^{-H(x)} = \Pr[X > H(x)], \tag{1.28}$$

where X has an exponential distribution with parameter 1.

From this point of view, it is to be expected that the exponential distribution will play a central role. Because of its remarkable properties, exponential distribution arise naturally in theoretical settings. It has many characterizations of both theoretical and practical importance. It is not surprising, then, that exponential distribution has been overused in applications; but that does not reduce its importance. There are many reasons why exponential distribution plays a central role within the class of lifetime probability distributions, more insight can be seen in Mann et al. (1974), Johnson et al. (1994), Balakrishnan and Basu (1995) and

Nelson (2004).

The best known characterization, perhaps, of the exponential distribution is its so called "lack of memory" property. Moreover, a distribution has a constant hazard rate if and only if it is an exponential distribution and a distribution F has a mean residual life independent of age if and only if it is an exponential distribution. More characterizations of the exponential distribution related to reliability and survival can be found in Basu (1965), Cowford (1966), Desu (1971), Galambos and Kotz (1978) and Azlarov and Volodin (1986) .

1.6.2 Gamma distribution

The single parameter involved in the exponential distribution serves both as a scale and as a frailty parameter. Moreover, if an age parameter or a Laplace transform parameter is introduced, the distribution remains an exponential distribution and only the parameter is changed. It is well observed that introduction of moment and convolution parameters to an exponential distribution both lead to a family called family of gamma distributions. Gamma distribution also plays an important role in areas of survival and reliability analysis, see Cox and Oakes (1984), Aalen (1988, 1994), Miller (2011), Elsayed (2012) and Chandra and Sen (2014).

A non-negative continuous random variable, X, is said to follow a gamma distribution with scale parameter $\theta(>0)$ and shape parameter $\lambda(>0)$ if its pdf is of the form

$$f(x) = \frac{\theta^{\lambda}}{\Gamma(\lambda)} x^{\lambda-1} e^{-\theta x}, x > 0, \tag{1.29}$$

where $\Gamma(a) = \int_0^{\infty} z^{a-1} e^{-z} dz$ is the gamma function.

Let us denote it by $X \sim gamma(\lambda, \theta)$.

Typically, gamma distributions with shape parameter as positive integers are termed as *Erlang distributions* and the *Erlang distribution* with shape parameter 1 simplifies to the exponential distribution. When λ is a half integer and $\theta = 1/2$, the gamma distribution is known as a chi-square distribution.
There are several ways to derive the gamma distribution from exponential distribution, viz., from a Poisson process, by introduction of moment parameter, by introduction of a convolution parameter and by mixtures; each is instructive in its own way. All of these are based in some way upon the exponential distribution.

The gamma density (1.29) was obtained by Pearson (1895) and is known as a Type III Pearson curve. Pearson derived the density from a differential equation; see Johnson et al. (1994). If X has the gamma density (1.29), then it can be shown that $1/X$ has the density g given by

$$g(x) = \frac{\theta^{\lambda}}{\Gamma(\lambda)} \frac{1}{x^{\lambda-1}} e^{-\theta/x}, x > 0. \tag{1.30}$$

This density was also obtained by Pearson (1895) and is known as a Type V Pearson curve. For further discussion of the Pearson curves, see also Elderton and Johnson (1969).
In general, the distribution function and the survival function of the gamma distribution do not have simple expression; they can be expressed only in terms of the incomplete gamma function when shape parameter is not an integer. However, properties of these functions can be determined in other analytical ways. Because the survival function of the gamma distribution can be given only in terms of the incomplete gamma function when shape parameter is not an integer, neither the hazard rate nor the reversed hazard rate can be expressed in closed form, more details can be found in Barlow and Proschan (1975).
The survival function S corresponding to the gamma density (1.29) is log-concave for $\lambda \geq 1$ and is log-convex on $[0, \infty)$ for $\lambda \leq 1$. The gamma cdf is log-concave for all λ. The hazard rate function of gamma distribution is increasing or constant

or decreasing depending on $\lambda > 1$ or $\lambda = 1$ or $\lambda < 1$, respectively. The reversed hazard rate of gamma density is decreasing for all λ because of the log-concavity of its cdf and residual life distribution converges in distribution to an exponential distribution whenever the hazard rate has a finite positive limit, more characterizations of gamma distribution can be found in Lukacs (1955), Gupta (1960), Engel and Zijlstra (1980), Wang (1981) and Johnson et al. (1994).

1.6.3 Finite mixtures of distributions

A wide range of observed phenomena, which do not normally yield to modeling through classical distributions because of their inherent complexity and heterogeneous nature, can be modeled with higher satisfaction by an important method called finite mixtures of distributions. The main reason for successful applications of finite mixture models in a vast range of fields in the biological, physical and social sciences lies in their flexibility and high degree of accuracy. The distribution of random quantity of interest is modeled as a mixture of a finite number of distributions, also called *component distributions*, with varying proportions in finite mixture model. Thus, a mixture model is capable to model quite complex situations incorporating an appropriate choice of its components to constitute explicitly the support of local areas of the true distribution. Moreover, It is quite capable of handling situations where a single parametric family might not be able to explain a satisfactory model for local variation in the observed data.

Newcomb (1886), can be named as pioneer, used the concept of finite mixture distribution while modeling outliers. However, Pearson (1894) has the credit for introduction of statistical modeling using finite mixtures of distributions when he applied the technique in an analysis of data, provided by Weldon (1892, 1893), related to crab morphometry. Finite mixtures of normal distributions in explaining the crab data was suggested by Pearson. He obtained estimates based on moments of the five parameters of the mixture of normal distributions as a solution of a ninth degree polynomial and it was a demanding task, computationally. Over the ensuing years, various attempts were made to simplify Pearson's (1894) moments-based approach to fitting of a normal mixture model and, thus, the utilization of mixture of normal distributions to model the different species of crab motivated ample use of finite mixture distributions in other applied areas also. Below is given a complete definition for mixture of probability distributions.

Definition 1.4. Let X be a random variable with family of probability distributions $\{g(x; \underset{\sim}{\theta}); \underset{\sim}{\theta} \in \Theta\}$, where Θ is the parameter space and is a subset of m^{th} dimensional Eucledian space $\Re^m$. Let $G(\underset{\sim}{\theta})$ be a cdf of $\underset{\sim}{\theta}$, the the pdf $f(x)$ defined by

$$f(x) = \int_{\theta \in \Theta} g(x; \underset{\sim}{\theta}) dG(\underset{\sim}{\theta}) \tag{1.31}$$

is called a general mixture density function. In (1.31), $G(\cdot)$ is called the mixing distribution.

When $G(\cdot)$ is discrete and assigns positive probability to a finite number of points θ_i $(i = 1, 2, \ldots, k)$, the density in (1.31) can be written in the form

$$f(x) = \sum_{i=1}^{K} \pi_i f_i(x; \theta_i), \tag{1.32}$$

where $f_i(x; \theta_i)$ is the pdf with parameter θ_i of the i^{th} component, $\pi_i' s, i = 1, 2, \ldots, K$, are the mixing proportions (or mixing weights or component priors). It is assumed that $\pi_i' s, i = 1, 2, \ldots, K$, are non-negative and $\sum_{i=1}^{K} \pi_i = 1$.

The equation (1.32) represents the pdf of finite mixture of densities.

1.6.3.1 Identifiability

Identifiability is a concept which plays an important role in the analysis of the finite mixture models. If there exists a one-to-one correspondence between the

mixing distribution and the resulting mixture, then the mixture is called identifiable. Unidentifiable mixtures cannot be expressed uniquely as functions of mixing distributions and component. For example. the finite mixture of uniform distributions is not identifiable because the mixture density can be represented in two different forms.
The concept of identifiability was introduced by Teicher (1960, 1961, 1963, and 1967) and he developed a theory to identify mixtures. He showed that a finite mixture of Poisson distributions is identifiable but mixtures of binomial distributions are not identifiable in certain cases. The identifiability of finite mixtures of negative binomial component distributions was showed by Yakowitz and Spragins (1968). An excellent and lucid account on the concept of identifiability of mixtures can be found in the work by Titterington et.a1. (1985). They pointed out that finite mixtures of continuous densities are mostly identifiable except uniform densities. More discussions on identifiability of finite mixtures can be found in Patil and Bildikar (1966), McLachlan and Basford (1988) and Maritz and Levin (1989). A comprehensive review on the topic can be perceived in Prakasa Rao (1992) and Lindsay (1995). Identifiability of a finite mixture of Gompertz densities was showed by Al-Hussaini et.al. (2000), Sankaran and Maya (2004, 2005) investigated the identifiability of beta finite mixtures and Pareto finite mixtures. Panteleeva et al. (2015) addressed that the Weibull mixtures is identifiable.

1.6.3.2 Parameter estimation in finite mixture models

There are number of methods that have been suggested for estimating the parameters in a finite mixture model over the years, e.g., Pearson (1894) applied the method of moments that requires to find the roots of a polynomial of ninth degree to derive the estimates of the five parameters involved in his model. An iterative method, developed subsequently by Cohen (1967), for solving the same problem requires solving only cubic polynomials. Fryer and Robertson (1972) and Tan and Chang (1972) showed that the method of moments is actually inferior to MLE in a mixture of two normal distributions. In fact, optimal solutions are not guaranteed by method of moments but was initially useful owing certain situations in which solutions by maximum likelihood method were unmanageable. However, with the invention of modern digital computers and sophisticated software programming, maximum likelihood method became prevalent for general mixture problems.

Another popular method for estimation of parameters of the mixture models is Bayesian method. An advantage is obtained by using prior information in Bayesian method over maximum likelihood because inference can be drawn even with smaller number of data points. For models with many parameters, MLE sometimes is ill-posed when the data set is not sufficiently large. However, in certain situations, Bayesian estimation of the parameters of mixture distributions demands lengthy computations. Further, the posterior inference can rarely be sampled directly because of their complicated forms, simplifying conjugate priors exist rarely, and there are no sufficient statistics available in many situations to simplify the analysis.

1.6.3.3 Fields of applications and recent developments

In many real life situations, finite mixture models are being utilized considerably for statistical analysis . A chronological survey on the field of applications is sough below.

A representative cross-section of the field of applications include the study on evening temperature distribution (Charlier and Wicksell, 1924), death times of mice (Muench, 1936), frequencies of comet (Schilling, 1947), association of chromosomes (Skellam, 1948), lifetime related to valves (Davis, 1952; Everitt and Hand, 1981) frequencies of water plankton (Cassie, 1962), heights of plants (Tanaka, 1962). response times (Cox, 1966), frequencies of death notice (Hasselblad, 1969), lengths of pike (Macdonald, 1971), traffic gaps (Ashton, 1971), Pollen grains (Usinger, 1975), concentrations of crop (Peters and Coberly, 1976), test scores in clinical studies (Symons, 1981), frequencies of certain crimes (Harris, 1983), fishery composition of mixed stocks (Miller, 1987), philatelic mixtures (Izenmann and Sommer, 1988) and completion of a particular task (Desmond and Chapmall, 1993).

Along with the above applications, mixture models are helpful in robustness studies (Hyrenius, 1950 and Tan, 1980), latent structure models and cluster analysis (Fielding, 1977; Symons, 1981 and McLachlan and Basford, 1988), approximation of different distributions (Dala1, 1978), generation of random variables (Peterson and Kronmal, 1982), density estimation based on kernel (Titterington, 1983), outlier analysis (Barnett and Lewis, 1984), prior density modeling (Diaconis and Ylvisaker, 1985) and models based on artificial neural networking (Rip1ey, 1994). Sankaran and Maya (2005) used properties of finite mixture of Pareto distributions in the context of income analysis.

In medical research also finite mixture densities are useful in many applications. Finite mixture densities are utilized to model age related to schizophrenia (Levine, 1981; McLachlan, 1987; McLachlan and Peel, 2000 and Everitt, 2003), to model mortality rate variations between different geographical areas (Betemps and Buncher, 1993), in survival data modeling (McGiffin et al., 1993; McLachlan and McGiffin, 1994 and McLachlan and Peel, 2000) and to identify brain activation regions in functional magnetic resonance imaging (Bullmore et al., 1996; Everitt and Bullmore, 1999 and Everitt, 1998).

In lifetime data analysis, the population of lifetimes are sometime disintegrated into sub-populations based on lifetimes of units under various periods of production, design differences, different raw materials utilized etc. In such situations, use of finite mixture of distributions is usual to model data. Accordingly, finite mixture of two exponential distributions for the analysis of failure times transmitter receivers of a single commercial airline was utilized by Mendenhall and Hader (1958). Mixture of exponential models was used by Cox (1959) for analysis of data on failure times by classifying the data on failure times into two sub-populations on the basis of identified and unidentified cause. For life testing of electron tubes, Kao (1959) utilized finite mixture of Weibull distributions. In reliability analysis, finite mixtures of inverse Gaussian distributions were studied by Ahmad (1982), Amoh (1983) and Al-Hussaini and Ahmad (1984). Different Properties and characterizations of finite mixture of exponential distributions can be found in Nassar and Mahmoud (1985) and Nassar (1988).

Characterizations of finite mixture of gamma distribution has been studied by Gharib (1995) and characterizations using concepts of reliability of finite mixture models have been addressed by Ahmad (1996). Al-Hussaini and Osman (1997) obtained the median of finite mixture of k components. Al-Hussaini (1999) used Bayesian method to predict observations under a mixture of two exponential components. Later, Al-Hussaini et al. (2000) studied the finite mixture of Gompertz densities as a lifetime model and order statistics based Bayesian predictive densities for finite mixture models has been discussed by Al-Hussaini (2001). In competing risk situation, finite mixture of distributions can be useful in modeling time to failure of a system (Crowder, 2001). Gamma distribution mixture and related applications can be found in Wiper et al. (2001). Behaviour of the hazard rate of finite mixture of distributions has been studied by Block et al. (2003). Jaheen (2003) has studied the aspects in Bayesian prediction under a mixture of two-component Gompertz life distributions. Efficient tools has been developed by Cross (2004) in reliability context using finite mixture of distributions. More applications and properties of finite mixture models in reliability theory can be found in Al-Hussaini and Sultan (2001). Later, Sultan et al. (2007) has studied properties and estimation aspects in the finite mixture of inverse Weibull distributions.

Afify (2011) studied classical estimation of mixed Rayleigh distribution. Kazmi et al. (2012) investigated on the Bayesian estimation for two-component mixture of Maxwell distributions. Mixture of Gamma distributions for Bayesian analysis in queuing theory has been investigated by Mohammadi et al. (2013). Mixture of the inverse Rayleigh distribution, its properties and estimation in Bayesian framework has been studied by Ali (2014). Ateya (2014) has investigated finite mixture of generalized exponential distributions based on censored data and Bayesian prediction under a finite mixture of generalized exponential lifetime model has been investigated by Mohamed (2014). Mohammed et al. (2015) recently used finite mixture model of exponential, gamma and Weibull distributions to analyze survival data in heterogeneous set up. For recent developments on finite mixtures of distribution, one can see Zhang and Huang (2015), Tahir et al. (2016) and Feroze (2016). Some results on information properties of mixture distributions is very recently addressed by Toomaj and Zarei (2017) and a method to generate a random sample from a finite mixture distribution has been proposed recently by Ghorbanzadeh et al. (2017).

With the above search of literature, below is presented the genesis of the thesis emphasizing on the motivation of the present research work, broader aims and objectives and scope of the research.

1.7 Genesis of the thesis

1.7.1 Motivation

Finite mixture distributions arising from the standard life distributions play, in most of the times, a better role in modeling real life phenomena as compared to the standard ones (cf. discussion in sub-section 1.6.3.3 of section 1.6) and knowing the importance of exponential and gamma distributions in lifetime modeling as discussed in section 1.6, the research work presented in this thesis is motivated in introducing newer probability distributions that might have added flexibility with regards to their survival properties and distributional form. Finite mixtures are used to derive new parametric families of distributions from old ones; this is done by using mixing distributions that have a common parameter; the mixture retains

that parameter so that it may yield a new parametric family.
In the present study, the role of finite mixture of exponential distribution with parameter θ and gamma distribution with scale parameter θ and shape parameter 3 with special kind of mixing proportions in the context of time-to-event situations has been examined.

1.7.2 Aims and objectives

The aim of this research work is to introduce and study more flexible probability distributions that are having additional flexibility over some popular life distributions available in the literature. The objectives are four-fold.

(i) To investigate the basic distributional, structural and survival (or reliability) properties of the newly proposed probability distributions.

(ii) To explore and study important survival and/or reliability properties that could make the proposed models competitive among other popular lifetime models.

(iii) To investigate the statistical inference procedures while estimating the unknown parameters involved in the proposed distributions.

(iv) To find applicability of the proposed distributions in the area of survival analysis and reliability studies.

1.7.3 Scope

- The study is confirmed in investigating newer probability distributions synthesized from popular standard life distributions, viz., exponential and gamma, and possible extensions or generalizations of them.

- The scope of the study is limited to the probability distributions of univariate random variables that are continuous and non-negative. The problems of identifiability of the finite mixture of distributions, presented in this thesis, are beyond the scope of the thesis.

- The application area of the study is restricted to time-to-event data sets arising from the fields of survival and reliability analysis.

1.8 Organization of the thesis

Rest of the thesis dissertation is organized as below.
In Chapter 2, a new probability distribution, named as *xgamma* distribution, is proposed, its different distributional properties, viz., shape, moments and related measures, and survival properties, viz., hazard rate function, MRL function, stochastic ordering are studied. Classical methods (method of moments and method of maximum likelihood) of estimating parameter of xgamma distribution are proposed for complete sample situation along with a simulation study and with a real lifetime data illustration. The proposed distribution is compared with exponential distribution with respect to certain properties and application.

Chapter 3 is dedicated in exploring and studying some additional distributional properties (such as characteristic and generating functions, important entropy measures, distributions of extreme order statistics) and additional survival properties (such as mean time to failure, ageing intensity, stress-strength reliability) of xgamma distribution. Classical as well as Bayesian methods of estimating parameter and important survival characteristics of xgamma distribution are investigated under progressively type-II censoring scheme. Simulation study and real life data illustration are presented to address the consequences of the methods described.

Considering the role of truncated distributions in the area of lifetime modeling, three truncated versions (double, lower and upper) of xgamma distribution is introduced in Chapter 4. Main emphasis is given in studying properties of upper truncated version of xgamma distribution. Different distributional properties, such as, moments and associated measures, entropy measures, distributions of extreme order statistics, and survival properties like, hazard rate function, reversed hazard rate function, are studied for upper truncated xgamma distribution. Method of maximum likelihood are proposed for the upper truncated version of xgamma distribution in complete sample situation. Real life data are analyzed to address the applicability of truncated versions and compared with popular life distributions.

Weighted distributions are very frequently studied in diverse area of applications. In Chapter 5, a weighted version of xgamma distribution is proposed taking a special non-negative weight function and, as a special case, length biased version of xgamma distribution is introduced and its different distributional and survival properties are studied in details. The main aim of the chapter is to find an application of weighted xgamma distribution in modeling lifetime data. Method of moments and method of maximum likelihood are proposed for estimating unknown parameter in length biased xgamma distribution in complete sample situation. Simulation study and real life data illustration are presented to understand the applicability of the proposed distribution and some popular life distributions are compared.

Chapter 6 deals with introducing and studying of two extensions or generalizations, viz., the *quasi xgamma* and the *two-parameter xgamma* distributions, of xgamma distribution. Different distributional (such as shape, moments and related measures, important entropy measures, order statistics distributions) and survival properties (such as hazard rate function, MRL function, stochastic orderings) are studied for both the extensions separately. Method of moments and method of maximum likelihood are proposed for estimating unknown parameters in each of the extension for complete sample situation supported with simulation studies, real lifetime data illustrations and comparisons with other life distributions.

Chapter 2

The xgamma distribution

The exponential and gamma are well known probability distributions used for modeling lifetime data. Both the distributions possess some interesting structural properties, for example, exponential distribution possesses memory less and constant hazard rate properties, see section 1.6 of Chapter 1 for discussion. Moreover, exponential distribution can be used in modeling time-to-event data or modeling waiting times as a special case of gamma distribution. Various extensions of both the distributions can be found in the literature for describing the uncertainty behind real life phenomena arising in the area of survival analysis (see Johnson et al., 1994; Lawless, 2002) and reliability engineering (see for more details Barlow and Proschan, 1975).

In this chapter, a new probability distribution, namely, xgamma distribution, is introduced and studied. A special finite mixture of exponential and gamma distributions are been considered for obtaining the form of the distribution, and hence, the name *xgamma* is proposed. The rest of the chapter is organized as follows.
The xgamma distribution is introduced in section 2.1. Shape of the distribution, moments and measures are investigated in the section 2.2 and in its dedicated subsections. Section 2.3 deals with the survival properties, such as, hazard rate and MRL functions, of xgamma distribution. In section 2.4, two classical methods of

estimation, viz., method of moments and method of maximum likelihood, are been proposed for complete sample case. An algorithm for generating random samples from xgamma distribution is described in section 2.6 along with a Monte-Carlo simulation study for investigating the behaviour of estimates. In section 2.7, a real data set on time-to-event is analyzed to address the possible application of xgamma distribution and comparison is made with exponential distribution. Finally, section 2.8 concludes the chapter with important findings.

2.1 Methodology and synthesis

As indicated above, a special finite mixture of exponential and gamma distributions is used to obtain a new probability distribution, called as xgamma distribution. we present below the synthesis of the distribution.

By considering $K = 2$ in (1.32), the pdf of a non-negative continuous random variable X can be re-written as

$$f(x) = \sum_{i=1}^{2} \pi_i f_i(x). \tag{2.1}$$

We consider $f_1(x)$ to follow an exponential distribution with parameter θ and $f_2(x)$ to follow a gamma distribution with scale parameter θ and shape parameter 3 i.e., $f_1(x) \sim exp(\theta)$ and $f_1(x) \sim gamma(3, \theta)$ with $\pi_1 = \frac{\theta}{(1+\theta)}$ and $\pi_2 = 1 - \pi_1$ in (2.1). So, we obtain the probability density function (pdf) of xgamma random variable, X, as

$$\begin{aligned} f(x) &= \frac{\theta}{(1+\theta)} \theta e^{-\theta x} + \frac{1}{(1+\theta)} \frac{\theta^3 x^{3-1} e^{-\theta x}}{\Gamma(3)}, \\ &= \frac{\theta^2}{(1+\theta)} \left(1 + \frac{\theta}{2} x^2\right) e^{-\theta x}, \end{aligned}$$

where $\Gamma(a) = \int_0^\infty z^{a-1}e^{-z}dz$ is the gamma function.

Then, the following definition can be given for the xgamma distribution with one parameter θ.

Definition 2.1. A non-negative continuous random variable, X, is said to follow an xgamma (XG) distribution with parameter θ if its pdf is of the form

$$f(x) = \frac{\theta^2}{(1+\theta)}\left(1+\frac{\theta}{2}x^2\right)e^{-\theta x}, x > 0, \theta > 0. \tag{2.2}$$

It is denoted by $X \sim xgamma(\theta)$ or by $X \sim XG(\theta)$.

Now, we find the cumulative distribution function (cdf) of $xgamma(\theta)$. For deriving cdf corresponding to (2.2), we consider

$$F(x) = 1 - \Pr(X > x) = 1 - \int_x^\infty f(t)dt.$$

Now,

$$\begin{aligned}\Pr(X > x) &= \int_x^\infty \frac{\theta^2}{(1+\theta)}\left(1+\frac{\theta}{2}t^2\right)e^{-\theta t}dt, \\ &= \frac{\theta^2}{(1+\theta)}\left[\int_x^\infty e^{-\theta t}dt + \frac{\theta}{2}\int_x^\infty t^2 e^{-\theta t}dt\right].\end{aligned}$$

We calculate,

$$\int_x^\infty e^{-\theta t}dt = \frac{e^{-\theta x}}{\theta}, \tag{2.3}$$

$$\int_x^\infty t e^{-\theta t}dt = \frac{xe^{-\theta x}}{\theta} + \frac{e^{-\theta x}}{\theta^2}. \tag{2.4}$$

$$\int_x^\infty t^2 e^{-\theta t}dt = \frac{x^2 e^{-\theta x}}{\theta} + \frac{2}{\theta}\left(\frac{xe^{-\theta x}}{\theta} + \frac{e^{-\theta x}}{\theta^2}\right). \tag{2.5}$$

So, using (2.3) and (2.5) we have

$$\begin{aligned}
\Pr(X > x) &= \frac{\theta^2}{(1+\theta)}\left[\frac{e^{-\theta x}}{\theta} + \frac{\theta}{2}\left\{\frac{x^2e^{-\theta x}}{\theta} + \frac{2}{\theta}\left(\frac{xe^{-\theta x}}{\theta} + \frac{e^{-\theta x}}{\theta^2}\right)\right\}\right], \\
&= \frac{\theta^2}{(1+\theta)}\left[\frac{e^{-\theta x}}{\theta} + \frac{x^2e^{-\theta x}}{2} + \frac{xe^{-\theta x}}{\theta} + \frac{e^{-\theta x}}{\theta^2}\right], \\
&= \frac{\theta^2}{(1+\theta)}\left[\frac{\theta e^{-\theta x} + \frac{x^2}{2}\theta^2 e^{-\theta x} + \theta x e^{-\theta x} + e^{-\theta x}}{\theta^2}\right], \\
&= \frac{e^{-\theta x}}{(1+\theta)}\left(1 + \theta + \theta x + \frac{\theta^2 x^2}{2}\right).
\end{aligned}$$

Hence, the cdf of $X \sim xgamma(\theta)$ is given by

$$F(x) = 1 - \frac{(1 + \theta + \theta x + \frac{\theta^2 x^2}{2})}{(1+\theta)}e^{-\theta x}, x > 0. \tag{2.6}$$

Figure 2.1 shows the pdf plots of xgamma distribution for some values of θ.

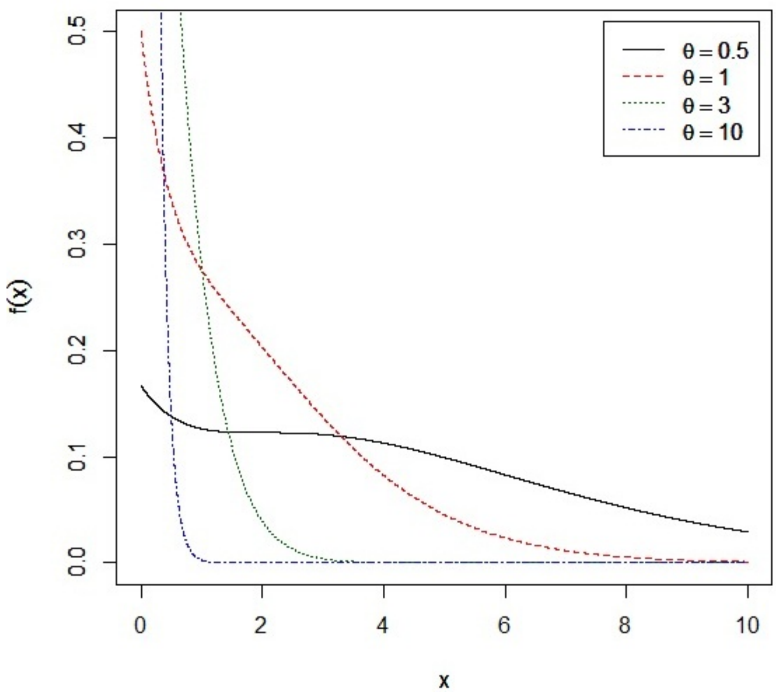

FIGURE 2.1: Probability density curves of xgamma distribution for some values of θ.

2.2 Shape, moments and related measures

This section is dedicated in studying shape, moments and other related measures of $xgamma(\theta)$.

2.2.1 Mode

To find the shape of a probability distribution, mode is an important measure. The first derivative of (2.2) with respect to x gives

$$\frac{d}{dx}f(x) = \frac{\theta^2}{(1+\theta)}\left(\theta x - \theta - \frac{\theta^2}{2}x^2\right)e^{-\theta x}. \tag{2.7}$$

And, the second derivative of (2.2) with respect to x gives

$$\frac{d^2}{dx^2}f(x) = \frac{\theta^2}{(1+\theta)}\left[(\theta - \theta^2 x)e^{-\theta x} - \left(\theta x - \theta - \frac{\theta^2}{2}x^2\right)\theta e^{-\theta x}\right]. \tag{2.8}$$

Equating (2.7) to 0, we find that, for $0 < \theta \leq 1/2$, (2.8) is negative if $x = \frac{1+\sqrt{1-2\theta}}{\theta}$. This implies

(i) for $\theta \leq 1/2$, $f(x) = 0$ implies $\frac{1+\sqrt{1-2\theta}}{\theta}$ is the unique critical point at which the pdf $f(x)$ is maximized,

(ii) for $\theta > 1/2$, $\frac{d}{dx}f(x) \leq 0$, i.e., $f(x)$ decreases in x.

Hence, the mode of xgamma distribution is given by

$$Mode(X) = \begin{cases} \frac{1+\sqrt{1-2\theta}}{\theta}, & \text{if } 0 < \theta \leq 1/2. \\ 0, & \text{otherwise.} \end{cases} \tag{2.9}$$

It is noted that $xgamma(\theta)$ is unimodal.

2.2.2 Non-central moments

Now, we find non-central moments for $xgamma(\theta)$.

The r^{th} order moment about origin of xgamma distribution is obtained as

$$\begin{aligned}
\mu'_r = E(X^r) &= \int_0^\infty x^r \frac{\theta^2}{(1+\theta)} \left(1 + \frac{\theta}{2}x^2\right) e^{-\theta x} dx, \\
&= \frac{\theta^2}{(1+\theta)} \left[\int_0^\infty x^r e^{-\theta x} dx + \frac{\theta}{2} \int_0^\infty x^{r+2} e^{-\theta x} dx\right], \\
&= \frac{\theta^2}{(1+\theta)} \left[\frac{\Gamma(r+1)}{\theta^r} + \frac{\theta}{2} \frac{\Gamma(r+3)}{\theta^{r+3}}\right],, \\
&= \frac{\theta^2}{(1+\theta)} \left[\frac{r!}{\theta^{r+1}} + \frac{(r+2)!}{2\theta^{r+2}}\right], \\
&= \frac{r![2\theta + (r+1)(r+2)]}{2\theta^r(1+\theta)}.
\end{aligned}$$

Here $\Gamma(a) = \int_0^\infty z^{a-1} e^{-z} dz$ is the gamma function.

Hence, we have,

$$\mu'_r = \frac{r![2\theta + (r+1)(r+2)]}{2\theta^r(1+\theta)} \text{ for } r = 1, 2, \ldots. \tag{2.10}$$

In particular, we get

$$\mu'_1 = \frac{(\theta+3)}{\theta(1+\theta)} = Mean(X) = \mu(\text{say}). \tag{2.11}$$

$$\mu'_2 = \frac{2(\theta+6)}{\theta^2(1+\theta)} \quad ; \quad \mu_3{}' = \frac{6(\theta+10)}{\theta^3(1+\theta)} \quad ; \quad \mu'_4 = \frac{24(\theta+15)}{\theta^4(1+\theta)}.$$

It is noted that for exponential distribution with parameter θ, the r^{th} order moment about origin is $\mu'_r = r!/\theta^r$.

2.2.3 Central moments and related measures

In this sub-section, we study central moments, coefficient of variation, measures of skewness and kurtosis for $xgamma(\theta)$.
The j^{th} order central moment of xgamma distribution can be obtained from the relation,

$$\mu_j = E[(X-\mu)^j] = \sum_{r=0}^{j} \binom{j}{r} \mu_r{}'(-\mu)^{j-r}. \tag{2.12}$$

In particular, we calculate,

$$\begin{aligned}
\mu_2 &= \mu_2' - \mu^2,\ \mu \text{ is given in (2.11)},\\
&= \frac{2(\theta+6)}{\theta^2(1+\theta)} - \left[\frac{(\theta+3)}{\theta(1+\theta)}\right]^2,\\
&= \frac{2(\theta+6)(\theta+1)-(\theta+3)^2}{\theta^2(1+\theta)^2},\\
&= \frac{(\theta^2+8\theta+3)}{\theta^2(1+\theta)^2},\\
&= Var(X) = \sigma^2(\text{say}).
\end{aligned} \tag{2.13}$$

The third order central moment is obtained by using

$$\begin{aligned}
\mu_3 &= \mu_3' - 3\mu_2'\mu + 2\mu^3,\\
&= \frac{6(\theta+10)}{\theta^3(1+\theta)} - \frac{6(\theta+6)(\theta+3)}{\theta^3(1+\theta)^2} + \frac{2(\theta+3)^3}{\theta^3(1+\theta)^3}
\end{aligned}$$

After simplification, we get,

$$\mu_3 = \frac{2(\theta^3+15\theta^2+9\theta+3)}{\theta^3(1+\theta)^3}. \tag{2.14}$$

Similarly, the forth order central moment is obtained by using

$$\begin{aligned}\mu_4 &= \mu_4' - 4\mu_3'\mu + 6\mu_2'\mu^2 - 3\mu^4,\\ &= \frac{24(\theta+15)}{\theta^4(1+\theta)} - \frac{24(\theta+10)(\theta+3)}{\theta^4(1+\theta)^2} + \frac{12(\theta+6)(\theta+3)^2}{\theta^4(1+\theta)^3} - \frac{3(\theta+3)^4}{\theta^4(1+\theta)^4}.\end{aligned}$$

On simplification, we get,

$$\mu_4 = \frac{3(5\theta^4 + 88\theta^3 + 310\theta^2 + 288\theta + 177)}{\theta^4(1+\theta)^4}. \tag{2.15}$$

The coefficients of variation (γ), skewness ($\sqrt{\beta_1}$) and kurtosis (β_2) for $xgamma(\theta)$ are obtained as

$$\gamma = \frac{\sqrt{(\theta^2 + 8\theta + 3)}}{(\theta+3)}, \tag{2.16}$$

$$\sqrt{\beta_1} = \sqrt{\frac{\mu_3^2}{\mu_2^3}} = \frac{2(\theta^3 + 15\theta^2 + 9\theta + 3)}{(\theta^2 + 8\theta + 3)^{3/2}} \tag{2.17}$$

and

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{3(5\theta^4 + 88\theta^3 + 310\theta^2 + 288\theta + 177)}{(\theta^2 + 8\theta + 3)^2}, \tag{2.18}$$

respectively.
The coefficients are increasing functions in θ (see Figure 2.2 for the graph of γ and $\sqrt{\beta_1}$ for varying θ).

The following points are noted from the current section.

(i) The mode of exponential distribution is always at 0 while the mode of xgamma can be varied as seen above. It is seen that if $X \sim xgamma(\theta)$, then $Mode(X) < Median(X) < Mean(X)$ which also holds good for exponential distribution.

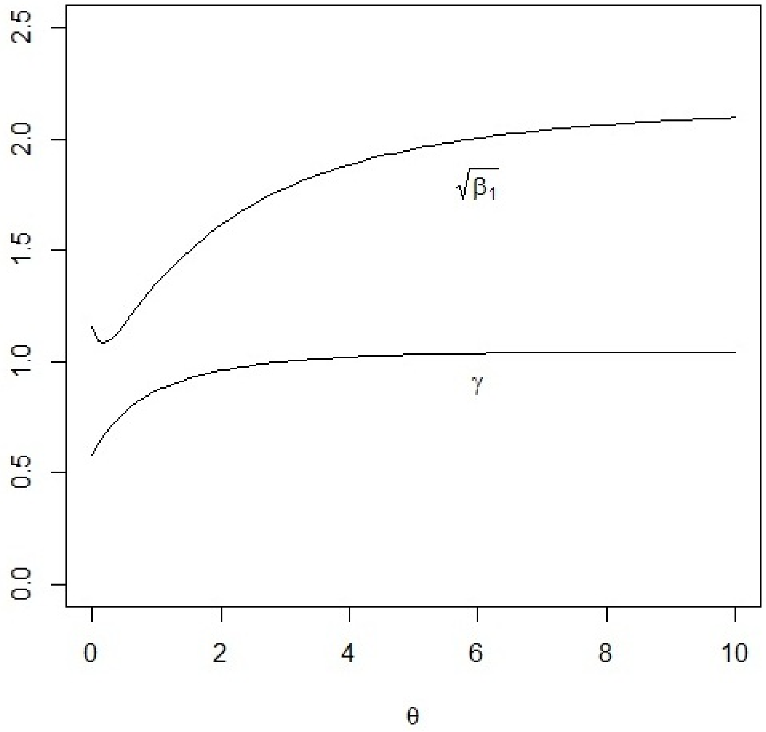

FIGURE 2.2: Plots for coefficients of variation and skewness

(ii) The values of γ, $\sqrt{\beta_1}$ and β_2 for exponential distribution are 1, 2 and 6, respectively. Hence, the xgamma distribution is more flexible than the exponential distribution in these aspects.

2.3 Survival properties

Among survival properties, hazard rate function, MRL function and stochastic order relations for $xgamma(\theta)$ are studied in this section.
For xgamma distribution, the survival function is given by

$$S(x) = \Pr(X > x) = \frac{(1 + \theta + \theta x + \frac{\theta^2 x^2}{2})}{(1 + \theta)} e^{-\theta x}. \tag{2.19}$$

2.3.1 Hazard rate or failure rate function

The hazard rate function (or failure rate function) for a continuous probability distribution with pdf $f(x)$, cdf $F(x)$ and survival function $S(x)$ is defined by

$$h(x) = \lim_{\triangle x \to 0} \frac{Pr(x < X < x + \triangle x | X > x)}{\triangle x} == \frac{f(x)}{S(x)}. \tag{2.20}$$

For xgamma distribution, the hazard rate (or failure rate) function is obtained as

$$h(x) = \frac{\theta^2(1 + \frac{\theta}{2}x^2)}{(1 + \theta + \theta x + \frac{\theta^2}{2}x^2)}, x > 0. \tag{2.21}$$

The hazard rate function in (2.21) possesses the following properties.

(i) $h(0) = \frac{\theta^2}{(1+\theta)} = f(0)$

(ii) $h(x)$ is an increasing function in $x > \sqrt{2/\theta}$ with $\theta^2/(1+\theta) < h(x) < \theta$.

Remark. For exponential distribution with parameter θ, $h(x) = \theta$ and so equation (2.21) shows flexibility of xgamma distribution over exponential distribution.

Figure 2.3 shows the hazard rate function of xgamma distribution for selected values of θ.

2.3.2 MRL function

As discussed in section 1.3.3 of Chapter 1, mean residual life (MRL) function is an important survival characteristic of a life distribution.

Using the equation in (1.17), i.e.,

$$m(x) = E[X - x | X > x] = \frac{1}{S(x)} \int_x^\infty S(t)dt,$$

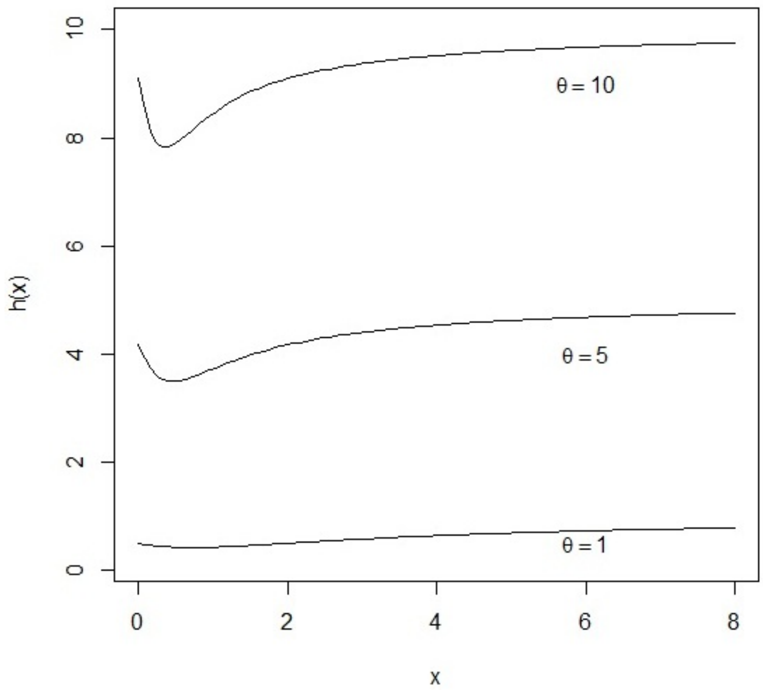

FIGURE 2.3: Hazard rate function plot of xgamma(θ) for some values of θ

for the xgamma distribution, the MRL function can be obtained as

$$m(x) = \frac{1}{(1+\theta)S(x)} \int_x^\infty \left(1 + \theta + \theta t + \frac{\theta^2 t^2}{2}\right) e^{-\theta t} dt,$$

Using (2.3), (2.4) and (2.5), we have

$$= \frac{1}{(1+\theta)S(x)} \left[(1+\theta)\int_x^\infty e^{-\theta t}dt + \theta \int_x^\infty te^{-\theta t}dt + \frac{\theta^2}{2}\int_x^\infty t^2 e^{-\theta t}dt\right],$$

$$= \frac{1}{(1+\theta+\theta x+\frac{\theta^2 x^2}{2})e^{-\theta x}} \left[\frac{3e^{-\theta x}}{\theta} + (2x+1)e^{-\theta x} + \frac{\theta x^2 e^{-\theta x}}{2}\right],$$

$$= \frac{1}{(1+\theta+\theta x+\frac{\theta^2 x^2}{2})} \left(\frac{3}{\theta} + 2x + 1 + \frac{\theta x^2}{2}\right),$$

$$= \frac{\left(1+\theta+\theta x+\frac{\theta^2 x^2}{2}\right) + (2+\theta x)}{\theta\left(1+\theta+\theta x+\frac{\theta^2 x^2}{2}\right)}.$$

Hence, the MRL function of $xgamma(\theta)$ is given by

$$m(x) = \frac{1}{\theta} + \frac{(2+\theta x)}{\theta(1+\theta+\theta x+\frac{\theta^2}{2}x^2)}. \tag{2.22}$$

The MRL function in (2.22) has the following properties.

(i) $m(0) = \mu = \frac{(\theta+3)}{\theta(1+\theta)}$.

(ii) $m(x)$ in decreasing in x and θ with $\frac{1}{\theta} < m(x) < \frac{(\theta+3)}{\theta(1+\theta)}$.

Note.

For the exponential distribution with parameter θ, MRL function is $1/\theta$ and hence equation (2.22) shows flexibility of xgamma distribution over the exponential distribution. Figure 2.4 shows the plot of MRL function of xgamma distribution for some values of θ.

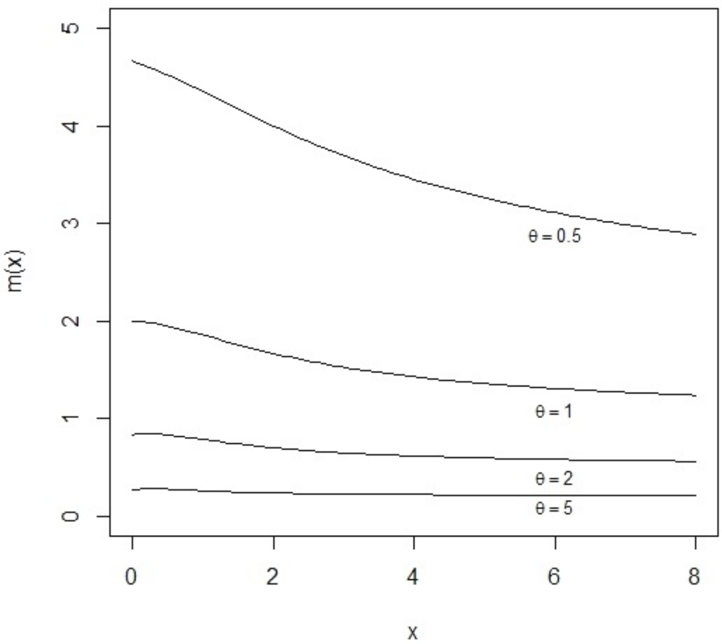

FIGURE 2.4: Plot for MRL function of xgamma(θ) for some values of θ

2.4 Stochastic ordering

Recall the basic definition on stochastic orderings given in the sub-section 1.3.5 of Chapter 1.

Definition 2.2. A continuous random variable X is said to be smaller than a random variable Y in the

(i) stochastic order ($X \leq_{ST} Y$) if $F_X(x) \geq F_Y(x)$ for all x.

(ii) hazard rate order ($X \leq_{HR} Y$) if $h_X(x) \geq h_Y(x)$ for all x.

(iii) mean residual life order ($X \leq_{MRL} Y$) if $m_X(x) \leq m_Y(x)$ for all x.

(iv) likelihood ratio order ($X \leq_{LR} Y$) if $\frac{f_X(x)}{f_Y(x)}$ decreases in x.

The following theorem shows that the xgamma random variables are ordered with respect to the strongest likelihood ratio ordering.

Theorem 2.3. *Let $X \sim$ xgamma(θ_1) and $Y \sim$ xgamma(θ_2). If $\theta_1 > \theta_2$ then $X \leq_{LR} Y$ and hence the other orderings.*

Proof. Note that,

$$\frac{f_X(x)}{f_Y(x)} = \frac{\theta_1^2(1+\theta_2)(2+\theta_1 x^2)}{\theta_2^2(1+\theta_1)(2+\theta_2 x^2)} e^{(\theta_2-\theta_1)x}.$$

Differentiating with respect to x, we have

$$\frac{d}{dx}\left(\frac{f_X(x)}{f_Y(x)}\right) = (\theta_2-\theta_1)\frac{\theta_1^2(1+\theta_2)}{\theta_2^2(1+\theta_1)} e^{(\theta_2-\theta_1)x}\left[\frac{(2+\theta_1 x^2)}{(2+\theta_2 x^2)} - \frac{4x}{(2+\theta_2 x^2)^2}\right],$$

which is negative for $\theta_1 > \theta_2$.

Hence,

$$\frac{f_X(x)}{f_Y(x)} \text{ decreases in } x \text{ and } X \leq_{LR} Y.$$

Now, by Shaked and Shanthikumar (1994), we have

$$X \leq_{LR} Y \Rightarrow X \leq_{HR} Y \Rightarrow X \leq_{MRL} Y \text{ and } X \leq_{HR} Y \Rightarrow X \leq_{ST} Y.$$

Hence the proof.

2.5 Parameter estimation

In this section method of moments and method of maximum likelihood are proposed for complete sample situation to estimate the unknown parameter θ in xgamma distribution.

2.5.1 Method of moments

Given a random sample $X_1, X_2, \ldots, X_n$ of size n from the xgamma distribution, the moment estimator for the parameter θ of xgamma distribution is obtained as follows.

We equate sample mean, $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ with first order moment about origin and we get

$$\bar{X} = \frac{(\theta+3)}{\theta(1+\theta)},$$

which provides a quadratic equation in θ as

$$\bar{X}\theta^2 + (\bar{X}-1)\theta - 3 = 0.$$

Solving it, we get the moment estimator, $\hat{\theta_M}$ (say), of θ as

$$\hat{\theta}_M = \frac{-(\bar{X}-1)+\sqrt{(\bar{X}-1)^2+12\bar{X}}}{2\bar{X}} \text{ for } \bar{X} > 0. \tag{2.23}$$

The following theorem shows that the moment estimator of θ is positively biased.

Theorem 2.4. *The moment estimator of xgamma distribution is positively biased, i.e.,* $E(\hat{\theta}_M) - \theta > 0$.

Proof. Let $\hat{\theta}_M = g(\bar{X})$ and $g(t) = \frac{-(t-1)+\sqrt{(t-1)^2+12t}}{2t}$. For $t > 0$, $g''(t) > 0$ and hence $g(t)$ is strictly convex.

Thus by the Jensen's inequality, we have

$$g[E(\bar{X})] < E[g(\bar{X})].$$

Now since

$$g[E(\bar{X})] = g(\mu) = g\left[\frac{(\theta+3)}{\theta(1+\theta)}\right] = \theta,$$

we have $E(\hat{\theta}_M) - \theta > 0$, and hence the proof.

It is noted that the sample raw moments are unbiased and consistent estimators of the corresponding population raw moments. They are also asymptotically normally distributed (CAN estimators) by the virtue of central limit theorem. Thus the moment estimator, $\hat{\theta}_M$, of θ for xgamma distribution is consistent, see Casella and Berger (2002).

2.5.2 Method of maximum likelihood

Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ be n observations on a random sample $X_1, X_2, \cdots, X_n$ of size n drawn from $xgamma(\theta)$. Then, the likelihood function is given by

$$L(\theta|\boldsymbol{x}) = \prod_{i=1}^{n} \frac{\theta^2}{(1+\theta)} \left(1 + \frac{\theta}{2} x_i^2\right) e^{-\theta x_i}. \tag{2.24}$$

The log-likelihood function is obtained as

$$l(\theta) = \ln L(\theta|\boldsymbol{x}) = 2n \ln\theta - n\ln(1+\theta) + \sum_{i=1}^{n} \ln\left(1 + \frac{\theta}{2} x_i^2\right) - \theta \sum_{i=1}^{n} x_i. \tag{2.25}$$

Taking the first (partial) derivative with respect to θ, we have the log-likelihood equation as

$$\frac{\partial \ln L(\theta|\boldsymbol{x})}{\partial\theta} = 0,$$

which implies

$$\sum_{i=1}^{n} \frac{x_i^2/2}{(1+\frac{\theta}{2}x_i^2)} + \frac{2n}{\theta} - \frac{n}{(1+\theta)} - \sum_{i=1}^{n} x_i = 0. \tag{2.26}$$

To obtain maximum likelihood estimator (MLE) of θ, $\hat{\theta}$(say), we can maximize (2.25) directly with respect to θ or can solve the non-linear equation, $\frac{\partial \ln L(\theta|\boldsymbol{x})}{\partial \theta} = 0$. It is seen that $\frac{\partial \ln L(\theta|\boldsymbol{x})}{\partial \theta} = 0$ cannot be solved analytically and hence numerical iteration technique, such as, *Newton-Raphson* algorithm, is applied to solve (2.26) for which (2.25) is maximized. The initial solution for such an iteration can be taken as $\theta_0 = \frac{n}{\sum_{i=1}^{n} x_i}$. Using this initial solution, we have,

$$\theta^{(i)} = \theta^{(i-1)} - \frac{l(\theta^{(i-1)}|\boldsymbol{x})}{l'(\theta^{(i-1)}|\boldsymbol{x})} \text{ for the } i^{\text{th}} \text{ iteration.}$$

We choose $\theta^{(i)}$ such that $\theta^{(i)} \cong \theta^{(i-1)}$.

Remark. Both the moment estimator and maximum likelihood estimator of exponential distribution is $\frac{1}{\bar{X}}$ which is also biased and consistent.

2.6 Simulation study

The inversion method for generating random data from the xgamma distribution fails because the equation $F(x) = u$, where u is an observation from the uniform distribution on $(0, 1)$, cannot be explicitly solved in x. However, the fact that the xgamma distribution is a special finite mixture of $exp(\theta)$ and $gamma(3, \theta)$ distributions can be used to construct a simulation algorithm.

To generate random data $X_i; i = 1, 2, \ldots, n$, from xgamma distribution with parameter θ, the following algorithm is proposed.

1. Generate $U_i \sim uniform(0, 1), i = 1, 2, \ldots, n$.

2. Generate $V_i \sim exp(\theta), i = 1, 2, \ldots, n$.

3. Generate $W_i \sim gamma(3, \theta), i = 1, 2, \ldots, n$.

4. If $U_i \leq \theta/(1+\theta)$, then set $X_i = V_i$, otherwise, set $X_i = W_i$.

A Monte-Carlo simulation study is carried out by considering $N = 10,000$ times for selected values of n and θ. Samples of sizes $20, 40$ and 100 are considered and values of θ are taken as $0.1, 0.5, 1.0, 1.5, 3$ and 6. The required numerical evaluations are carried out using R software. The following two measures are computed.

(i) Bias of the simulated estimates $\hat{\theta}_i, i = 1, 2, \ldots, N$:

$$\frac{1}{N}\sum_{i=1}^{N}(\hat{\theta}_i - \theta).$$

(ii) Mean Square Error (MSE) of the simulated estimates $\hat{\theta}_i, i = 1, 2, \ldots, N$:

$$\frac{1}{N}\sum_{i=1}^{N}(\hat{\theta}_i - \theta)^2.$$

The results of the simulation study are shown in Table 2.1. In Table 2.1, for each selected value of θ, the corresponding values relating to xgamma distribution are presented in first row and that relating to exponential distribution in second row. The following important observations are made from the simulation study.

(i) The bias is positive in case of xgamma distribution (as shown in the Theorem 2.4). Moreover, bias and MSE decreases as n increases and increases when θ increases.

(ii) In terms of bias and MSE, the parameter θ under the xgamma distribution is efficiently estimated compared to that of the exponential distribution.

Table 2.1: Simulation table showing average bias and MSE of the estimators

θ	Model	$n = 20$		$n = 40$		$n = 100$	
		Bias	MSE	Bias	MSE	Bias	MSE
0.1	Xgamma	0.00193	0.00020	0.00078	0.00009	0.00034	0.00004
	Exponential	-0.06375	0.00409	-0.06420	0.00414	-0.06438	0.00415
0.5	Xgamma	0.01182	0.00595	0.00539	0.00275	0.00199	0.00106
	Exponential	-0.27887	0.07934	-0.28258	0.08057	-0.28455	0.08125
1.0	Xgamma	0.02655	0.02750	0.01347	0.01290	0.00517	0.00499
	Exponential	-0.48093	0.24223	-0.49040	0.24558	-0.49631	0.24828
1.5	Xgamma	0.04411	0.07234	0.02804	0.03395	0.00871	0.01221
	Exponential	-0.63116	0.43490	-0.64478	0.43260	-0.65975	0.44133
3.0	Xgamma	0.12181	0.36497	0.05765	0.15694	0.02204	0.05985
	Exponential	-0.88938	1.04281	-0.94784	1.00708	-0.98001	1.0019
6.0	Xgamma	0.27864	1.75155	0.14144	0.78423	0.05511	0.28684
	Exponential	-1.06299	2.59379	-1.19641	2.09340	-1.28001	1.88262

2.7 Application

In this section, a real data set is analyzed as an illustration to show that the xgamma distribution can be a better model than one based on the exponential distribution.

Data on relief times (in hours) of 20 patients receiving an analgesic (Gross and Clark, 1975) are used for the purpose. Both the xgamma and exponential distributions are fitted to this data set. Maximum likelihood estimates (MLEs) for both the cases are calculated for the data. For model selection, negative log-likelihood value, Akaike information criterion (AIC), see Akaike (1974), consistent Akaike information criteria (cAIC) and Bayesian information criterion (BIC), see Schwarz (1978), are considered. We note,

$$AIC = -2\ln L + 2k, \tag{2.27}$$

$$cAIC = AIC + 2k(k+1)/(n-k-1), \tag{2.28}$$

$$BIC = k\ln(n) - 2\ln L, \tag{2.29}$$

where $\ln L$ denotes the log-likelihood function evaluated at the maximum likelihood estimate, k is the number of parameters and n is the sample size.

Lower the values of negative log-likelihood value, AIC, cAIC and BIC, better is the model. The required numerical evaluations are carried out using R software. Table 2.3 provides the MLEs with corresponding standard errors of estimates in parentheses and model selection criteria for the model parameters.

From Table 2.3, it is clear that the values of the AIC, cAIC and BIC are smaller for the xgamma distribution as compared to the exponential model and it follows that the xgamma distribution provides better fit to the data. So the new distribution seems to be a competitive model in describing time-to-event data set.

Table 2.2: Data on relief times of 20 patients receiving an analgesic.

1.1	1.4	1.3	1.7	1.9	1.8	1.6	2.2	1.7	2.7
4.1	1.8	1.5	1.2	1.4	3.0	1.7	2.3	1.6	2.0

Table 2.3: Estimates of parameters and model selection criteria for relief times data

Model	Estimate(Std. Error)	-Log-likelihood	AIC	cAIC	BIC
Exponential(θ)	$\hat{\theta}$=0.52632(0.11769)	32.84	67.67	67.90	68.67
Xgamma(θ)	$\hat{\theta}$=1.10747(0.16943)	31.51	65.02	65.24	66.01

2.8 Conclusion

The xgamma distribution, a special finite mixture of exponential distribution with parameter θ and gamma distribution with scale θ and shape 3, is introduced and studied in this chapter. Various mathematical and structural properties of the distribution are been studied including the shape, moments, measures of skewness and kurtosis. Important survival properties like, hazard rate function and MRL function are derived and their properties are been discussed.

The method of moments and method of maximum likelihood are been proposed for estimating the unknown parameter of xgamma distribution. In order to illustrate

the applicability of the xgamma distribution, a real lifetime data is analyzed and xgamma distribution is compared with exponential distribution.
The following important findings are observed in this chapter.

1. The one parameter xgamma distribution is unimodal and different values of the parameter regulates the shape of the distribution.

2. The hazard rate function is initially decreasing and then increasing and having added flexibility over the constant hazard rate of exponential distribution. Moreover, MRL function is decreasing for entire range of xgamma random variable.

3. xgamma random variables are ordered with strongest likelihood ratio ordering and thereby other stochastic orderings.

4. It is seen that the moment estimator of the parameter in xgamma distribution is positively biased.

5. Comprehensive algorithm for generating random samples from xgamma distribution is proposed. A simulation study confirms that estimates of the parameter behave satisfactorily for larger sample sizes.

6. Real lifetime data illustration shows that xgamma provides better fit to a lifetime data set as compared to the exponential distribution and has the potential as a competent life distribution for modeling time-to-event data sets.

Chapter 3

Survival estimation in xgamma distribution

In the Chapter 2, we have introduced the xgmma distribution and have studied some distributional and survival properties of it. We have also found that xgmma distribution has certain flexibility over exponential distribution and could be utilized as a potential life distribution in describing time-to-even data set.
There are many scenarios in life-testing and reliability experiments in which units are lost or removed from experimentation before failure. The loss may occur unintentionally, or it may have been designed so in the study. More often, however, the removal of units from experimentation is pre-planned and intentional, and is done so in order to free up testing facilities for other experimentation, to save time and cost, or to exploit the straightforward analysis that is termed as *progressive censoring* of the experimental units (see section 1.4 of Chapter 1 for discussion). In the present chapter, the objective is two-fold.

(i) Firstly, some further essential distributional properties, such as, characteristic and generating functions, important entropy measures, distributions of order statistics; and some additional survival and/or reliability properties, such as, mean time to failure, ageing intensity and stress-strength reliability, are studied for xgamma distribution.

(ii) Secondly, since progressively type-II right censored sampling scheme is more generalized and it includes complete sample and conventional type-II samples as special cases, the problem of classical as well as Bayesian estimation for the parameter of xgamma distribution and its important survival characteristics under progressively type-II right censoring scheme are been considered.

The rest of the chapter is organized as follows:
Some additional distributional and survival/reliability properties of $xgamma(\theta)$ are studied in section 3.1 and in its delegate subsections. Section 3.2 describes progressively type-II right censoring scheme and sample generation algorithm from such scheme. Maximum likelihood (ML) method of estimation and method of Bayesian estimation, considering progressively type-II right censoring scheme, of the parameter of xgamma and important survival characteristics are been described in section 3.3 and section 3.4, respectively. In section 3.5, a Monte-Carlo simulation study is carried out to compare the estimates described in the previous sections and results of the simulation study are depicted. Section 3.6 deals with a real data analysis for illustration of the methods and to establish the suitability of xgamma model among some standard lifetime models. Finally, the section 3.7 concludes by mentioning overall chapter findings and noting some open research problems for future study.

3.1 Additional properties of $xgamma(\theta)$

In this section some additional distributional and survival/reliability properties of xgamma distribution are been investigated.
It is noted that in a very close fashion with the construction of Lindley distribution

(Lindley, 1958), the xgamma distribution is a special finite mixture of the exponential distribution with mean $1/\theta$ and gamma distribution having scale parameter θ and shape=3, with mixing proportions $\theta/(1+\theta)$ and $1/(1+\theta)$, respectively.
The following theorem shows that xgamma random variate is stochastically larger than those of exponential and Lindley.

Theorem 3.1. *Let X_E, X_L and X_{XG} denote the exponential, the Lindley and the xgamma random variables with parameter θ, respectively. Then, $X_E <_{ST} X_L <_{ST} X_{XG}$.*

Proof. For two random variables X and Y, $X <_{ST} Y$ if $S_X(x) < S_Y(x)$ for all $x(> 0)$.
Here $S_X(t) = 1 - P(X \leq x) = 1 - F_X(x)$, the corresponding survival function.
Now, $S_{X_E}(x) = e^{-\theta x}$, $S_{X_L}(x) = \frac{(1+\theta+\theta x)}{(1+\theta)} e^{-\theta x}$ and $S_{X_{XG}}(x) = \frac{\left(1+\theta+\theta x+\frac{\theta^2}{2}x^2\right)}{(1+\theta)} e^{-\theta x}$.
For given θ, $S_{X_E}(x) < S_{X_L}(x) < S_{X_{XG}}(x)$ for all $x(> 0)$. Hence the proof.

3.1.1 Characteristic and generating functions

In this sub-section, we derive the characteristic function, moment generating function and cumulant generating function for xgamma distribution with parameter θ. The characteristic function of $X \sim xgamma(\theta)$ is derived as

$$\phi_X(t) = E(e^{itX}) = \int_0^\infty \frac{\theta^2}{(1+\theta)} \left(1 + \frac{\theta}{2}x^2\right) e^{-(\theta-it)x} dx,$$

$$= \frac{\theta^2}{(1+\theta)} \left[\int_0^\infty e^{-(\theta-it)x} dx + \frac{\theta}{2} \int_0^\infty x^2 e^{-(\theta-it)x} dx\right],$$

$$= \frac{\theta^2}{(1+\theta)} \left[\frac{1}{(\theta-it)} + \frac{\theta\Gamma(3)}{2(\theta-it)^3}\right],$$

Here $\Gamma(a) = \int_0^\infty z^{a-1} e^{-z} dz$ is the gamma function.

$$= \frac{\theta^2}{(1+\theta)} \left[\frac{1}{(\theta-it)} + \frac{\theta}{(\theta-it)^3}\right]; t \in \Re, i = \sqrt{-1}. \tag{3.1}$$

Next, we derive the moment generating function of $xgamma(\theta)$.

The moment generating function of X can be obtained as

$$\begin{aligned} M_X(t) = E(e^{tX}) &= \int_0^\infty \frac{\theta^2}{(1+\theta)}\left(1+\frac{\theta}{2}x^2\right)e^{-(\theta-t)x}dx, \\ &= \frac{\theta^2}{(1+\theta)}\left[\int_0^\infty e^{-(\theta-t)x}dx + \frac{\theta}{2}\int_0^\infty x^2 e^{-(\theta-t)x}dx\right], \\ &= \frac{\theta^2}{(1+\theta)}\left[\frac{1}{(\theta-t)} + \frac{\theta\Gamma(3)}{2(\theta-t)^3}\right], \\ &\text{Here } \Gamma(a) = \int_0^\infty z^{a-1}e^{-z}dz \text{ is the gamma function.} \\ &= \frac{\theta^2}{(1+\theta)}\left[\frac{1}{(\theta-t)} + \frac{\theta}{(\theta-t)^3}\right]; t \in \Re. \end{aligned} \tag{3.2}$$

The cumulant generating function of X is obtained by taking natural logarithm on $M_X(t)$ and is given by

$$\begin{aligned} K_X(t) &= \ln[M_X(t)], \\ &= \ln\frac{\theta^2}{(1+\theta)(\theta-t)} + \ln\left[1+\frac{\theta}{(\theta-t)^2}\right]; t \in \Re. \end{aligned} \tag{3.3}$$

3.1.2 Entropy measures

An entropy of a random variable X is a measure of variation of the uncertainty. A popular entropy measure is Rényi entropy, see 1.2.4 for more discussion on entropy measures. Now, if $X \sim xgamma(\theta)$, using (1.10) then we calculate,

$$\int_0^\infty f^\gamma(x)dx = \frac{\theta^{2\gamma}}{(1+\theta)^\gamma}\int_0^\infty \left(1+\frac{\theta}{2}x^2\right)^\gamma e^{-\gamma\theta x}dx \text{ for } \gamma > 0 (\neq 1).$$

Now, by power series expansion, we have,

$$\left(1+\frac{\theta}{2}x^2\right)^\gamma = \sum_{j=0}^{\gamma}\binom{\gamma}{j}\left(\frac{\theta x^2}{2}\right)^j.$$

Hence,

$$\int_0^\infty f^\gamma(x)dx = \frac{\theta^{2\gamma}}{(1+\theta)^\gamma}\sum_{j=0}^{\gamma}\binom{\gamma}{j}\left(\frac{\theta}{2}\right)^j\int_0^\infty x^{2j}e^{-\gamma\theta x}dx,$$

$$= \frac{\theta^{2\gamma}}{(1+\theta)^\gamma}\sum_{j=0}^{\gamma}\binom{\gamma}{j}\frac{\theta^j\Gamma(2j+1)}{2^j(\gamma\theta)^{2j+1}},$$

Here $\Gamma(a) = \int_0^\infty z^{a-1}e^{-z}dz$ is the gamma function.

$$= \frac{\theta^{2\gamma}}{(1+\theta)^\gamma}\sum_{j=0}^{\gamma}\binom{\gamma}{j}\frac{\Gamma(2j+1)}{2^j\theta^{j+1}\gamma^{2j+1}} \text{ for } \gamma > 0(\neq 1)$$

So, the final form of Rényi entropy is obtained as

$$H_R(\gamma) = \frac{1}{1-\gamma}\ln\left[\int_0^\infty f^\gamma(x)dx\right],$$

$$= \frac{1}{1-\gamma}\left[2\gamma\ln\theta - \gamma\ln(1+\theta)\right] + \frac{1}{1-\gamma}\ln\left[\sum_{j=0}^{\gamma}\binom{\gamma}{j}\frac{\Gamma(2j+1)}{2^j\theta^{j+1}\gamma^{2j+1}}\right]. \quad (3.4)$$

Tsallis entropy is a generalization of the standard Boltzmann–Gibbs entropy. Using (1.12), i.e.,

$$S_q(X) = \frac{1}{q-1}\ln\left[1 - \int_0^\infty f^q(x)dx\right] \text{ for } q > 0\,(\neq 1),$$

when $X \sim xgamma(\theta)$, we find $\int_0^\infty f^q(x)dx$. Hence,

$$\int_0^\infty f^q(x)dx = \frac{\theta^{2q}}{(1+\theta)^q}\int_0^\infty\left(1+\frac{\theta}{2}x^2\right)^q e^{-q\theta x}dx,$$

Using expansion $\left(1+\frac{\theta}{2}x^2\right)^q = \sum_{j=0}^{q}\binom{q}{j}\left(\frac{\theta x^2}{2}\right)^j$,

$$= \frac{\theta^{2q}}{(1+\theta)^q}\sum_{j=0}^{q}\binom{q}{j}\left(\frac{\theta}{2}\right)^j\int_0^\infty x^{2j}e^{-q\theta x}dx,$$

$$= \frac{\theta^{2q}}{(1+\theta)^q}\sum_{j=0}^{q}\binom{q}{j}\frac{\theta^j\Gamma(2j+1)}{2^j(q\theta)^{2j+1}},$$

Here $\Gamma(a) = \int_0^\infty z^{a-1}e^{-z}dz$ is the gamma function.

So, the final form of Tsallis entropy is given by

$$S_q(X) = \frac{1}{q-1} \ln \left[1 - \frac{\theta^{2q}}{(1+\theta)^q} \sum_{j=0}^{q} \binom{q}{j} \frac{\Gamma(2j+1)}{2^j \theta^{j+1} \gamma^{2j+1}} \right] \text{ for } q > 0 (\neq 1). \tag{3.5}$$

Now, we find Shannon measure of entropy.

Using (1.11), i.e.,

$$H(f) = E[-\ln f(x)] = -\int_0^\infty \ln f(x) f(x) dx,$$

Shannon measure of entropy can be calculated. For $xgamma(\theta)$, Shannon measure of entropy can be derived as below.

$$\begin{aligned} H(f) &= -\int_0^\infty \ln f(x) f(x) dx, \\ &= -\int_0^\infty \ln \left[\frac{\theta^2}{(1+\theta)} \left(1 + \frac{\theta}{2} x^2\right) e^{-\theta x} \right] f(x) dx, \\ &= -\left[\int_0^\infty \ln \frac{\theta^2}{(1+\theta)} f(x) dx + \int_0^\infty \ln \left(1 + \frac{\theta}{2} x^2\right) f(x) dx - \theta \int_0^\infty x f(x) dx \right], \\ &= -\left[\ln \frac{\theta^2}{(1+\theta)} + \frac{\theta^2}{(1+\theta)} \int_0^\infty \ln \left(1 + \frac{\theta}{2} x^2\right) \left(1 + \frac{\theta}{2} x^2\right) e^{-\theta x} dx - \frac{(\theta+3)}{(1+\theta)} \right]. \end{aligned} \tag{3.6}$$

Now,

$$\begin{aligned} &\frac{\theta^2}{(1+\theta)} \int_0^\infty \ln \left(1 + \frac{\theta}{2} x^2\right) \left(1 + \frac{\theta}{2} x^2\right) e^{-\theta x} dx \\ &= \frac{\theta^2}{(1+\theta)} \int_0^\infty \ln \left(1 + \frac{\theta}{2} x^2\right) e^{-\theta x} dx + \frac{\theta^2}{(1+\theta)} \frac{\theta}{2} \int_0^\infty \ln \left(1 + \frac{\theta}{2} x^2\right) x^2 e^{-\theta x} dx. \end{aligned}$$

Putting the expansion,

$$\ln \left(1 + \frac{\theta}{2} x^2\right) = \sum_{j=1}^{\infty} (-1)^{j+1} \frac{(\frac{\theta}{2} x^2)^j}{j},$$

we have,

$$\frac{\theta^2}{(1+\theta)}\int_0^\infty \ln\left(1+\frac{\theta}{2}x^2\right)\left(1+\frac{\theta}{2}x^2\right)e^{-\theta x}dx$$
$$=\frac{\theta^2}{(1+\theta)}\sum_{j=1}^{\infty}\frac{(-1)^{j+1}\theta^j}{j2^j}\int_0^\infty x^{2j}e^{-\theta x}dx+\frac{\theta^3}{2(1+\theta)}\sum_{j=1}^{\infty}\frac{(-1)^{j+1}\theta^j}{j2^j}\int_0^\infty x^{2j+2}e^{-\theta x}dx,$$
$$=\frac{\theta^2}{(1+\theta)}\sum_{j=1}^{\infty}\frac{(-1)^{j+1}\theta^j}{j2^j}\frac{\Gamma(2j+1)}{\theta^{2j+1}}+\frac{\theta^3}{2(1+\theta)}\sum_{j=1}^{\infty}\frac{(-1)^{j+1}\theta^j}{j2^j}\frac{\Gamma(2j+3)}{\theta^{2j+3}}.$$

Here $\Gamma(a)=\int_0^\infty z^{a-1}e^{-z}dz$ is the gamma function.

Now, from (3.6), with small arrangement, the final form of Shannon entropy is given by

$$H(f)=\frac{(3+\theta)}{(1+\theta)}-\ln\frac{\theta^2}{(1+\theta)}-\frac{1}{(1+\theta)}\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j\theta^{2j}2^j}\left[\theta\Gamma(2j+1)+\frac{1}{2}\Gamma(2j+3)\right]. \tag{3.7}$$

3.1.3 Distributions of order statistics

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n drawn from $xgamma(\theta)$ and $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ be the order statistics.

The extreme order statistics, $X_{1:n}$ and $X_{n:n}$ represent the life of series and parallel systems and have important applications in probability and statistics.

The pdf of smallest order statistic, $X_{1:n} = Min\{X_1, X_2, \ldots, X_n\}$, is obtained as

$$f_{X_{1:n}}(x)=n[1-F(x)]^{n-1}f(x),$$
$$=\frac{n\theta^2}{(1+\theta)^n}\left(1+\theta+\theta x+\frac{\theta^2}{2}x^2\right)^{n-1}\left(1+\frac{\theta}{2}x^2\right)e^{-n\theta x}, x>0. \tag{3.8}$$

The pdf of largest order statistic, $X_{n:n} = Max\{X_1, X_2, \ldots, X_n\}$, is found as

$$\begin{aligned} f_{X_{n:n}}(x) &= n[F(x)]^{n-1} f(x), \\ &= \frac{n\theta^2}{(1+\theta)} \left[1 - \frac{\left(1+\theta+\theta x+\frac{\theta^2}{2}x^2\right)}{(1+\theta)} e^{-\theta x}\right]^{n-1} \left(1+\frac{\theta}{2}x^2\right) e^{-\theta x}, x > 0. \end{aligned} \tag{3.9}$$

3.1.4 Survival characteristics

An important phenomena in reliability theory is 'ageing' which is an inherent property of a unit (be it a living organism or a mechanical system of components). Ageing is usually characterized by the failure rate function, MRL function, mean time to failure (MTTF), etc.

For any $t(> 0)$, we recall the survival function (sf) of $X \sim xgamma(\theta)$ is given by

$$S(t) = \Pr(X > t) = 1 - F(t) = \frac{\left(1+\theta+\theta t+\frac{\theta^2}{2}t^2\right)}{(1+\theta)} e^{-\theta t}. \tag{3.10}$$

For any $t(> 0)$, the failure rate function of $xgamma(\theta)$ is

$$h(t) = \frac{\theta^2 \left(1+\frac{\theta}{2}t^2\right)}{\left(1+\theta+\theta t+\frac{\theta^2}{2}t^2\right)}. \tag{3.11}$$

The MTTF is given as

$$MTTF = \int_0^\infty S(t)dt = \frac{(\theta+3)}{\theta(1+\theta)}. \tag{3.12}$$

The following theorem shows that xgamma distribution is decreasing failure rate (DFR) in distribution for a particular range of t.

Theorem 3.2. *The cdf of xgamma distribution in (2.6) is DFR for* $t < \sqrt{2/\theta}$.

Proof. We state the Lemma 5.9 by Barlow and Proschan (1975).

Lemma 3.3. *If* f *is a density on* $[0, \infty)$ *for which* $\ln f(x)$ *is convex on* $[0, \infty)$, *then corresponding distribution function* F *is DFR.*

Now, we show that the logarithm of the pdf in (2.2) is convex and try to find out the range. For $xgamma(\theta)$, $\ln f(t)$ is defined and is twice differentiable with respect to t in $(0, \infty)$. We have,

$$\ln f(t) = 2\ln\theta - \ln(1+\theta) + \ln\left(1 + \frac{\theta}{2}t^2\right) - \theta t. \tag{3.13}$$

The second derivative of (t) with respect to t is given by

$$\frac{d^2}{dt^2}\ln f(t) = \frac{\theta\left(1 - \frac{\theta}{2}t^2\right)}{\left(1 + \frac{\theta}{2}t^2\right)^2},$$

which is positive when $t < \sqrt{2/\theta}$ and so, $\ln f(t)$ is convex for $t < \sqrt{2/\theta}$ and thereby, corresponding cdf is DFR for $t < \sqrt{2/\theta}$. Hence the proof.

It is noted that the xgamma distribution is increasing failure rate (IFR) in distribution for $t > \sqrt{2/\theta}$.

3.1.5 Ageing intensity

It is clear that a unimodal failure rate can be effectively viewed as either approximately decreasing or approximately increasing or approximately constant. So, the representation of ageing of a system by failure rate is qualitative and, therefore, a new notion, called ageing intensity (AI) has been developed by Jiang et al. (2003).

Ageing intensity of a random variable X, denoted by $L_X(t)$, quantitatively evaluates the ageing property of a system. It is defined as

$$L_X(t) = \frac{h(t)}{Z(t)}, \text{where defined,} \tag{3.14}$$

$$= \frac{-tf(t)}{S(t)\ln S(t)} \text{ for } t > 0, \tag{3.15}$$

where $f(\cdot)$ and $S(\cdot)$ are respectively the pdf and survival function of the random variable X. $Z(t)$ is the failure rate average, i.e., $Z(t) = \frac{1}{t}\int_0^t h(u)du$.
The larger the value of AI function, stronger the tendency of ageing of the associated random variable. Also, the failure rate function uniquely determines the AI function, but not conversely.
Different properties of AI function are extensively studied by Nanda et al. (2007) and reliability analysis using AI function is studied by Bhattacharjee et al. (2013). The following definition on ageing intensity is noted.

Definition 3.4. A random variable X is said to be increasing (decreasing) in ageing intensity or IAI (DAI) if the corresponding AI function $L_X(t)$ is increasing (decreasing) in $t(> 0)$.

When $X \sim xgamma(\theta)$, we derive the AI function as given below.
We have,

$$\ln S(t) = \ln\left(1 + \theta + \theta t + \frac{\theta^2}{2}t^2\right) - \theta t - \ln(1 + \theta).$$

From (3.15), we have then,

$$\begin{aligned} L_X(t) &= \frac{-t\frac{\theta^2}{(1+\theta)}\left(1 + \frac{\theta}{2}t^2\right)e^{-\theta t}}{\left[\frac{\left(1+\theta+\theta t+\frac{\theta^2}{2}t^2\right)}{(1+\theta)}e^{-\theta t}\right]\ln S(t)}, \\ &= \frac{-t\theta^2\left(1 + \frac{\theta}{2}t^2\right)}{\left(1 + \theta + \theta t + \frac{\theta^2}{2}t^2\right)\left[\ln\left(1 + \theta + \theta t + \frac{\theta^2}{2}t^2\right) - \theta t - \ln(1 + \theta)\right]}. \end{aligned}$$

Hence, the AI function for $xgamma(\theta)$ is given by

$$L_X(t) = \frac{\theta^2 t(1 + \frac{\theta}{2}t^2)}{(1 + \theta + \theta t + \frac{\theta^2 t^2}{2})[\ln(1 + \theta) + \theta t - \ln(1 + \theta + \theta t + \frac{\theta^2 t^2}{2})]}, t > 0. \quad (3.16)$$

It is been observed that, for $t > 0$, $L_X(t)$ in (3.16) is initially decreasing then increasing and again decreasing (see Figure 3.1) depending on the values of θ. The optimum value (a maximum or a minimum) of the AI function for $xgamma(\theta)$ is inversely proportional with θ.

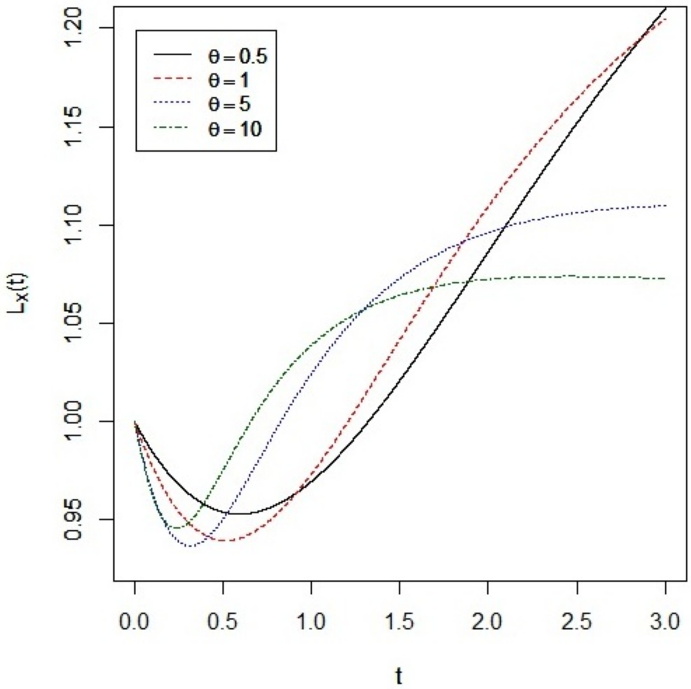

FIGURE 3.1: Plot of AI function of xgamma distribution for different θ.

3.1.6 Stress-strength reliability

The stress-strength model describes the life of a component which has a random strength Y that is subjected to a random stress X. The component fails at the instant that the stress applied to it exceeds the strength, and the component will function satisfactorily whenever $Y > X$. So, stress-strength reliability is defined as

$$R = \Pr(Y > X) = \int_0^\infty S_Y(x) f_X(x) dx. \tag{3.17}$$

Let $X \sim xgamma(\theta_1)$ and $Y \sim xgamma(\theta_2)$ be independent random variables. Then stress-strength reliability is obtained as below.

$$\begin{aligned}
&\int_0^\infty S_Y(x) f_X(x) dx, \\
&= \frac{\theta_1^2}{(1+\theta_1)(1+\theta_2)} \int_0^\infty \left(1 + \frac{\theta_1}{2}x^2\right)\left(1 + \theta + \theta x + \frac{\theta^2}{2}x^2\right) e^{-(\theta_1+\theta_2)x} dx, \\
&= \frac{\theta_1^2}{(1+\theta_1)(1+\theta_2)} \int_0^\infty \left(1 + \theta + \theta x + \frac{\theta^2}{2}x^2\right) e^{-(\theta_1+\theta_2)x} dx \\
&+ \frac{\theta_1^3}{2(1+\theta_1)(1+\theta_2)} \int_0^\infty x^2 \left(1 + \theta + \theta x + \frac{\theta^2}{2}x^2\right) e^{-(\theta_1+\theta_2)x} dx.
\end{aligned} \tag{3.18}$$

We have the expressions for the integrals,

$$\int_0^\infty \left(1 + \theta + \theta x + \frac{\theta^2}{2}x^2\right) e^{-(\theta_1+\theta_2)x} dx = \frac{(1+\theta_2)}{(\theta_1+\theta_2)} + \frac{\theta_2}{(\theta_1+\theta_2)^2} + \frac{\theta_2^2}{(\theta_1+\theta_2)^3}. \tag{3.19}$$

$$\int_0^\infty x^2 \left(1 + \theta + \theta x + \frac{\theta^2}{2}x^2\right) e^{-(\theta_1+\theta_2)x} dx = \frac{2(1+\theta_2)}{(\theta_1+\theta_2)^3} + \frac{6\theta_2}{(\theta_1+\theta_2)^4} + \frac{12\theta_2^2}{(\theta_1+\theta_2)^5}. \tag{3.20}$$

Hence, using (3.19) and (3.20), from (3.18), we get on simplification,

$$\begin{aligned}
R = &\frac{\theta_1^2}{(1+\theta_1)(1+\theta_2)} \left[\frac{1+\theta_2}{\theta_1+\theta_2} + \frac{\theta_2}{(\theta_1+\theta_2)^2} + \frac{\theta_2^2 + \theta_1(1+\theta_2)}{(\theta_1+\theta_2)^3} + \frac{3\theta_1\theta_2}{(\theta_1+\theta_2)^4}\right] \\
&+ \frac{6\theta_1^3\theta_2^2}{(1+\theta_1)(1+\theta_2)(\theta_1+\theta_2)^5}.
\end{aligned} \tag{3.21}$$

We note that, when $\theta_1 = \theta_2 = \theta$, i.e., when stress and strength are i.i.d $xgamma(\theta)$, then $R = \frac{1}{2}$.

Now, estimation aspects (classical as well as Bayesian) of the unknown parameter and survival characteristics (vide subsection 3.1.4) of xgamma distribution under progressively type-II censored scheme are studied in the consecutive sections below.

3.2 Progressively type-II right censored scheme

As discussed in section 1.4 in Chapter 1, progressive type-II right censoring scheme is more generalized censoring scheme that includes conventional type-II censoring and complete sample (non-censored) as special cases. We describe below the operational process of progressively type-II censored scheme.

Suppose, n identical items/units/subjects are put on a life testing experiment and progressively type-II censoring scheme $\underline{R} = (R_1, R_2, \ldots, R_m)$ is pre-fixed such that after the first failure R_1 surviving units are removed from the remaining $(n-1)$ live units and after second failure R_2 surviving units are removed from the remaining $(n - R_1 - 2)$ live units, and so on. This procedure is continued all R_m remaining items are removed after the m^{th} failure.
It is clear that $\sum_{i=1}^{m} R_i + m = n$. It is to be noted that, if $R_1 = R_2 = \cdots = R_m = 0$ then the progressive censoring scheme is reduced to complete sampling scheme and if $R_1 = R_2 = \cdots = R_{m-1} = 0$ and $R_m = n - m$ then the scheme reduces to conventional type-II censoring.

Next, an algorithm is described on generating random sample from xgamma distribution under progressively type-II right censoring scheme.

3.2.1 Generation of random sample

To generate a progressive type-II right censored sample from xgamma distribution, the algorithm given in Balakrishnan and Sandhu (1995) is utilized. The following steps are followed for sample generation.

(i) Generate m independent $uniform(0,1)$ observations $w_1, w_2, \ldots, w_m$.

(ii) Set $v_i = w_i^{1/\left(1+\sum_{j=m-i+1}^{m} R_j\right)}$ for $i = 1, 2, \ldots, m$.

(iii) Set $u_i = 1 - v_m.v_{m-1}.\ldots.v_{m-i+1}$ for $i = 1, 2, \ldots, m$. Then $u_1, u_2, \ldots, u_m$ is progressively type-II sample from the $uniform(0,1)$ distribution.

(iv) Now, setting $x_i = F^{-1}(u_i)$, where $F(\cdot)$ is the cdf of $xgamma(\theta)$ given in equation (2.6), where x_i can be obtained by solving the non-linear equation $1 - u_i - \frac{\left(1+\theta+\theta x_i+\frac{\theta^2}{2}x_i^2\right)}{(1+\theta)} e^{-\theta x_i} = 0$.
Then we have, $(x_1, x_2, \ldots, x_m)$ as a required progressively type-II right censored sample from $xgamma(\theta)$ with censoring scheme $\underline{R} = (R_1, R_2, \ldots, R_m)$.

3.3 Maximum likelihood estimation

In this section we obtain the maximum likelihood estimators of the parameter and survival characteristics of the xgamma distribution considering progressively type-II censoring scheme.
Let $\tilde{x} = (x_1, x_2, \ldots, x_m)$ be a progressively type-II right censoring sample from $xgamma(\theta)$ with progressive censoring scheme $\underline{R}$. On the basis of progressively type-II right censored sample, using (2.2) and (2.19), the likelihood function can be obtained as

$$\begin{aligned} L(\theta, \tilde{x}) &= C \prod_{i=1}^{m} f(x_i)\{S(x_i)\}^{R_i}, \\ &= C \prod_{i=1}^{m} \left[\frac{\theta^2}{(1+\theta)} \left(1 + \frac{\theta}{2} x_i^2\right) e^{-\theta x_i} \left\{ \frac{\left(1+\theta+\theta x_i + \frac{\theta^2}{2} x_i^2\right)}{(1+\theta)} e^{-\theta x_i} \right\}^{R_i} \right], \end{aligned} \tag{3.22}$$

where $C = n(n - R_1 - 1)(n - R_1 - R_2 - 2)\ldots(n - R_1 - R_2 - \cdots - R_m - m + 1)$.

3.3.1 Procedure of estimation

The likelihood equation in (3.22) can be written as

$$L(\theta,\tilde{x}) = C\frac{\theta^{2m}}{(1+\theta)^m}\prod_{i=1}^{m}\left(1+\frac{\theta}{2}x_i^2\right)e^{-\theta(1+R_i)x_i}\left[\frac{\left(1+\theta+\theta x_i+\frac{\theta^2}{2}x_i^2\right)}{(1+\theta)}\right]^{R_i}. \tag{3.23}$$

hence, the log-likelihood function becomes

$$\begin{aligned}\ln L(\theta,\tilde{x}) = \ln C + 2m\ln\theta - m\ln(1+\theta) - \theta\sum_{i=1}^{m}(1+R_i)x_i + \sum_{i=1}^{m}\ln\left(1+\frac{\theta}{2}x_i^2\right)\\ + \sum_{i=1}^{m}R_i\ln\frac{\left(1+\theta+\theta x_i+\frac{\theta^2}{2}x_i^2\right)}{(1+\theta)}.\end{aligned} \tag{3.24}$$

The first derivative of (3.24) with respect to θ gives

$$\begin{aligned}\frac{\partial}{\partial\theta}\ln L(\theta,\tilde{x}) = \frac{2m}{\theta} - \frac{m}{(1+\theta)} + \sum_{i=1}^{m}\frac{x_i^2/2}{(1+\frac{\theta}{2}x_i^2)} - \sum_{i=1}^{m}(1+R_i)x_i\\ + \sum_{i=1}^{m}R_i x_i\frac{(1+\theta x_i+\frac{\theta^2}{2}x_i)}{(1+\theta)(1+\theta+\theta x_i+\frac{\theta^2}{2}x_i^2)}.\end{aligned} \tag{3.25}$$

The second derivative of (3.24) with respect to θ gives

$$\begin{aligned}\frac{\partial^2}{\partial\theta^2}\ln L(\theta,\tilde{x}) = -\frac{2m}{\theta^2} + \frac{m}{(1+\theta)^2} - \sum_{i=1}^{m}\frac{x_i^4/4}{\left(1+\frac{\theta}{2}x_i^2\right)^2}\\ - \sum_{i=1}^{m}R_i x_i\frac{(1+2\theta+\theta^2/2)\frac{x_i^3\theta^2}{2} + (2\theta+1)(1+\theta x_i^2) + (1+\theta x_i)}{(1+\theta)^2(1+\theta+\theta x_i+\frac{\theta^2}{2}x_i^2)^2}.\end{aligned} \tag{3.26}$$

3.3.2 MLEs of θ and survival characteristics

The maximum likelihood estimator (MLE) of θ is the solution of the likelihood equation

$$\frac{\partial}{\partial\theta}\ln L(\theta,\tilde{x})=0,$$

which implies, using (3.25)

$$\frac{2m}{\theta}-\frac{m}{(1+\theta)}+\sum_{i=1}^{m}\frac{x_i^2/2}{\left(1+\frac{\theta}{2}x_i^2\right)}-\sum_{i=1}^{m}(1+R_i)x_i + \sum_{i=1}^{m}R_i x_i\frac{(1+\theta x_i+\frac{\theta^2}{2}x_i)}{(1+\theta)(1+\theta+\theta x_i+\frac{\theta^2}{2}x_i^2)}=0 \quad (3.27)$$

The MLE of θ can not be obtained from equation (3.27) by direct analytic solution, so for any censoring scheme, we use numerical procedure, such as, *Newton-Raphson* for given values of $(n,m,\underline{R},\tilde{x})$.

Let the MLE of θ thus obtained be $\hat{\theta}$. Once $\hat{\theta}$ is obtained, the MLEs of $S(t)$, $h(t)$ and $MTTF$ can be derived utilizing the invariance property of maximum likelihood and are given by

$$\widehat{S(t)}=\frac{(1+\hat{\theta}+\hat{\theta}t+\frac{\hat{\theta}^2}{2}t^2)}{(1+\hat{\theta})}e^{-\hat{\theta}t}, \quad (3.28)$$

$$\widehat{h(t)}=\frac{\hat{\theta}^2\left(1+\frac{\hat{\theta}}{2}t^2\right)}{\left(1+\hat{\theta}+\hat{\theta}t+\frac{\hat{\theta}^2}{2}t^2\right)} \quad (3.29)$$

and

$$\widehat{MTTF}=\frac{\hat{\theta}+3}{\hat{\theta}(1+\hat{\theta})}, \quad (3.30)$$

respectively.

3.3.3 Observed Fisher's information

The observed Fisher's information is given by

$$I(\hat{\theta}) = -\frac{\partial^2}{\partial\theta^2} \ln L(\theta, \tilde{x})|_{(\theta=\hat{\theta})}. \tag{3.31}$$

The asymptotic variance of $\hat{\theta}$ is

$$Var(\hat{\theta}) = \frac{1}{I(\hat{\theta})}.$$

We note that the $xgamma(\theta)$ belongs to one parameter exponential family of distributions. Hence, the sampling distribution of $(\theta - \hat{\theta})/\sqrt{Var(\hat{\theta})}$ is approximately $N(0, 1)$.

Hence, for large sample, $100(1-\alpha)\%$ confidence interval for θ can be obtained as

$$\hat{\theta} \pm z_{\alpha/2}\sqrt{Var(\hat{\theta})}. \tag{3.32}$$

We also estimate the coverage probability as

$$Pr\left[\left|\frac{(\theta-\hat{\theta})}{\sqrt{Var(\hat{\theta})}}\right| \le z_{\alpha/2}\right], \tag{3.33}$$

where z_p is such that $p = \int_{z_p}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$.

3.4 Bayesian estimation

In this section we obtain and discuss about the Bayesian estimation for the parameter θ and survival characteristics, namely, survival function, hazard rate function and HTTF, of $xgamma(\theta)$ considering progressively type-II censored sample.

We suppose that the prior distribution of the unknown parameter θ is a two parameter gamma distribution with shape parameter β and scale parameter α. So, the prior pdf of θ is given by

$$g(\theta) = \frac{\alpha^\beta}{\Gamma(\beta)} e^{-\alpha\theta} \theta^{\beta-1}, \theta > 0, \alpha > 0, \beta > 0. \tag{3.34}$$

Observing progressively type-II right censoring data and using (3.22), the posterior distribution of θ is given by

$$\Pi(\theta|\tilde{x}) = \frac{L(\tilde{x}, \theta)g(\theta)}{\displaystyle\int_\theta L(\tilde{x}, \theta)g(\theta)d\theta}. \tag{3.35}$$

Now, using (3.22) and (3.35), the posterior distribution of θ is obtained as

$$\Pi(\theta|\tilde{x}) = C_1^{-1} \frac{\theta^{2m+\beta-1}}{(1+\theta)^m} e^{-\theta[\alpha + \sum_{i=1}^m (1+R_i)x_i]} \prod_{i=1}^m \left(1 + \frac{\theta}{2} x_i^2\right) \prod_{i=1}^m \left\{\frac{(1+\theta+\theta x_i + \theta^2 x_i^2/2)}{1+\theta}\right\}^{R_i}, \tag{3.36}$$

where

$$C_1 = \int_0^\infty \frac{\theta^{2m+\beta-1}}{(1+\theta)^m} e^{-\theta[\alpha + \sum_{i=1}^m (1+R_i)x_i]} \prod_{i=1}^m \left(1 + \frac{\theta}{2} x_i^2\right) \prod_{i=1}^m \left\{\frac{(1+\theta+\theta x_i + \theta^2 x_i^2/2)}{1+\theta}\right\}^{R_i} d\theta.$$

Although C_1 is not in a closed form, we can evaluate it numerically for the given values of $(\alpha, \beta, n, m, \underline{R}, \tilde{x})$.

We choose the squared error loss function (SELF) considering that the decisions become gradually more damaging for large errors. So, we consider SELF, $L(\theta^*, \theta) = (\theta^* - \theta)^2$, where θ^* is the Bayes estimator of θ.

The Bayes estimator of θ, θ^*, is the posterior mean and is given by

$$\theta^* = E(\theta|\tilde{x}) = C_1^{-1} \int_0^\infty \frac{\theta^{2m+\beta}}{(1+\theta)^m} e^{-\theta[\alpha+\sum_{i=1}^m (1+R_i)x_i]} \prod_{i=1}^m \left(1 + \frac{\theta}{2} x_i^2\right) \prod_{i=1}^m \left\{\frac{(1+\theta+\theta x_i + \theta^2 x_i^2/2)}{1+\theta}\right\}^{R_i} d\theta. \quad (3.37)$$

It is clear that the above estimator is not in closed form, but we can always evaluate it numerically using given values of $(\alpha, \beta, n, m, \underline{R}, \tilde{x})$ and a given mission time $t(> 0)$.

The respective Bayes estimators, $S^*(t), h^*(t)$ and $MTTF^*$ can directly be obtained by using (3.10), (3.11) and (3.12) and replacing θ by θ^*.

3.5 Simulation study

A Monte-Carlo simulation study is carried out to obtain the estimates of the parameter and survival characteristics of $xgamma(\theta)$ developed in the previous sections. Considering progressive type-II censored sample, MLEs and Bayes estimates are obtained for two sets of values for θ, α and β.
Three different sample sizes are considered, i.e., $n = 20$, $n = 30$ and $n = 50$ are taken for the simulation study.
The following steps are included in the simulation study.

1. Choose the values of (α, β, t, n, m) and the censoring scheme $\underline{R}$.

2. Take $\theta = \beta/\alpha$ (the mean of the prior distribution of θ).

3. Compute the actual values of $S(t), h(t)$ and $MTTF$.

4. Generate a progressively type-II censored sample of size n with m failures using the algorithm in sub-section 3.2.1. For each n, 4 values of m is considered

so that the percentage of failure information, $(m/n)100$, is $40\%, 50\%, 80\%$ and 100%.

The scheme with $n = m$ and $R_i = 0, \quad \forall i = 1, 2, \ldots, m$ denotes complete sample. The scheme with $R_i = 0, \forall\, i = 1, 2, \ldots, m-1; R_m = n - m$ denotes conventional type-II censored sample (i.e., $n - m$ units are removed at the m^{th} failure).

5. Compute maximum likelihood and Bayes estimates of $\theta, S(t), h(t), MTTF$ according to section 3.3.2 and section 3.4. Also, compute the confidence interval and corresponding probabilities of coverage for θ as in section 3.3.3.

6. Repeat the above steps (1 to 4) for $N = 10,000$ times for the values of of $(\theta, \alpha, \beta, t)$ with each censoring scheme.

   Compute the expected value (EV) and expected risk (ER) of the estimates obtained in the above step 4 using the formulas

$$EV = \frac{1}{N} \sum \hat{\phi}(\theta) \quad \text{and} \quad ER = \frac{1}{N} \sum (\hat{\phi}(\theta) - \phi(\theta))^2,$$

   where $\hat{\phi}(\theta)$ is an estimate of $\phi(\theta)$. ER is the mean square error (MSE) in case of MLEs.

The results of the Monte-Carlo simulation study are reported in Tables 3.1-3.4. Table 3.1 represents MLEs, mean square errors (MSE) of the parameter and survival characteristics for different censoring schemes (CS), $\theta = 1$ and a given mission time $t = 0.80$; whereas the respective Bayes estimates along with expected risk (ER) are presented in Table 3.2.

Table 3.3 represents MLEs, mean square errors (MSE) of the parameter and survival characteristics for different censoring schemes (CS), $\theta = 2$ and a given mission time $t = 0.40$; whereas the respective Bayes estimates along with expected risk (ER) are presented in Table 3.4.

TABLE 3.1: MLEs when $\theta = 1.0, t = 0.80, S(t) = 0.7010, h(t) = 0.4231, MTTF = 2.00, N = 10,000.$

n	m	CS	$\hat{\theta}$		CP	$\widehat{S(t)}$		$\widehat{h(t)}$		$\widehat{HTTF}$	
			EV	MSE		EV	MSE	EV	MSE	EV	MSE
20	8	(12,0*7)	1.0794	0.0882	0.97	0.6765	0.0097	0.4798	0.0384	1.9799	0.3586
20	8	(4*3,0*5)	1.0630	0.0972	0.96	0.6829	0.0100	0.4700	0.0438	2.0274	0.3665
20	8	(0*5,4*3)	0.9793	0.0830	0.89	0.7120	0.0095	0.4180	0.0344	2.2606	0.5628
20	8	(0*7,12)	0.9866	0.0789	0.91	0.7093	0.0088	0.4220	0.0333	2.2239	0.4954
20	10	(10,0*9)	1.0700	0.0743	0.95	0.6792	0.0081	0.4728	0.0325	1.9725	0.2891
20	10	(5*2,0*8)	1.0649	0.0767	0.94	0.6812	0.0084	0.4698	0.0335	1.9918	0.3070
20	10	(0*8,5*2)	0.9723	0.0574	0.89	0.7132	0.0071	0.4116	0.0227	2.2282	0.4359
20	10	(0*9,10)	0.9531	0.0533	0.89	0.7110	0.0067	0.3995	0.0207	2.2741	0.4518
20	16	(4,0*15)	1.0327	0.0365	0.95	0.6907	0.0044	0.4467	0.0150	2.0041	0.1835
20	16	(2*2,0*14)	1.0449	0.0375	0.95	0.6864	0.0045	0.4543	0.0154	1.9741	0.1787
20	16	(0*14,2*2)	0.9801	0.0324	0.92	0.7095	0.0041	0.4140	0.0127	2.1376	0.2199
20	16	(0*15,4)	0.9831	0.0318	0.93	0.7084	0.0040	0.4158	0.0126	2.1285	0.2196
20	20	(0*20)	1.0340	0.0361	0.95	0.6902	0.0044	0.4475	0.0147	2.0009	0.1885
30	12	(18,0*11)	1.0502	0.0560	0.95	0.6854	0.0065	0.4591	0.0236	1.9963	0.2518
30	12	(6*3,0*9)	1.0518	0.0584	0.95	0.6849	0.0069	0.4605	0.0244	2.0006	0.2728
30	12	(0*9,6*3)	0.9580	0.0482	0.88	0.7181	0.0060	0.4020	0.0188	2.2439	0.3866
30	12	(0*11,18)	0.9598	0.0497	0.89	0.7175	0.0060	0.4031	0.0199	2.2348	0.3633
30	15	(15,0*14)	1.0432	0.0410	0.96	0.6871	0.0050	0.4536	0.0167	1.9886	0.2076
30	15	(5*3,0*12)	1.0425	0.0434	0.95	0.6875	0.0052	0.4533	0.0180	1.9918	0.2048
30	15	(0*12,5*3)	0.9430	0.0343	0.89	0.7228	0.0044	0.3915	0.0129	2.2483	0.3013
30	15	(0*14,15)	0.9464	0.0358	0.89	0.7217	0.0046	0.3936	0.0136	2.2408	0.2954
30	24	(6,0*23)	1.0229	0.0222	0.95	0.6936	0.0028	0.4394	0.0088	1.9981	0.1220
30	24	(2*3,0*21)	1.0233	0.0231	0.95	0.6935	0.0029	0.4397	0.0092	2.0004	0.1313
30	24	(0*21,2*3)	0.9735	0.0209	0.92	0.7114	0.0027	0.4089	0.0080	2.1259	0.1519
30	24	(0*23,6)	0.9679	0.0201	0.91	0.7133	0.0026	0.4053	0.0076	2.1394	0.1545
30	30	(0*30)	1.0148	0.0165	0.96	0.6963	0.0021	0.4338	0.0065	2.0058	0.0960
50	20	(30,0*19)	1.0287	0.0281	0.95	0.6918	0.0035	0.4435	0.0113	1.9969	0.1490
50	20	(5*6,0*14)	1.0253	0.0298	0.94	0.6931	0.0037	0.4415	0.0120	2.0093	0.1591
50	20	(0*14,5*6)	0.9313	0.0267	0.87	0.7268	0.0035	0.3835	0.0099	2.2566	0.2408
50	20	(0*19,30)	0.9203	0.0286	0.86	0.7308	0.0038	0.3770	0.0104	2.2945	0.2834
50	25	(25,0*24)	1.0224	0.0191	0.96	0.6936	0.0024	0.4388	0.0075	1.9925	0.1081
50	25	(5*5,0*20)	1.0162	0.0204	0.95	0.6959	0.0026	0.4351	0.0081	2.0115	0.1168
50	25	(0*20,5*5)	0.9309	0.0221	0.87	0.7268	0.0030	0.3827	0.0080	2.2451	0.2059
50	25	(0*24,25)	0.9195	0.0231	0.86	0.7309	0.0031	0.3759	0.0083	2.2787	0.2272
50	40	(10,0*39)	1.0142	0.0138	0.94	0.6964	0.0018	0.4332	0.0054	2.0002	0.0794
50	40	(5*2,0*38)	1.0157	0.0132	0.95	0.6958	0.0017	0.4341	0.0052	1.9950	0.0753
50	40	(0*38,5*2)	0.9605	0.0129	0.90	0.7158	0.0017	0.4000	0.0048	2.1378	0.1019
50	40	(0*39,10)	0.9631	0.0124	0.92	0.7148	0.0016	0.4016	0.0046	2.1295	0.0974
50	50	(0*50)	1.0163	0.0111	0.95	0.6955	0.0014	0.4343	0.0044	1.9875	0.0620

Table 3.2: Bayes Estimates when $\theta = 1.0, t = 0.80, \alpha = 2, \beta = 2, S(t) = 0.7010, h(t) = 0.4231, MTTF = 2.00, N = 10,000$.

n	m	CS	θ^*		$S^*(t)$		$h^*(t)$		$HTTF^*$	
			EV	ER	EV	ER	EV	ER	EV	ER
20	8	(12,0*7)	1.0646	0.0624	0.6835	0.0067	0.4743	0.0272	2.1268	0.3332
20	8	(4*3,0*5)	1.0599	0.0665	0.6854	0.0070	0.4717	0.0295	2.1410	0.3298
20	8	(0*5,4*3)	1.0590	0.0680	0.6858	0.0072	0.4712	0.0299	2.1498	0.3562
20	8	(0*7,12)	1.0706	0.0652	0.6816	0.0068	0.4782	0.0289	2.1090	0.3073
20	10	(10,0*9)	1.0596	0.0553	0.6844	0.0060	0.4695	0.0242	2.0921	0.2695
20	10	(5*2,0*8)	1.0578	0.0577	0.6852	0.0062	0.4686	0.0252	2.1017	0.2843
20	10	(0*8,5*2)	1.0609	0.0545	0.6839	0.0061	0.4703	0.0235	2.0897	0.2765
20	10	(0*9,10)	1.0446	0.0487	0.6894	0.0055	0.4596	0.0207	2.1235	0.2782
20	16	(4,0*15)	1.0299	0.0316	0.6932	0.0038	0.4474	0.0131	2.0815	0.1801
20	16	(2*2,0*14)	1.0421	0.0325	0.6890	0.0038	0.4550	0.0135	2.0508	0.1703
20	16	(0*14,2*2)	1.0306	0.0321	0.6928	0.0038	0.4478	0.0133	2.0769	0.1751
20	16	(0*15,4)	1.0350	0.0321	0.6915	0.0037	0.4505	0.0134	2.0664	0.1746
20	20	(0*20)	1.0307	0.0312	0.6929	0.0037	0.4479	0.0129	2.0796	0.1849
30	12	(18,0*11)	1.0441	0.0450	0.6891	0.0051	0.4583	0.0192	2.0965	0.2416
30	12	(6*3,0*9)	1.0510	0.0482	0.6869	0.0055	0.4629	0.0204	2.0864	0.2544
30	12	(0*9,6*3)	1.0516	0.0456	0.6864	0.0052	0.4630	0.0195	2.0757	0.2309
30	12	(0*11,18)	1.0560	0.0464	0.6851	0.0051	0.4657	0.0201	2.0627	0.2154
30	15	(15,0*14)	1.0396	0.0351	0.6810	0.0042	0.4539	0.0145	2.0704	0.2005
30	15	(5*3,0*12)	1.0423	0.0372	0.6894	0.0043	0.4557	0.0157	2.0652	0.1943
30	15	(0*12,5*3)	1.0391	0.0330	0.6900	0.0039	0.4534	0.0137	2.0628	0.1762
30	15	(0*14,15)	1.0452	0.0349	0.6882	0.0041	0.4573	0.0147	2.0499	0.1727
30	24	(6,0*23)	1.0232	0.0207	0.6960	0.0028	0.4411	0.0083	2.0536	0.1255
30	24	(2*3,0*21)	1.0229	0.0212	0.6947	0.0027	0.4412	0.0085	2.0513	0.1305
30	24	(0*21,2*3)	1.0264	0.0218	0.6935	0.0026	0.4433	0.0088	2.0395	0.1178
30	24	(0*23,6)	1.0226	0.0206	0.6952	0.0026	0.4408	0.0082	2.0490	0.1196
30	30	(0*30)	1.0151	0.0156	0.6981	0.0025	0.4352	0.0061	2.0511	0.1049
50	20	(30,0*19)	1.0284	0.0256	0.6944	0.0032	0.4451	0.0104	2.0617	0.1490
50	20	(5*6,0*14)	1.0318	0.0271	0.6919	0.0033	0.4476	0.0111	2.0499	0.1503
50	20	(0*14,5*6)	1.0303	0.0234	0.6933	0.0031	0.4462	0.0096	2.0487	0.1322
50	20	(0*19,30)	1.0228	0.0236	0.6955	0.0031	0.4417	0.0095	2.0686	0.1458
50	25	(25,0*24)	1.0221	0.0176	0.6953	0.0025	0.4402	0.0070	2.0443	0.1107
50	25	(5*5,0*20)	1.0196	0.0192	0.6962	0.0025	0.4388	0.0077	2.0534	0.1159
50	25	(0*20,5*5)	1.0303	0.0199	0.6921	0.0026	0.4455	0.0081	2.0249	0.1103
50	25	(0*24,25)	1.0224	0.0195	0.6955	0.0025	0.4404	0.0077	2.0465	0.1131
50	40	(10,0*39)	1.0177	0.0200	0.6989	0.0026	0.4359	0.0068	2.0364	0.0820
50	40	(5*2,0*38)	1.0202	0.0199	0.6993	0.0037	0.4372	0.0067	2.0342	0.0929
50	40	(0*38,5*2)	1.0168	0.0149	0.6970	0.0019	0.4356	0.0055	2.0263	0.0738
50	40	(0*39,10)	1.0206	0.0142	0.6976	0.0043	0.4377	0.0051	2.0242	0.0682
50	50	(0*50)	1.0160	0.0107	0.6960	0.0014	0.4350	0.0042	2.0130	0.0609

TABLE 3.3: MLEs when $\theta = 2.0, t = 0.40, S(t) = 0.6171, h(t) = 1.1262, MTTF = 0.8333, N = 10,000$.

n	m	CS	$\hat{\theta}$ EV	$\hat{\theta}$ MSE	CP	$\widehat{S(t)}$ EV	$\widehat{S(t)}$ MSE	$\widehat{h(t)}$ EV	$\widehat{h(t)}$ MSE	$\widehat{HTTF}$ EV	$\widehat{HTTF}$ MSE
20	8	(12,0*7)	2.2019	0.4974	0.97	0.5889	0.0132	1.2865	0.2913	0.8244	0.0751
20	8	(4*3,0*5)	2.1586	0.4352	0.96	0.5960	0.0122	1.2536	0.2521	0.8415	0.0767
20	8	(0*5,4*3)	2.0978	0.5119	0.92	0.6094	0.0141	1.2110	0.2970	0.8943	0.1065
20	8	(0*7,12)	2.0523	0.4909	0.92	0.6177	0.0136	1.1773	0.2845	0.9177	0.1132
20	10	(10,0*9)	2.1506	0.4057	0.94	0.5969	0.0116	1.2472	0.2341	0.8420	0.0757
20	10	(5*2,0*8)	2.1488	0.3567	0.96	0.5962	0.0101	1.2445	0.2064	0.8296	0.0510
20	10	(0*8,5*2)	2.0019	0.3391	0.94	0.6243	0.0106	1.1371	0.1911	0.9210	0.0942
20	10	(0*9,10)	1.9768	0.3135	0.92	0.6287	0.0101	1.1181	0.1755	0.9303	0.0914
20	16	(4,0*15)	2.0977	0.1861	0.96	0.6023	0.0061	1.2031	0.1041	0.8281	0.0399
20	16	(2*2,0*14)	2.0909	0.1942	0.95	0.6039	0.0063	1.1983	0.1090	0.8331	0.0411
20	16	(0*14,2*2)	1.9616	0.1645	0.92	0.6283	0.0058	1.1029	0.0899	0.9023	0.0512
20	16	(0*15,4)	1.9581	0.1585	0.93	0.6289	0.0055	1.1002	0.0870	0.9013	0.0476
20	20	(0*20)	2.1001	0.2033	0.95	0.6023	0.0064	1.2052	0.1149	0.8285	0.0398
30	12	(18,0*11)	2.1259	0.2682	0.95	0.5987	0.0084	1.2259	0.1518	0.8304	0.0550
30	12	(6*3,0*9)	2.1272	0.2809	0.95	0.5987	0.0085	1.2270	0.1601	0.8303	0.0537
30	12	(0*9,6*3)	1.9659	0.2362	0.90	0.6291	0.0082	1.1083	0.1293	0.9227	0.0821
30	12	(0*11,18)	1.9747	0.2376	0.93	0.6274	0.0081	1.1146	0.1313	0.9137	0.0707
30	15	(15,0*14)	2.1145	0.2023	0.96	0.5991	0.0066	1.2159	0.1132	0.8230	0.0427
30	15	(5*3,0*12)	2.0885	0.1896	0.96	0.6042	0.0063	1.1965	0.1058	0.8350	0.0435
30	15	(0*12,5*3)	1.9568	0.1902	0.91	0.6298	0.0067	1.1003	0.1040	0.9131	0.0628
30	15	(0*14,15)	1.9261	0.1874	0.90	0.6357	0.0067	1.0777	0.1018	0.9306	0.0659
30	24	(6,0*23)	2.0538	0.1137	0.95	0.6092	0.0039	1.1689	0.0630	0.8345	0.0257
30	24	(2*3,0*21)	2.0593	0.1153	0.96	0.6082	0.0039	1.1730	0.0639	0.8323	0.0266
30	24	(0*21,2*3)	1.9514	0.1105	0.90	0.6291	0.0040	1.0940	0.0599	0.8929	0.0362
30	24	(0*23,6)	1.9427	0.0921	0.93	0.6303	0.0035	1.0871	0.0495	0.8926	0.0309
30	30	(0*30)	2.0624	0.1231	0.96	0.6078	0.0042	1.1755	0.0683	0.8322	0.0276
50	20	(30,0*19)	2.0996	0.1536	0.95	0.6012	0.0050	1.2036	0.0861	0.8184	0.0311
50	20	(5*6,0*14)	2.0826	0.1580	0.94	0.6046	0.0052	1.1913	0.0885	0.8289	0.0336
50	20	(0*14,5*6)	1.9257	0.1496	0.89	0.6350	0.0055	1.0763	0.0809	0.9196	0.0523
50	20	(0*19,30)	1.9158	0.1257	0.90	0.6363	0.0047	1.0684	0.0673	0.9190	0.0463
50	25	(25,0*24)	2.0644	0.1113	0.96	0.6071	0.0038	1.1766	0.0617	0.8283	0.0250
50	25	(5*5,0*20)	2.0736	0.1140	0.95	0.6054	0.0040	1.1835	0.0632	0.8240	0.0253
50	25	(0*20,5*5)	1.9044	0.1093	0.89	0.6382	0.0042	1.0596	0.0583	0.9205	0.0417
50	25	(0*24,25)	1.9172	0.1191	0.88	0.6359	0.0044	1.0692	0.0642	0.9150	0.0419
50	40	(10,0*39)	2.0327	0.0616	0.95	0.6121	0.0022	1.1520	0.0337	0.8333	0.0155
50	40	(5*2,0*38)	2.0420	0.0662	0.96	0.6104	0.0023	1.1590	0.0364	0.8290	0.0155
50	40	(0*38,5*2)	1.9222	0.0575	0.92	0.6335	0.0022	1.0710	0.0306	0.8934	0.0206
50	40	(0*39,10)	1.9200	0.0592	0.90	0.6340	0.0023	1.0695	0.0315	0.8954	0.0216
50	50	(0*50)	2.0234	0.0612	0.95	0.6139	0.0022	1.1452	0.0335	0.8382	0.0157

Table 3.4: Bayes Estimates when $\theta = 2.0, t = 0.40, \alpha = 3, \beta = 3, S(t) = 0.6171, h(t) = 1.1262, MTTF = 0.8333, N = 10,000$.

n	m	CS	θ^*		$S^*(t)$		$h^*(t)$		$HTTF^*$	
			EV	ER	EV	ER	EV	ER	EV	ER
20	8	(12,0*7)	2.0930	0.1608	0.6079	0.0050	1.2054	0.09087	0.8886	0.0442
20	8	(4*3,0*5)	2.0780	0.1496	0.6105	0.0047	1.1941	0.0841	0.8952	0.0445
20	8	(0*5,4*3)	2.1119	0.1763	0.6047	0.0054	1.2198	0.1001	0.8818	0.0450
20	8	(0*7,12)	2.0853	0.1662	0.6095	0.0052	1.2000	0.0939	0.8955	0.0473
20	10	(10,0*9)	2.0751	0.1669	0.6109	0.0053	1.1915	0.0935	0.8925	0.0499
20	10	(5*2,0*8)	2.0814	0.1445	0.6091	0.0045	1.1954	0.0815	0.8810	0.0385
20	10	(0*8,5*2)	2.0823	0.1542	0.6093	0.0049	1.1965	0.0868	0.8846	0.0420
20	10	(0*9,10)	2.0681	0.1432	0.6117	0.0046	1.1857	0.0804	0.8897	0.0402
20	16	(4,0*15)	2.0679	0.1136	0.6098	0.0037	1.1829	0.0631	0.8646	0.0299
20	16	(2*2,0*14)	2.0628	0.1163	0.6113	0.0039	1.1791	0.0647	0.8690	0.0312
20	16	(0*14,2*2)	2.0435	0.1103	0.6141	0.0037	1.1651	0.0612	0.8770	0.0309
20	16	(0*15,4)	2.0446	0.1045	0.6141	0.0035	1.1655	0.0582	0.8750	0.0284
20	20	(0*20)	2.0682	0.1190	0.6100	0.0039	1.1832	0.0666	0.8654	0.0299
30	12	(18,0*11)	2.0754	0.1347	0.6095	0.0044	1.1901	0.0752	0.8762	0.0380
30	12	(6*3,0*9)	2.0808	0.1377	0.6086	0.0044	1.1941	0.0773	0.8731	0.0365
30	12	(0*9,6*3)	2.0657	0.1302	0.6114	0.0043	1.1829	0.0725	0.8819	0.0388
30	12	(0*11,18)	2.0731	0.1273	0.6098	0.0041	1.1883	0.0715	0.8749	0.0329
30	15	(15,0*14)	2.0790	0.1191	0.6081	0.0039	1.1914	0.0664	0.8623	0.0311
30	15	(5*3,0*12)	2.0622	0.1125	0.6112	0.0038	1.1789	0.0624	0.8707	0.0321
30	15	(0*12,5*3)	2.0790	0.1206	0.6086	0.0039	1.1914	0.0675	0.8636	0.0307
30	15	(0*14,15)	2.0561	0.1148	0.6123	0.0038	1.1745	0.0640	0.8742	0.0312
30	24	(6,0*23)	2.0425	0.0822	0.6133	0.0029	1.1622	0.0453	0.8597	0.0216
30	24	(2*3,0*21)	2.0472	0.0834	0.6121	0.0029	1.1658	0.0461	0.8570	0.0220
30	24	(0*21,2*3)	2.0498	0.0882	0.6130	0.0032	1.1674	0.0485	0.8593	0.0238
30	24	(0*23,6)	2.0462	0.0742	0.6130	0.0027	1.1645	0.0407	0.8568	0.0198
30	30	(0*30)	2.0486	0.0884	0.6120	0.0030	1.1670	0.0489	0.8575	0.0228
50	20	(30,0*19)	2.0787	0.1020	0.6080	0.0035	1.1894	0.0568	0.8506	0.0244
50	20	(5*6,0*14)	2.0716	0.1055	0.6093	0.0036	1.1844	0.0587	0.8553	0.0261
50	20	(0*14,5*6)	2.0550	0.1036	0.6123	0.0035	1.1723	0.0575	0.8643	0.0269
50	20	(0*19,30)	2.0505	0.0867	0.6125	0.0030	1.1686	0.0480	0.8621	0.0231
50	25	(25,0*24)	2.0535	0.0823	0.6116	0.0029	1.1700	0.0453	0.8536	0.0230
50	25	(5*5,0*20)	2.0659	0.0853	0.6098	0.0031	1.1790	0.0469	0.8484	0.0212
50	25	(0*20,5*5)	2.0557	0.0832	0.6124	0.0032	1.1713	0.0455	0.8552	0.0218
50	25	(0*24,25)	2.0681	0.0910	0.6091	0.0031	1.1810	0.0505	0.8477	0.0214
50	40	(10,0*39)	2.0394	0.0718	0.6176	0.0045	1.1564	0.0343	0.8540	0.0196
50	40	(5*2,0*38)	2.0475	0.0758	0.6159	0.0046	1.1625	0.0365	0.8497	0.0195
50	40	(0*38,5*2)	2.0492	0.0803	0.6192	0.0059	1.1623	0.0361	0.8545	0.0220
50	40	(0*39,10)	2.0368	0.0627	0.6158	0.0034	1.1552	0.0312	0.8505	0.0170
50	50	(0*50)	2.0330	0.0754	0.6199	0.0049	1.1513	0.0353	0.8596	0.0208

The following observations are noted from the results of simulation study.

(i) The MLEs and Bayes estimates of θ are approximately unbiased and as sample size n increases, the average bias and MSE decrease for complete samples.

(ii) For all the censoring scheme, the bias and MSE of estimates are quite small, though the bias increases as the failure information decreases.

(iii) The widths of the confidence intervals based on MLEs are getting sharper with the increase of sample size n and failure information m. The coverage probabilities are very close to the corresponding nominal levels.

(iv) In this case Bayes estimates are uniformly better than the MLEs showing that additional prior information about θ provide an improvement.

3.6 Real data illustration

In this section, a real data set representing the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, observed and reported by Bjerkedal (1960) is utilized to illustrate the methods described in the previous sections. The data is presented in Table 3.5.

TABLE 3.5: Data on survival times of 72 guinea pigs infected with virulent tubercle bacilli

10	33	44	56	59	72	74	77	92	93	96	100
100	102	105	107	107	108	108	108	109	112	113	115
116	120	121	122	122	124	130	134	136	139	144,	146
153	159	160	163	163	168	171	172	176	183	195	196
197	202	213	215	216	222	230	231	240	245	251	253
254	254	278	293	327	342	347	361	402	432	458	555

The data set is positively skewed (skewness: 1.34), with mean survival time of 176.82 days, standard deviation of 103.45 days and is unimodal.
Here, six survival models, namely exponential, gamma, Weibull, lognormal, Lindley (Lindley (1958)) and xgamma have been considered. Method of maximum likelihood is used to estimate the parameters of the above six distributions. To identify the best fit of above models, (i) negative log-likelihood, (ii) AIC (iii) BIC are considered. Statistical software R is used for data analysis.
The maximum likelihood estimates (MLEs) of the model parameters with corresponding standard errors (Std. Error) of estimates in parentheses, negative log-likelihood, AIC and BIC values are shown in Table 3.6.

It is clear from Table 3.6 that xgamma is fitting the above data quite satisfactorily. According to model selection criterion, viz. AIC, we found that xgamma is the best model followed by gamma, Weibull, Lindley, lognormal and exponential, respectively.
In an almost similar fashion, according to BIC criteria, we can see that xgamma, again, is the best choice followed by gamma, Lindley, Weibull, lognormal and exponential, respectively.
Moreover, another advantage of using xgamma instead of lognormal or gamma or Weibull for modeling survival data is that xgamma distribution has only one unknown parameter, like Lindley distribution, and hence estimation procedures become more convenient in view of computational ease.
In Table 3.7, MLEs of θ along with corresponding 95% confidence intervals are calculated for various progressively type-II censoring schemes based on the above data. MLEs of survival functions, failure rates (for mission time $t = 10$) and the mean time to failure are also tabulated correspondingly. The percentages of failure information taken here as $28\%, 42\%, 56\%$ and 100%.

Table 3.6: The MLEs of parameter(s), negative log-likelihood, AIC and BIC values for different survival models.

Survival Model	Estimate(Std. Error)	-log-likelihood	AIC	BIC
Exponetial(θ)	$\hat{\theta}$=0.0057(0.00064)	444.61	891.22	893.50
Gamma(α, β)	$\hat{\alpha}$=0.0174(0.00295) $\hat{\beta}$=3.0824(0.48259)	425.80	855.60	860.16
Weibull(α, λ)	$\hat{\alpha}$=199.5869(13.629) $\hat{\lambda}$=1.8252(0.15870)	427.36	858.72	863.28
Lognormal(μ, σ)	$\hat{\mu}$=5.0043(0.07413) $\hat{\sigma}$=0.6290(0.05242)	429.09	862.19	866.74
Lindley(θ)	$\hat{\theta}$=0.0113(0.00093)	429.28	860.56	862.83
Xgamma(θ)	$\hat{\theta}$=0.0168(0.00114)	425.59	853.18	855.46

Table 3.7: The MLEs of θ with 95% confidence intervals (CIs) and MLEs of survival characteristics for selected censoring schemes.

n	m	CS	$\hat{\theta}$	95% CI	$\hat{S}(t=10)$	$\hat{h}(t=10)$	$\widehat{HTTF}$
72	20	(2*19,14)	0.01339	(0.01051,0.01628)	0.99799	0.00026	222.03
72	20	(14,2*19)	0.01753	(0.01366,0.02139)	0.99646	0.00048	169.19
72	20	(0*18,26*2)	0.01499	(0.01179,0.01819)	0.99745	0.00033	198.19
72	20	(26*2,0*18)	0.01632	(0.01290,0.01974)	0.99163	0.00044	178.03
72	20	(1*18,17*2)	0.01575	(0.01237,0.01913)	0.99717	0.00037	188.52
72	20	(17*2,1*18)	0.01932	(0.01491,0.02373)	0.99565	0.00060	153.29
72	30	(1*29,13)	0.01596	(0.01302,0.01889)	0.99701	0.00038	186.04
72	30	(13,1*29)	0.01734	(0.01405,0.02063)	0.99654	0.00046	171.05
72	30	(0*28,21*2)	0.01594	(0.01303,0.01884)	0.99710	0.00038	186.04
72	30	(21*2,0*28)	0.02072	(0.01647,0.02496)	0.99496	0.00070	142.82
72	30	(1*28,7*2))	0.01526	(0.01244,0.01806)	0.99735	0.00035	194.63
72	30	(7*2,1*28)	0.01806	(0.01465,0.02146)	0.99623	0.00051	164.17
72	40	(0*39,32)	0.01573	(0.01317,0.01830)	0.99718	0.00037	188.70
72	40	(32,0*39)	0.01809	(0.01483,0.02134)	0.99622	0.00051	163.92
72	40	(1*32,0*8)	0.01034	(0.01002,0.01076)	0.99905	0.00032	189.14
72	40	(0*8,1*32)	0.01429	(0.01202,0.01657)	0.99769	0.00030	207.95
72	72	(0*72)	0.01678	(0.01453,0.01903)	0.99677	0.00043	176.81

3.7 Conclusion

In this chapter, some additional distributional properties and properties related to survival and/or reliability are been extensively addressed. Ageing intensity function of the xgamma distribution is also studied so as to quantify the ageing phenomena. The unknown parameter in xgamma distribution is estimated using maximum likelihood method and Bayesian method under progressively type-II right censoring scheme. Simulation study is been carried out to compare the estimates described under the two estimation procedures and various progressively type-II censoring schemes. The following important findings are found in this chapter.

1. xgamma random variable is stochastically larger than those of Lindley and exponential under the similar parameter.

2. The xgamma distribution do not possess single monotone ageing intensity property for the entire range of the parameter, rather it depends highly on the values of the unknown parameter.

3. Stress-strength reliability is 50% if stress and strength are identically and independently distributed (i.i.d) xgamma distribution.

4. It is recommended to use Bayes estimate for parameter provided a prior information is available; otherwise the method of maximum likelihood would be a better choice.

5. Real data illustration shows that xgamma is a competent life distribution as compared to exponential, gamma, Weibull, log-normal and Lindley distributions.

Apart from searching possible other areas of application for xgamma distribution, below are pointed some open research problems for future investigation.

Open research problems:

- Study of Bayesian estimation for the parameter in xgamma distribution under different loss functions.
- System reliability models for xgamma lives could reveal some interesting failure properties.

Chapter 4

Truncated xgamma distributions

In Chapter 2, we have introduced the xgamma distribution and some of its basic distributional and survival properties have been studied. Further, some important additional properties related to the distributional form and survival and/or reliability have also been investigated and the xgamma distribution has been established in the list of other standard and popular life distributions; and Bayesian estimation of the parameter and other survival characteristics have also been investigated under progressively type-II censored situation in Chapter 3.

Sometimes it is necessary to assume the range of the random variable associated with a life distribution as finite instead of having a range from 0 to ∞. This is assumed or considered either that might be a necessity of designed life or it might be required to do so from simply a mathematical point of view. In this chapter, we therefore introduce the truncated versions of xgamma distribution. Below are described some situations where truncated distributions have successfully been applied in the literature.

In manufacturing industries, final products often pass through a screening inspection before being sent to the customers. It is, then, a normal practice that if a product's performance falls within certain tolerance limits, it is being judged as conforming and thereby sent to the customer. A product is usually rejected and therefore scrapped or reworked if it fails to conform. In this case, the actual distribution to the customer is truncated. As a standard practice, truncated distributions are applied, in particular, for such industrial applications and settings (see for more details Cho and Govindaluri, 2002; Kapur and Cho, 1994, 1996; Phillips and Cho, 1998, 2000; Khasawneh et al., 2004, 2005, and references therein).

As an another example, truncated distributions are well applied in multistage production process in which inspection is done at each stage and only conforming items are sent to the next stage. Very useful application can also be found in accelerated life testing situations. In practice, the concept of a truncated distribution plays an important role in analyzing a variety of production processes, process optimization and quality improvement techniques.

To model intensity statistics in the study of atomic heterogeneity, truncated distributions are utilized, see Mukhopadhyay et al. (2000) for more insight. In an another situation, measurements match well with a truncated distribution with much better fit over smaller file or request sizes for high-performance Ethernet, see for more details Field et al. (2004).

Truncated versions of several standard continuous probability distributions are been studied by different authors and applied successfully to numerous real life situations, see for example, Hegde and Dahiya (1989), Mittal and Dahiya (1989), Nadarajah (2008), Zaninetti and Ferraro (2008) and, Zhang and Xie (2011).

The rest of the chapter is organized as follows.
In section 4.1, we introduce truncated versions (upper, lower and double) of xgamma distribution and special attention is paid to the upper truncated version. Section 4.2 deals with basic structural and distributional properties of upper truncated xgamma distribution. Sections 4.4 and 4.5 deal with the studies of entropy measures and distributions of order statistics, respectively. Different survival and/or reliability properties are studied in section 4.6. The unknown parameters

for upper truncated xgamma distributions are estimated by the method of maximum likelihood in section 4.7. In section 4.8, a real life data set is analyzed to illustrate the application of the proposed distribution. Finally, section 4.9 presents the overall conclusion of the chapter.

4.1 Truncated versions of xgamma distribution

If X is a non-negative continuous random variable, then for double truncated version of X, we have the following definition.

Definition 4.1. A non-negative continuous random variable, X, is said to follow a double truncated distribution (DTD) over the interval $[\alpha, \beta]$ if it has the cdf as

$$G(x) = \frac{F(x) - F(\alpha)}{F(\beta) - F(\alpha)}, \alpha \leq x \leq \beta, \tag{4.1}$$

where $F(\cdot)$ denote the cdf of the baseline distribution, α and β are points of truncation.
The corresponding pdf is given by

$$g(x) = \frac{f(x)}{F(\beta) - F(\alpha)}, \alpha \leq x \leq \beta, \tag{4.2}$$

where $f(\cdot)$ is the pdf of baseline distribution.

The following three cases are recognized from the above definition.

(i) When $\alpha \to 0$ and $\beta \to \infty$, we have the baseline lifetime distribution with support $(0, \infty)$.

(ii) When $\alpha \to 0$, we have upper truncated distribution (UTD) of the baseline distribution.

(iii) When $\beta \to \infty$, we have lower truncated distribution (LTD) of the baseline distribution.

Applying the above definition and taking the baseline distribution as xgamma distribution with parameter $\theta(>0)$, the following definitions for truncated versions of xgamma distribution can be placed.

4.1.1 Double truncated xgamma distribution

Definition 4.2. A continuous random variable, X, is said to follow a double truncated xgamma (DTXG) distribution with parameters α, β and θ if its pdf is of the form

$$g(x;\alpha,\beta,\theta) = \frac{\theta^2}{(1+\theta)} \frac{\left(1+\frac{\theta}{2}x^2\right)e^{-\theta x}}{[F(\beta)-F(\alpha)]}, \; \alpha \leq x \leq \beta, \alpha > 0, \beta > 0, \theta > 0, \tag{4.3}$$

where $F(\alpha)$ and $F(\beta)$ can be obtained from (2.6) by putting $x = \alpha$ and $x = \beta$, respectively.
It is denoted by $X \sim DTXG(\alpha, \beta, \theta)$.

4.1.2 Lower truncated xgamma distribution

When $\beta \to \infty$, from (4.3), the following definition for lower truncated xgamma distribution is made.

Definition 4.3. A continuous random variable, X, is said to follow a lower truncated xgamma (LTXG) distribution with parameters α and θ if its pdf is of the form

$$g(x;\alpha,\theta) = \frac{\theta^2\left(1+\frac{\theta}{2}x^2\right)e^{-\theta(x-\alpha)}}{\left(1+\theta+\theta\alpha+\frac{\theta^2\alpha^2}{2}\right)}, \; x \geq \alpha, \alpha > 0, \theta > 0. \tag{4.4}$$

It is denoted by $X \sim LTXG(\alpha, \theta)$.

4.1.3 Upper truncated xgamma distribution

The upper truncated version of xgamma distribution has the following definition.

Definition 4.4. A continuous random variable, X, is said to follow an upper truncated xgamma (UTXG) distribution with parameters β and θ if its pdf is of the form

$$g(x;\beta,\theta) = K(\beta,\theta)\left(1+\frac{\theta}{2}x^2\right)e^{-\theta x}, 0 \leq x \leq \beta, \beta > 0, \theta > 0, \qquad (4.5)$$

where $K(\beta,\theta) = \frac{\theta^2}{(1+\theta)\left(1-e^{-\theta\beta}\right)-\theta\beta\left(1+\frac{\theta\beta}{2}\right)e^{-\theta\beta}}$, a function of β and θ.

It is denoted by $X \sim UTXG(\beta,\theta)$.

The cdf of $X \sim UTXG(\beta,\theta)$ is obtained from (4.1) by putting $F(0) = 0$ as $\alpha \to 0$. Hence the cdf is given by

$$G(x;\beta,\theta) = \Pr(X \leq x) = \frac{F(x)}{F(\beta)}, 0 \leq x \leq \beta, \qquad (4.6)$$

where $F(x)$ is the cdf of xgamma distribution in (2.6) and $F(\beta)$ can be obtained from (2.6) by putting $x = \beta$.

Hereafter, main concentration is given in studying different structural and survival properties of $UTXG(\beta,\theta)$.
The properties of upper truncated xgamma is been studied owing the fact that designed lifetimes of equipment or units are usually having finite upper limit and it is, therefore, quite rational to think that upper range of the random variable, describes life, is actually finite. Sometimes, the upper truncated version of the base distribution can provide better fit to lifetime data.

The following theorem shows that $UTXG(\beta,\theta)$ is unimodal.

Theorem 4.5. *For* $0 < \theta \leq \frac{1}{2}$*, the pdf of* $UTXG(\beta, \theta)$ *attains a maximum at* $x = \frac{1+\sqrt{1-2\theta}}{\theta}$ *and for* $\theta > \frac{1}{2}$*, it decreases in* x*.*

Proof. Taking first derivative of (4.5) with respect to x, we get

$$\frac{d}{dx}g(x;\beta,\theta) = K(\beta,\theta)\left[\theta x e^{-\theta x} - \theta\left(1+\frac{\theta}{2}x^2\right)e^{-\theta x}\right],$$

which is positive for $0 < \theta \leq 1/2$ and $g(x;\beta,\theta)$ attains a maximum value at $x = \frac{1+\sqrt{1-2\theta}}{\theta}$. For $\theta > \frac{1}{2}$, $\frac{d}{dx}g(x;\beta,\theta)$ is negative and hence $g(x;\beta,\theta)$ decreases in x.

Hence the proof.

So, the mode of $X \sim UTXG(\beta,\theta)$ is given by

$$Mode(X) = \begin{cases} \frac{1+\sqrt{1-2\theta}}{\theta}\ , & \text{if } 0 < \theta \leq 1/2. \\ 0\ , & \text{otherwise.} \end{cases} \tag{4.7}$$

In the subsequent sections, we study different properties of $UTXG(\beta,\theta)$.

4.2 Moments and associated measures

In this section, we find the moments and related measures of $UTXG(\beta,\theta)$. First we find the non-central moments.

The r^{th} order non-central moment of $X \sim UTXG(\beta,\theta)$ can be obtained as

$$\begin{aligned}\mu_r' = E(X^r) &= \int_0^\beta x^r g(x;\beta,\theta)dx,\\ &= K(\beta,\theta)\int_0^\beta x^r\left(1+\frac{\theta}{2}x^2\right)e^{-\theta x}dx,\end{aligned}$$

Hence, we have,

$$
\begin{aligned}
\mu_r' &= K(\beta,\theta)\left[\int_0^\beta x^r e^{-\theta x}dx + \frac{\theta}{2}\int_0^\beta x^{r+2}e^{-\theta x}dx\right],\\
&= \frac{K(\beta,\theta)}{\theta^{r+1}}\left[\frac{1}{\theta^{r+1}}\gamma(r+1,\theta\beta) + \frac{1}{2\theta^{r+2}}\gamma(r+3,\theta\beta)\right],
\end{aligned}
$$

Here $\gamma(a,x) = \int_0^x z^{a-1}e^{-z}dz$ is the lower incomplete gamma function.

$$
= \frac{K(\beta,\theta)}{\theta^{r+1}}\left[\gamma(r+1,\theta\beta) + \frac{1}{2\theta}\gamma(r+3,\theta\beta)\right] \text{ for } r = 1,2,\ldots. \tag{4.8}
$$

In particular, we have,

$$
\mu_1' = E(X) = \frac{K(\beta,\theta)}{\theta^2}\left[\gamma(2,\theta\beta) + \frac{1}{2\theta}\gamma(4,\theta\beta)\right] = \mu\,(say). \tag{4.9}
$$

$$
\mu_2' = E(X^2) = \frac{K(\beta,\theta)}{\theta^3}\left[\gamma(3,\theta\beta) + \frac{1}{2\theta}\gamma(5,\theta\beta)\right]. \tag{4.10}
$$

So, we have the second order central moment as

$$
\begin{aligned}
Var(X) &= \sigma^2 \text{ (say)},\\
&= \mu_2' - \mu^2,\\
&= \frac{K(\beta,\theta)}{\theta^3}\left[\left\{\gamma(3,\theta\beta) + \frac{1}{2\theta}\gamma(5,\theta\beta)\right\} - \frac{K(\beta,\theta)}{\theta}\left\{\gamma(2,\theta\beta) + \frac{1}{2\theta}\gamma(4,\theta\beta)\right\}^2\right]
\end{aligned}
\tag{4.11}
$$

It is clear from the expressions of μ and σ^2 that those are not in simple forms. Hence, we compute the values of μ and σ^2 for some selected values of β and θ to understand the changing behaviour of them with varying values of the parameters.

Table 4.1 shows the values of μ and σ^2 for selected values of β and θ.

TABLE 4.1: Mean and variance values of $UTXG(\beta,\theta)$ for selected values of θ and β.

		θ							
$\beta\downarrow$		0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00
5	μ	2.6285	2.3682	2.0293	1.6935	1.3996	1.1606	0.9735	0.8288
	σ^2	2.1392	2.0934	1.9335	1.6620	1.3447	1.0473	0.8035	0.6175
10	μ	5.2821	3.9136	2.7518	1.9870	1.5097	1.1998	0.9870	0.8333
	σ^2	7.8938	6.8105	4.6456	2.8830	1.8275	1.2252	0.8661	0.6389
15	μ	7.3244	4.4884	2.8494	1.9997	1.5111	1.2000	0.9870	0.8333
	σ^2	12.3538	10.6627	5.4461	2.9961	1.8408	1.2267	0.8662	0.6389
20	μ	8.7030	4.6336	2.8567	1.9999	1.5111	1.2000	0.9870	0.8333
	σ^2	17.0923	12.3157	5.5432	2.9999	1.8410	1.2267	0.8663	0.6389

4.3 Characteristic and generating functions

In this section, we derive the characteristic, moment generating and cumulant generating functions for $X \sim UTXG(\beta,\theta)$.

For any $t \in \Re$, the characteristic function of X is derived as

$$\begin{aligned}
\phi_X(t) &= E\left[e^{itX}\right], \\
&= K(\beta,\theta)\int_0^{\beta} e^{itx}\left(1+\frac{\theta}{2}x^2\right)e^{-\theta x}dx, \\
&= K(\beta,\theta)\int_0^{\beta}\left(1+\frac{\theta}{2}x^2\right)e^{-(\theta-it)x}dx, \\
&= K(\beta,\theta)\left[\int_0^{\beta} e^{-(\theta-it)x}dx+\frac{\theta}{2}\int_0^{\beta} x^2e^{-(\theta-it)x}dx\right], \\
&= K(\beta,\theta)\left[\frac{1}{(\theta-it)}\gamma\{1,(\theta-it)\beta\}+\frac{\theta}{2(\theta-it)^3}\gamma\{3,(\theta-it)\beta\}\right],
\end{aligned}$$

where $\gamma(a,x)=\int_0^x z^{a-1}e^{-z}dz$ is the lower incomplete gamma function.

Hence the characteristic function of $UTXG(\beta,\theta)$ is given by

$$\phi_X(t) = K(\beta,\theta)\left[\frac{1}{(\theta-it)}\gamma\{1,(\theta-it)\beta\}+\frac{\theta}{2(\theta-it)^3}\gamma\{3,(\theta-it)\beta\}\right]. \quad (4.12)$$

Now, we find the moment generating function for $UTXG(\beta,\theta)$.

The moment generating function can be obtained as

$$\begin{aligned}
M_X(t) &= E\left[e^{tX}\right], \\
&= K(\beta,\theta)\int_0^{\beta} e^{tx}\left(1+\frac{\theta}{2}x^2\right)e^{-\theta x}dx, \\
&= K(\beta,\theta)\int_0^{\beta}\left(1+\frac{\theta}{2}x^2\right)e^{-(\theta-t)x}dx, \\
&= K(\beta,\theta)\left[\int_0^{\beta} e^{-(\theta-t)x}dx+\frac{\theta}{2}\int_0^{\beta} x^2e^{-(\theta-t)x}dx\right], \\
&= K(\beta,\theta)\left[\frac{1}{(\theta-t)}\gamma\{1,(\theta-t)\beta\}+\frac{\theta}{2(\theta-t)^3}\gamma\{3,(\theta-t)\beta\}\right], \qquad (4.13)
\end{aligned}$$

where $\gamma(a,x)=\int_0^x z^{a-1}e^{-z}dz$ is the lower incomplete gamma function.

The cumulant generating function of X is obtained by taking natural logarithm of $M_X(t)$ and is given by

$$\begin{aligned}
K_X(t) &= \ln K(\beta,\theta)+\ln\left[\frac{1}{(\theta-t)}\gamma\{1,(\theta-t)\beta\}+\frac{\theta}{2(\theta-t)^3}\gamma\{3,(\theta-t)\beta\}\right], \\
&= \ln\left[\frac{\theta^2}{(1+\theta)\left(1-e^{-\theta\beta}\right)-\theta\beta\left(1+\frac{\theta\beta}{2}\right)e^{-\theta\beta}}\right] \\
&\quad+\ln\left[\frac{1}{(\theta-t)}\gamma\{1,(\theta-t)\beta\}+\frac{\theta}{2(\theta-t)^3}\gamma\{3,(\theta-t)\beta\}\right], t\in\Re. \qquad (4.14)
\end{aligned}$$

4.4 Entropy measures

We first find the Rényi entropy. Rényi entropy is defined as

$$H_R(\delta)=\frac{1}{1-\delta}\ln\left[\int_0^{\infty} f^{\delta}(x)dx\right] \text{ for } \delta>0(\neq 1). \qquad (4.15)$$

When $X \sim UTXG(\beta,\theta)$, we derive,

$$\int_0^\beta g^\delta(x;\beta,\theta)dx = [K(\beta,\theta)]^\delta \int_0^\beta \left(1+\frac{\theta}{2}x^2\right)^\delta e^{-\delta\theta x}.$$

Now using the expansion,

$$\left(1+\frac{\theta}{2}x^2\right)^\delta = \sum_{j=0}^{\delta} \binom{\delta}{j} \left(\frac{\theta x^2}{2}\right)^j,$$

we have,

$$\begin{aligned}
&\int_0^\beta g^\delta(x;\beta,\theta)dx \\
&= [K(\beta,\theta)]^\delta \int_0^\beta \sum_{j=0}^{\delta} \binom{\delta}{j} \left(\frac{\theta x^2}{2}\right)^j e^{-\delta\theta x}dx, \\
&= [K(\beta,\theta)]^\delta \sum_{j=0}^{\delta} \binom{\delta}{j} \left(\frac{\theta}{2}\right)^j \int_0^\beta x^{2j} e^{-\delta\theta x}dx, \\
&= [K(\beta,\theta)]^\delta \sum_{j=0}^{\delta} \binom{\delta}{j} \left(\frac{\theta}{2}\right)^j \frac{1}{(\delta\theta)^{2j+1}} \gamma(2j+1,\delta\theta\beta),
\end{aligned}$$

Here $\gamma(a,x) = \int_0^x z^{a-1}e^{-z}dz$ is the lower incomplete gamma function.

$$= [K(\beta,\theta)]^\delta \sum_{j=0}^{\delta} \binom{\delta}{j} \frac{1}{2^j\delta^{2j+1}\theta^{j+1}} \gamma(2j+1,\delta\theta\beta). \tag{4.16}$$

From (4.15), using (4.16), the final form of Rényi entropy is obtained as

$$H_R(\delta) = \frac{\delta}{1-\delta}K(\beta,\theta) + \frac{1}{1-\delta}\ln\left[\sum_{j=0}^{\delta}\binom{\delta}{j}\frac{1}{2^j\delta^{2j+1}\theta^{j+1}}\gamma(2j+1,\delta\theta\beta)\right]. \tag{4.17}$$

Next, we calculate Tsallis entropy (also called q-entropy) when $X \sim UTXG(\beta,\theta)$. Tsallis entropy is defined as

$$S_q(X) = \frac{1}{q-1}\ln\left[1-\int_0^\infty f^q(x)dx\right] \text{ for } q>0\,(\neq 1). \tag{4.18}$$

Now, when $X \sim UTXG(\beta, \theta)$, to derive Tsallis entropy, we calculate

$$\int_0^\beta g^q(x; \beta, \theta)dx$$

in a very similar fashion as in (4.16) by simply replacing δ with q.
So, we have

$$\int_0^\beta g^q(x; \beta, \theta)dx = [K(\beta, \theta)]^q \sum_{j=0}^{q} \binom{q}{j} \frac{1}{2^j q^{2j+1} \theta^{j+1}} \gamma(2j+1, q\theta\beta).$$

Hence, from (4.18), the final form of Tsallis entropy is given by

$$S_q(X) = \frac{1}{q-1} \ln \left[1 - [K(\beta, \theta)]^q \sum_{j=0}^{q} \binom{q}{j} \frac{1}{2^j q^{2j+1} \theta^{j+1}} \gamma(2j+1, q\theta\beta) \right]. \quad (4.19)$$

Now, we derive Shannon entropy for $UTXG(\beta, \theta)$. Shannon measure of entropy is defined as

$$H(f) = E[-\ln f(x)] = -\int_0^\infty \ln f(x) f(x) dx. \quad (4.20)$$

When $X \sim UTXG(\beta, \theta)$, we have,

$$E[-\ln g(x; \beta, \theta)]$$
$$= -\int_0^\beta \ln g(x; \beta, \theta) g(x; \beta, \theta) dx,$$

Let us denote $g(x; \beta, \theta)$ as $g(x)$ for simplicity.

$$= -\left[\int_0^\beta \ln \left\{ K(\beta, \theta) \left(1 + \frac{\theta}{2}x^2\right) e^{-\theta x} \right\} g(x; \beta, \theta) dx \right],$$
$$= -\left[\int_0^\beta \ln K(\beta, \theta) g(x) dx + \int_0^\beta \ln \left(1 + \frac{\theta}{2}x^2\right) g(x) dx - \theta \int_0^\beta x g(x) dx \right],$$

Since, $\int_0^\beta g(x)dx = 1$ and $\int_0^\beta x g(x) dx = E(X)$, we have

$$= -\left[\ln K(\beta, \theta) + \int_0^\beta \ln \left(1 + \frac{\theta}{2}x^2\right) g(x) dx - \theta E(X) \right]. \quad (4.21)$$

Now we calculate $\int_0^\beta \ln\left(1+\frac{\theta}{2}x^2\right) g(x)dx$.

$$
\begin{aligned}
&\int_0^\beta \ln\left(1+\frac{\theta}{2}x^2\right) g(x)dx,\\
&= K(\beta,\theta)\int_0^\beta \ln\left(1+\frac{\theta}{2}x^2\right)\left(1+\frac{\theta}{2}x^2\right)e^{-\theta x}dx,\\
&= K(\beta,\theta)\int_0^\beta \ln\left(1+\frac{\theta}{2}x^2\right)e^{-\theta x}dx\\
&+\frac{\theta K(\beta,\theta)}{2}\int_0^\beta \ln\left(1+\frac{\theta}{2}x^2\right)x^2e^{-\theta x}dx,
\end{aligned}
$$

Using the expansion $\ln\left(1+\frac{\theta}{2}x^2\right)=\sum_{j=1}^{\infty}(-1)^{j+1}\frac{\left(\frac{\theta x^2}{2}\right)^j}{j}$, we have,

$$
\begin{aligned}
&= K(\beta,\theta)\sum_{j=1}^{\infty}(-1)^{j+1}\frac{\left(\frac{\theta}{2}\right)^j}{j}\int_0^\beta x^{2j}e^{-\theta x}dx\\
&+\frac{\theta K(\beta,\theta)}{2}\sum_{j=1}^{\infty}(-1)^{j+1}\frac{\left(\frac{\theta}{2}\right)^j}{j}\int_0^\beta x^{2j+2}e^{-\theta x}dx,\\
&= K(\beta,\theta)\sum_{j=1}^{\infty}(-1)^{j+1}\frac{\theta^j}{j2^j\theta^{2j+1}}\gamma(2j+1,\theta\beta)\\
&+\frac{\theta K(\beta,\theta)}{2}\sum_{j=1}^{\infty}(-1)^{j+1}\frac{\theta^j}{j2^j\theta^{2j+3}}\gamma(2j+3,\theta\beta),
\end{aligned}
$$

Here $\gamma(a,x)=\int_0^x z^{a-1}e^{-z}dz$ is the lower incomplete gamma function.

$$
= K(\beta,\theta)\sum_{j=1}^{\infty}(-1)^{j+1}\frac{1}{j2^j\theta^{2j+1}}\left[\gamma(2j+1,\theta\beta)+\frac{1}{2\theta}\gamma(2j+3,\theta\beta)\right].
$$

Now, using (4.21) and substituting the value of $E(X)$ from (4.9), the final form of Shannon entropy is obtained as

$$
\begin{aligned}
H(g) =&\frac{K(\beta,\theta)}{\theta}\left[\gamma(2,\theta\beta)+\frac{1}{2\theta}\gamma(4,\theta\beta)\right]-\ln K(\beta,\theta)\\
&-K(\beta,\theta)\sum_{j=0}^{\infty}(-1)^{j+1}\frac{1}{j2^j\theta^{j+1}}\left[\gamma(2j+1,\theta\beta)+\frac{1}{2\theta}\gamma(2j+3,\theta\beta)\right]. \quad (4.22)
\end{aligned}
$$

4.5 Distributions of order statistics

The distributions of order statistics play important role in obtaining system reliabilities (be it biological or mechanical) when the components are connected in series or parallel configurations.

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n drawn from $X \sim UTXG(\beta, \theta)$. Denote $X_{j:n}$ as the j^{th} order statistic. Then $X_{1:n}$ and $X_{n:n}$ denote respectively the smallest and largest order statistics for a sample of size n drawn from $UTXG(\beta, \theta)$. We use (4.5) and (4.6) for deriving the probability density functions of $X_{1:n}$ and $X_{n:n}$.

The pdf of $X_{1:n}$ is derived as

$$\begin{aligned} f_{X_{1:n}}(x;\beta,\theta) &= n[1-G(x;\beta,\theta)]^{n-1} g(x;\beta,\theta), \\ &= \frac{n\theta^2}{(1+\theta)\{F(\beta)\}^n}\left(1+\frac{\theta}{2}x^2\right) e^{-\theta x}\{F(\beta)-F(x)\}^{n-1}, 0 \le x \le \beta. \end{aligned} \tag{4.23}$$

The pdf of $X_{n:n}$ is obtained as

$$\begin{aligned} f_{X_{n:n}}(x;\beta,\theta) &= n[G(x;\beta,\theta)]^{n-1} g(x;\beta,\theta), \\ &= \frac{n\theta^2}{(1+\theta)\{F(\beta)\}^n}\left(1+\frac{\theta}{2}x^2\right) e^{-\theta x}\{F(x)\}^{n-1}, 0 \le x \le \beta. \end{aligned} \tag{4.24}$$

Here $F(x)$ is given in (2.6) and $F(\beta)$ can be obtained by putting $x = \beta$ in (2.6).

4.6 Survival properties

In this section we study important survival properties of a random variable X following $UTXG(\beta, \theta)$.

The survival or reliability function of X is given by

$$S(x;\beta,\theta) = \Pr(X > x) = \frac{F(\beta) - F(x)}{F(\beta)}, 0 \leq x \leq \beta, \tag{4.25}$$

where $F(x)$ is given in (2.6) and $F(\beta)$ can be obtained by putting $x = \beta$ in (2.6).

4.6.1 Hazard rate or failure rate function

The hazard rate (or failure rate) function is obtained as

$$h(x) = \frac{g(x;\beta,\theta)}{S(x;\beta,\theta)} = \frac{\theta^2}{(1+\theta)} \frac{\left(1 + \frac{\theta}{2}x^2\right) e^{-\theta x}}{[F(\beta) - F(x)]}, 0 \leq x \leq \beta, \tag{4.26}$$

where $F(x)$ is given in (2.6) and $F(\beta)$ can be obtained by putting $x = \beta$ in (2.6).

Now, we investigate aging property of the failure rate function in (4.26). The distribution in (4.5) is increasing failure rate (IFR) or decreasing failure rate (DFR) depending on a particular range of X. We have the following theorem.

Theorem 4.6. *The distribution, $UTXG(\beta,\theta)$, is IFR (DFR) if $x > (<)\sqrt{\frac{2}{\theta}}$ for all $\theta > 0$.*

Proof. If a continuous non-negative random variable X has pdf $f(x)$, we define

$$\eta(x) = -\frac{f(x)}{f'(x)},$$

where $f'(x)$ is the first derivative of $f(x)$ with respect to x.
Then, $f(x)$ is IFR (DFR) according as $\eta(x)$ is increasing (decreasing) in x (see Gupta, 2001 for the characterization). When $X \sim UTXG(\beta,\theta)$, let us consider

$$\eta_{UTXG}(x) = -\frac{g(x;\beta,\theta)}{g'(x;\beta,\theta)},$$

where $g'(x;\beta,\theta)$ is the first derivative of $g(x;\beta,\theta)$ with respect to x, and, when $X \sim xgamma(\theta)$, let us take

$$\eta_{XG}(x) = -\frac{f(x;\theta)}{f'(x;\theta)},$$

where $f(x;\theta)$ is the pdf of xgamma distribution in (2.2) and $f'(x;\theta)$ is the first derivative of $f(x;\theta)$ with respect to x.

Then we have,

$$\eta_{UTXG}(x) = \eta_{XG}(x) = \theta - \frac{\theta x}{\left(1 + \frac{\theta}{2}x^2\right)},$$

which gives after taking first derivative with respect to x,

$$\eta'_{XG}(x) = \frac{\theta\left(\frac{\theta}{2}x^2\right)}{\left(1 + \frac{\theta}{2}x^2\right)^2}$$

and is positive (negative) if $x > (<)\sqrt{\frac{2}{\theta}}$ for all $\theta > 0$. Hence the proof.

4.6.2 Reversed hazard rate function

The reversed hazard rate function of X is given by

$$r(x) = \frac{g(x;\beta,\theta)}{G(x;\beta,\theta)} = \frac{\theta^2}{(1+\theta)} \frac{\left(1 + \frac{\theta}{2}x^2\right) e^{-\theta x}}{F(x)}, 0 \leq x \leq \beta, \tag{4.27}$$

where $F(x)$ is given in (2.6).

We note that, if we calculate the reversed hazard rate function for $xgamma(\theta)$, we get by using (2.2) and (2.6),

$$r(x) = \frac{f(x)}{F(x)} = \frac{\theta^2}{(1+\theta)} \frac{\left(1 + \frac{\theta}{2}x^2\right) e^{-\theta x}}{F(x)}, x > 0. \tag{4.28}$$

Hence, the expression for reversed hazard rate function of $xgamma(\theta)$ in (4.28) is almost same as that of $UTXG(\beta,\theta)$ obtained in (4.27) expect that the range of the latter is restricted upto β.

4.7 Parameter estimation

In this section, we propose maximum likelihood of estimation for the unknown parameters the upper truncated xgamma distribution under complete sample situation.

Let us denote $X_1, X_2, \ldots, X_n$ be a random sample of size n and $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ be a particular realization on that.

4.7.1 Method of maximum likelihood

When $X \sim UTXG(\beta, \theta)$, let us take, on a similar fashion as earlier, $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ to denote a particular realization on a random sample of size n from it.

The likelihood function, in this case, is given by

$$L(\beta, \theta|\boldsymbol{x}) = \prod_{i=1}^{n} K(\beta, \theta) \left(1 + \frac{\theta}{2} x_i^2\right) e^{-\theta x_i}. \tag{4.29}$$

The MLE of β is $\hat{\beta} = max\{X_1, X_2, \ldots, X_n\} = X_{n:n}$, the largest order statistic. Given $\hat{\beta}$, the log-likelihood function is obtained using (4.29) as

$$\begin{aligned} \ln L(\hat{\beta}, \theta|\boldsymbol{x}) = 2n \ln \theta + \sum_{i=1}^{n} \ln \left(1 + \frac{\theta}{2} x_i^2\right) - \theta \sum_{i=1}^{n} x_i \\ - n \ln \left[(1+\theta)\left(1 - e^{-\theta\hat{\beta}}\right) - \theta\hat{\beta}\left(1 + \frac{\theta\hat{\beta}}{2}\right) e^{-\theta\hat{\beta}}\right]. \end{aligned} \tag{4.30}$$

The MLE of θ, $\hat{\theta}$ (say), is then the solution of the log-likelihood equation (taking first derivative of (4.30) and equating with 0) given by

$$\frac{2n}{\theta} + \sum_{i=1}^{n} \frac{x_i^2}{2\left(1 + \frac{\theta}{2} x_i^2\right)} - \sum_{i=1}^{n} x_i - \frac{n\left[\left(1 - e^{-\theta\hat{\beta}}\right) + \theta\hat{\beta}\left(1 + \frac{\theta\hat{\beta}^2}{2}\right) e^{-\theta\hat{\beta}}\right]}{\left[(1+\theta)\left(1 - e^{-\theta\hat{\beta}}\right) - \theta\hat{\beta}\left(1 + \frac{\theta\hat{\beta}}{2}\right) e^{-\theta\hat{\beta}}\right]} = 0, \tag{4.31}$$

which is a non-linear equation in θ and can not be solved analytically. We adopt numerical method like, *Newton-Raphson*, for solving (4.31) to obtain $\hat{\theta}$.

4.8 Application

In this section applicability of the truncated version of xgamma distribution is illustrated by a real data analysis.
Strength data of glass of aircraft window reported by Fuller et al. (1994) are been considered for the purpose. The data is presented in Table 4.2.

TABLE 4.2: Data on Glass strength of aircraft window.

18.83	20.80	21.657	23.03	23.23	24.05	24.321	25.5	25.52	25.80
26.69	26.770	26.78	27.05	27.67	29.90	31.11	33.20	33.73	33.76
33.890	34.76	35.75	35.91	36.98	37.08	37.09	39.58	44.045	45.29
45.381									

The data are fitted with exponential distribution with rate λ, Lindley distribution with parameter θ, xgamma(θ) and UTXG(β, θ) distributions. Maximum likelihood estimates are obtained for the unknown parameters in each model. As model comparison criteria, we have considered negative log-likelihood values, AIC and BIC.
Lower the value of AIC and/or BIC, better is the model. The result of the data analysis is shown in Table 4.3. Statistical software R is utilized for data analysis. In Table 4.3, estimates along with the standard error (Std. Error) of estimation in parentheses, model selection criteria are shown.
From Table 4.3, it is clear that $UTXG(\beta, \theta)$ distribution provides best fit for the given data set and shows improved description over xgamma distribution in application, as expected. Hence, we can conclude that the truncated version of xgamma distribution can be a better way to justify real data modeling as compared to xgamma and other popular life distributions in particular situations.

Table 4.3: Estimates of the parameters and model selection criteria for glass strength data.

Distributions	Estimate(Std. Error)	-Log-likelihood	AIC	BIC
Exponential(λ)	$\hat{\lambda}$=0.0324(0.0058)	137.26	276.53	277.96
Lindley(θ)	$\hat{\theta}$=0.0629(0.0080)	126.99	255.99	257.42
Xgamma(θ)	$\hat{\theta}$=0.0937(0.0098)	122.27	246.55	247.98
UTXG(β, θ)	$\hat{\beta}$=45.381 $\hat{\theta}$=0.0349(0.0166)	109.66	221.33	222.76

4.9 Conclusion

In this chapter, truncated versions, called as upper truncated, lower truncated and double truncated, of xgamma distribution are introduced. Particularly, the different distributional and survival properties of the upper truncated xgamma distribution are been studied in details. The maximum likelihood method is suggested for estimating the unknown parameters of upper truncated xgamma distribution.

The following important findings are observed in this chapter.

1. Upper truncated xgamma distribution is unimodal. Moreover, the distribution is sometimes IFR and sometimes DFR depending on the particular range of the concerned random variable.
2. Real data illustration shows that upper truncated version of xgamma distribution can be better alternative in modeling lifetime data sets compared to the some other popular lifetime models.

Although, the upper truncated xgamma distribution, studied in this chapter, has showed added flexibility in modeling lifetime data, it shows certain restrictions in the form of moment expressions and in the basic distributional form. The applicability of such truncated version of xgamma lives is also restricted to specific situations where designed lifetime is deliberately finite in its upper range.

Moreover, it is not straightforward to construct a simulation algorithm by applying available methods in literature for generating random samples from upper truncated xgamma distribution, the expression for the MRL function of the distribution is difficult to derive in an user friendly form and hence studies of the important properties in comparison to xgamma model are quite paralyzed in the connection. However, the upper truncated xgamma distribution can be utilized in real data analysis where lifetime is truncated for some specific purpose where direct xgamma life is not appropriate.

Chapter 5

Weighted xgamma distribution

In the previous chapters, we have introduced and studied the xgamma distribution and its truncated versions mainly the upper truncated one. It is observed in Chapter 4 that the upper truncated version of xgamma distribution provides some flexibility over xgamma for modeling time-to-event data set. However, as mentioned in the last paragraph of Chapter 4, the applicability of upper truncated xgamma distribution is very specific. In this chapter, we study a weighted version of xgamma model and try to find an application in lifetime data. The study of weighted distributions is useful for two main purposes, it could provide a new understanding of the baseline distribution on which weight is considered and it might provide a method of extending the baseline distribution for added flexibility in fitting data. We discuss below the concept and applicability of weighted distributions.

The concept of weighted distributions can be traced back to Fisher (1934) in the study of the effect of methods of ascertainment upon estimation of frequencies. While extending the basic ideas of Fisher, Rao (1965, 1985) saw the need for a unifying concept by identifying various sampling situations that can be modeled by what he termed as weighted distributions. Zelen (1974) introduced weighted distributions to represent what he broadly perceived as length-biased sampling in the context of cell kinetics and the early detection of disease.

In a series of articles with other co-authors, Patil has extensively pursued weighted distributions for purposes of encountered data analysis, equilibrium population analysis subject to harvesting and predation, meta-analysis incorporating publication bias and heterogeneity, modeling clustering and extraneous variation, etc., see for more details on these applications Dennis and Patil (1984), Laird et al. (1988), Patil (1981, 1991, 1996), Patil and Ord (1976), Patil and Rao (1978), Patil and Taillie (1988), Patil et al. (1993), Taillie et al. (1995) and references therein. More references can be seen in Patil (1997).

In this chapter, the weighted version of xgamma distribution as a generalization of xgamma distribution (see Chapter 2 and Chapter 3) is studied, with special reference study is made to its length biased version. The method of moments and method of maximum likelihood estimation are proposed to estimate the unknown parameter of the length biased xgamma distribution. The length biased xgamma distribution is applied for modeling time-to-event data set and compared with other life distributions for complete sample situation.

5.1 Synthesis of weighted xgamma distribution

The form of the pdf of weighted distribution, by definition (see Patil, 1988), is given by

$$f(x) = \frac{w(x) f_0(x)}{E[w(X)]}, \tag{5.1}$$

where $w(x)$ is weight function which in non-negative and $f_0(x)$ is a probability density function.

To synthesize the weighted version of xgamma distribution, we take $w(x) = x^r$ for $r = 1, 2, 3, \ldots,$ and $f_0(x)$ is taken as the pdf of xgamma distribution as in (2.2).

Note that, taking $w(x) = x^r$ for $r = 1, 2, 3, \ldots$, $E[w(X)]$ is nothing but the r^{th} order non-central (raw) moment of xgamma distribution given in (2.10), i.e.,

$$E(X^r) = \frac{r!}{\theta^r(1+\theta)}\left[\theta + \frac{(1+r)(2+r)}{2}\right], \text{ for } r = 1, 2, 3, \ldots.$$

Now using (5.1) and putting the expressions for $w(x)$, $E[w(X)]$ and $f_0(x)$, the pdf of r^{th} order moment weighted version of xgamma distribution can be derived as

$$\begin{aligned} f(x) &= \frac{x^r \frac{\theta^2}{(1+\theta)}\left(1 + \frac{\theta}{2}x^2\right)e^{-\theta x}}{\frac{r!}{\theta^r(1+\theta)}\left[\theta + \frac{(1+r)(2+r)}{2}\right]}, \\ &= \frac{2\theta^{r+2}}{r![2\theta + (1+r)(2+r)]}\left(x^r + \frac{\theta}{2}x^{r+2}\right)e^{-\theta x}. \end{aligned}$$

We have the following definition for the weighted xgamma distribution.

Definition 5.1. A non-negative continuous random variable, X, is said to follow weighted xgamma (WXG) distribution with parameters r and θ if its pdf is of the form

$$f(x) = \frac{2\theta^{r+2}}{r![2\theta + (1+r)(2+r)]}\left(x^r + \frac{\theta}{2}x^{r+2}\right)e^{-\theta x}, x > 0, \theta > 0, r = 1, 2, 3, \ldots. \tag{5.2}$$

It is denoted by $X \sim WXG(r, \theta)$.

The Figure 5.1 shows the plot of density functions for weighted xgamma distribution for different values of r and θ.

Non-central moments

Now, we find the non-central moments for $WXG(r, \theta)$.

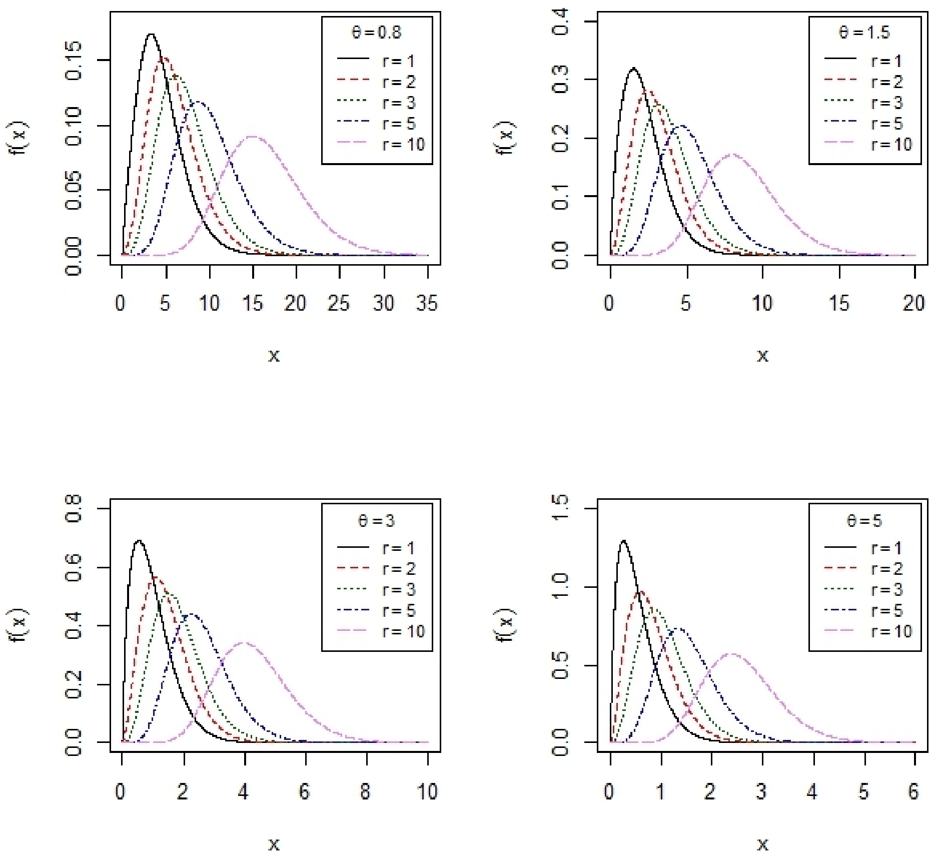

Figure 5.1: Probability density curves of weighted xgamma distribution for different values of θ and r

The k^{th}, for $k = 1, 2, 3, \ldots$, order non-central moment of $WXG(r, \theta)$ can be obtained as

$$\begin{aligned}\mu_k' = E[X^k] &= \int_0^\infty x^k f(x) dx, \\ &= \frac{2\theta^{r+2}}{r![2\theta + (1+r)(2+r)]} \int_0^\infty x^k \left(x^r + \frac{\theta}{2} x^{r+2}\right) e^{-\theta x} dx, \\ &= \frac{2\theta^{r+2}}{r![2\theta + (1+r)(2+r)]} \left[\int_0^\infty x^{r+k} e^{-\theta x} dx + \frac{\theta}{2} \int_0^\infty x^{r+k+2} e^{-\theta x} dx\right].\end{aligned}$$

Hence, we have,

$$\mu'_k = \frac{2\theta^{r+2}}{r![2\theta + (1+r)(2+r)]}\left[\frac{\Gamma(r+k+1)}{\theta^{r+k+1}} + \frac{\theta}{2}\frac{\Gamma(r+k+3)}{\theta^{r+k+3}}\right],$$

Here $\Gamma(a) = \int_0^\infty z^{a-1}e^{-z}dz$ is the gamma function.

$$= \frac{2\theta^{r+2}}{r![2\theta + (1+r)(2+r)]}\left[\frac{(r+k)!}{\theta^{r+k+1}} + \frac{(r+k+2)!}{2\theta^{r+k+2}}\right],$$

$$= \frac{2\theta^{r+2}(r+k)!}{r![2\theta + (1+r)(2+r)]\theta^{r+k+1}}\left[1 + \frac{(r+k+2)(r+k+1)}{2\theta}\right],$$

$$= \frac{(r+k)!}{r![2\theta + (1+r)(2+r)]\theta^{k-1}}\left[\frac{2\theta + (r+k+2)(r+k+1)}{\theta}\right],$$

$$= \frac{(r+k)!}{r!\theta^k}\frac{[2\theta + (1+r+k)(2+r+k)]}{[2\theta + (r+1)(r+2)]}. \tag{5.3}$$

In particular, by putting $k = 1$ in (5.3), the mean of $WXG(r, \theta)$ distribution is obtained as

$$E(X) = \frac{(r+1)!}{r!\theta}\frac{[2\theta + (r+2)(r+3)]}{[2\theta + (r+1)(r+2)]},$$

$$= \frac{(r+1)}{\theta}\frac{[2\theta + (r+2)(r+3)]}{[2\theta + (r+1)(r+2)]}. \tag{5.4}$$

Similarly, putting $k = 2$ in (5.3), the second order raw moment for $WXG(r, \theta)$ is obtained as

$$\mu'_2 = E(X^2) = \frac{(r+1)(r+2)}{\theta^2}\frac{[2\theta + (r+3)(r+4)]}{[2\theta + (r+1)(r+2)]}. \tag{5.5}$$

Next, we find the expressions for cdf, survival function and hazard rate function of $WXG(r, \theta)$.

Cumulative distribution function

The cdf of $WXG(r,\theta)$ can be obtained as

$$\begin{aligned}
F(x) &= P(X \leq x), \\
&= \int_0^x \frac{2\theta^{r+2}}{r![2\theta+(1+r)(2+r)]}\left(t^r + \frac{\theta}{2}t^{r+2}\right)e^{-\theta t}dt, \\
&\text{Putting } \theta t = u, \text{ we have,} \\
&= \frac{2\theta^{r+2}}{r![2\theta+(1+r)(2+r)]}\left[\int_0^{\theta x}\left(\frac{u}{\theta}\right)^r e^{-u}\frac{du}{\theta} + \frac{\theta}{2}\int_0^{\theta x}\left(\frac{u}{\theta}\right)^{r+2} e^{-u}\frac{du}{\theta}\right], \\
&= \frac{2\theta^{r+2}}{r![2\theta+(1+r)(2+r)]}\left[\frac{1}{\theta^{r+1}}\int_0^{\theta x} u^r e^{-u}du + \frac{1}{2\theta^{r+2}}\int_0^{\theta x} u^{r+2}e^{-u}du\right], \\
&= \frac{2\theta}{r![2\theta+(1+r)(2+r)]}\left[\int_0^{\theta x} u^{r+1-1}e^{-u}du + \frac{1}{2\theta}\int_0^{\theta x} u^{r+3-1}e^{-u}du\right], \\
&= \frac{2\theta}{r![2\theta+(1+r)(2+r)]}\left[\gamma(r+1,\theta x) + \frac{1}{2\theta}\gamma(r+3,\theta x)\right], \qquad (5.6)
\end{aligned}$$

where $\gamma(a,x) = \int_0^x u^{a-1}e^{-u}du$ is the lower incomplete gamma function.

Survival function

The survival function of $WXG(r,\theta)$ can be derived as

$$\begin{aligned}
S(x) &= \Pr(X > x), \\
&= \int_x^\infty \frac{2\theta^{r+2}}{r![2\theta+(1+r)(2+r)]}\left(u^r + \frac{\theta}{2}u^{r+2}\right)e^{-\theta u}du, \\
&\text{Putting } \theta t = u, \text{ we have,} \\
&= \frac{2\theta^{r+2}}{r![2\theta+(1+r)(2+r)]}\left[\int_{\theta x}^{\infty}\left(\frac{u}{\theta}\right)^r e^{-u}\frac{du}{\theta} + \frac{\theta}{2}\int_{\theta x}^{\infty}\left(\frac{u}{\theta}\right)^{r+2} e^{-u}\frac{du}{\theta}\right], \\
&= \frac{2\theta^{r+2}}{r![2\theta+(1+r)(2+r)]}\left[\frac{1}{\theta^{r+1}}\int_{\theta x}^{\infty} u^r e^{-u}du + \frac{1}{2\theta^{r+2}}\int_{\theta x}^{\infty} u^{r+2}e^{-u}du\right], \\
&= \frac{2\theta}{r![2\theta+(1+r)(2+r)]}\left[\int_{\theta x}^{\infty} u^{r+1-1}e^{-u}du + \frac{1}{2\theta}\int_{\theta x}^{\infty} u^{r+3-1}e^{-u}du\right], \\
&= \frac{2\theta}{r![2\theta+(1+r)(2+r)]}\left[\Gamma(r+1,\theta x) + \frac{1}{2\theta}\Gamma(r+3,\theta x)\right], \qquad (5.7)
\end{aligned}$$

where $\Gamma(a,x) = \int_x^\infty u^{a-1}e^{-u}du$ is the upper incomplete gamma function.

Hazard rate or failure rate function

The failure rate or hazard rate function of $WXG(r,\theta)$ is obtained as

$$h(x)=\frac{f(x)}{S(x)}=\frac{\theta^{r+1}\left(x^r+\frac{\theta}{2}x^{r+2}\right)e^{-\theta x}}{\left[\Gamma(r+1,\theta x)+\frac{1}{2\theta}\Gamma(r+3,\theta x)\right]}\;;x>0,r=1,2,3,\ldots. \quad (5.8)$$

The main emphasis is given in studying the length biased version of xgamma distribution hereafter. The rest of the chapter is organized as follows.

The length biased version for xgamma distribution is described along with its moments and related measures in section 5.2. Distributions of order statistics for length biased xgamma distribution are derived in section 5.3. Important entropy measures are described in section 5.4 and different survival properties are studied in section 5.5 for length biased version of xgamma distribution. Section 5.6 deals with the methods of estimation for the unknown parameter in length biased xgamma model for complete sample case. An algorithm for generating random samples from length biased xgamma along with a Monte-Carlo simulation study is presented in section 5.7. Real data illustration is described in section 5.8 for studying the application of length biased xgamma model. Finally, the section 5.9 summaries the chapter mentioning important finding and some open research problems for future investigation.

5.2 The length biased xgamma distribution

This section deals with the length biased version of xgamma distribution. The length biased version of xgamma distribution is obtained as a special case of weight xgamma distribution discussed in the previous section.

If we put $r=1$ in (5.2), then we obtain so called length biased version of the xgamma distribution. We have the following definition for the length biased xgamma distribution.

Definition 5.2. A non-negative continuous random variable, X, is said to follow the length biased xgamma (LBXG) distribution with parameter θ if its pdf is of the form

$$f(x) = \frac{\theta^3}{(\theta+3)}\left(x + \frac{\theta}{2}x^3\right)e^{-\theta x}, x > 0, \theta > 0. \tag{5.9}$$

It is denoted by $X \sim LBXG(\theta)$.

We note that the length biased xgamma distribution is a special mixture of $gamma(2, \theta)$ and $gamma(4, \theta)$ with mixing proportions $\theta/(3+\theta)$ and $3/(3+\theta)$, respectively.
The probability density plots of $LBXG(\theta)$ for different values of θ is shown in Figure 5.2.

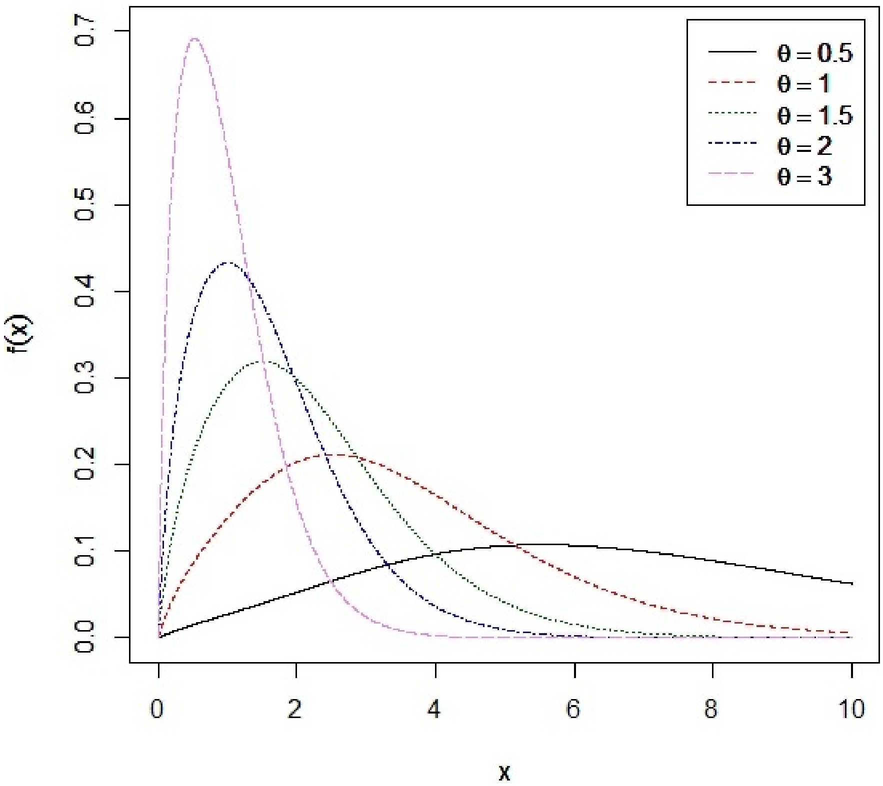

FIGURE 5.2: Probability density function of length biased xgamma distribution for different values of θ.

Now, for finding cdf of $LBXG(\theta)$, we calculate the followings.

$$F(x) = \Pr(X \leq x) = 1 - \Pr(X > x).$$

Now, we find $\Pr(X > x)$.

$$\begin{aligned}\Pr(X > x) &= \int_x^\infty \frac{\theta^3}{(\theta+3)}\left(t + \frac{\theta}{2}t^3\right)e^{-\theta t}dt,\\ &= \frac{\theta^3}{(\theta+3)}\left[\int_x^\infty te^{-\theta t}dt + \frac{\theta}{2}\int_x^\infty t^3e^{-\theta t}dt\right].\end{aligned} \tag{5.10}$$

Now, integrating by parts, we can have,

$$\begin{aligned}\int_x^\infty t^3e^{-\theta t}dt &= \frac{x^3e^{-\theta x}}{\theta} + \frac{3}{\theta}\int_x^\infty t^2e^{-\theta t}dt,\\ &= \frac{x^3e^{-\theta x}}{\theta} + \frac{3}{\theta}\left\{\frac{x^2e^{-\theta x}}{\theta} + \frac{2}{\theta}\left(\frac{xe^{-\theta x}}{\theta} + \frac{e^{-\theta x}}{\theta^2}\right)\right\},\\ &= \frac{x^3e^{-\theta x}}{\theta} + \frac{3x^2e^{-\theta x}}{\theta^2} + \frac{6xe^{-\theta x}}{\theta^3} + \frac{6e^{-\theta x}}{\theta^4}.\end{aligned} \tag{5.11}$$

So, using (2.4) and (5.11), from (5.10), we have,

$$\begin{aligned}&\Pr(X > x)\\ &= \frac{\theta^3}{(\theta+3)}\left[\frac{xe^{-\theta x}}{\theta} + \frac{e^{-\theta x}}{\theta^2} + \frac{\theta}{2}\left(\frac{x^3e^{-\theta x}}{\theta} + \frac{3x^2e^{-\theta x}}{\theta^2} + \frac{6xe^{-\theta x}}{\theta^3} + \frac{6e^{-\theta x}}{\theta^4}\right)\right],\\ &= \frac{\theta^3}{(\theta+3)}\left[\frac{xe^{-\theta x}}{\theta} + \frac{e^{-\theta x}}{\theta^2} + \frac{x^3e^{-\theta x}}{2} + \frac{3x^2e^{-\theta x}}{2\theta} + \frac{3xe^{-\theta x}}{\theta^2} + \frac{3e^{-\theta x}}{\theta^3}\right],\\ &= \frac{\theta^3e^{-\theta x}}{(\theta+3)}\left[\frac{2\theta^2x + 2\theta + \theta^3x^3 + 3\theta^2x^2 + 6\theta x + 6}{2\theta^3}\right],\\ &= \frac{e^{-\theta x}}{(\theta+3)}\left[(3+\theta) + (3+\theta)\theta x + \frac{3}{2}\theta^2x^2 + \frac{1}{2}\theta^3x^3\right].\end{aligned}$$

Hence, the cdf of $X \sim LBXG(\theta)$ is given by

$$F(x) = 1 - \frac{\left[(3+\theta) + (3+\theta)\theta x + \frac{3}{2}\theta^2x^2 + \frac{1}{2}\theta^3x^3\right]}{(\theta+3)}e^{-\theta x}, x > 0. \tag{5.12}$$

The characteristic function of $LBXG(\theta)$ is obtained as

$$\begin{aligned}
\phi_X(t) &= E\left[e^{itX}\right], \\
&= \frac{\theta^3}{(\theta+3)} \int_0^\infty e^{itx}\left(x+\frac{\theta}{2}x^3\right)e^{-\theta x}dx, \\
&= \frac{\theta^3}{(\theta+3)}\left[\int_0^\infty x e^{-(\theta-it)x}dx + \frac{\theta}{2}\int_0^\infty x^3 e^{-(\theta-it)x}dx\right], \\
&= \frac{\theta^3}{(\theta+3)}\left[\frac{\Gamma(2)}{(\theta-it)^2} + \frac{\theta\Gamma(4)}{2(\theta-it)^4}\right],
\end{aligned}$$

Here $\Gamma(a) = \int_0^\infty z^{a-1}e^{-z}dz$ is the gamma function.

$$\begin{aligned}
&= \frac{\theta^3}{(\theta+3)}\left[\frac{1}{(\theta-it)^2} + \frac{3\theta}{(\theta-it)^4}\right], \\
&= \frac{\theta^3}{(\theta+3)}\left[(\theta-it)^{-2} + 3\theta(\theta-it)^{-4}\right]; i=\sqrt{-1}, t\in\Re. \qquad (5.13)
\end{aligned}$$

5.2.1 Moments and associated measures

Now, we find the moments and measures related to moments of $LBXG(\theta)$.

The k^{th} order raw moment, μ_k' for $k = 1, 2, 3, \ldots$, of length-biased xgamma distribution can be obtained either directly using the pdf in (5.9) or by substituting $k = 1, 2, 3, \ldots$ in (5.3) after putting $r = 1$.

Hence, we have

$$\mu_k' = E(X^k) = \frac{(k+1)!\,[2\theta + (2+k)(3+k)]}{2\theta^k(\theta+3)} \quad \text{for} \quad k = 1, 2, 3, \ldots. \qquad (5.14)$$

In particular,

$$E(X) = \frac{2(\theta+6)}{\theta(\theta+3)} \quad ; \quad E(X^2) = \frac{6(\theta+10)}{\theta^2(\theta+3)}. \qquad (5.15)$$

So, we have the expression for second order central moment or the population variance for X as

$$\begin{aligned} Var(X) = \mu_2 &= \mu_2' - \mu_1'^2, \\ &= \frac{6(\theta+10)}{\theta^2(\theta+3)} - \left[\frac{2(\theta+6)}{\theta(\theta+3)}\right]^2, \\ &= \frac{6(\theta+10)(\theta+3) - 4(\theta+6)^2}{\theta^2(\theta+3)^2}, \\ &\text{On simplification, we have,} \\ &= \frac{2(\theta^2+15\theta+18)}{\theta^2(\theta+3)^2} \end{aligned} \tag{5.16}$$

so that the coefficient of variation (CV) becomes

$$\gamma = \frac{\sqrt{Var(X)}}{E(X)} = \frac{\sqrt{2(\theta^2+15\theta+18)}}{2(\theta+6)}. \tag{5.17}$$

The moment generating function of X is derived as

$$\begin{aligned} M_X(t) &= E\left[e^{tX}\right], \\ &= \frac{\theta^3}{(\theta+3)} \int_0^\infty e^{tx}\left(x + \frac{\theta}{2}x^3\right) e^{-\theta x} dx, \\ &= \frac{\theta^3}{(\theta+3)} \left[\int_0^\infty x e^{-(\theta-t)x} dx + \frac{\theta}{2}\int_0^\infty x^3 e^{-(\theta-t)x} dx\right], \\ &= \frac{\theta^3}{(\theta+3)} \left[\frac{\Gamma(2)}{(\theta-t)^2} + \frac{\theta\Gamma(4)}{2(\theta-t)^4}\right], \\ &\text{Here } \Gamma(a) = \int_0^\infty z^{a-1}e^{-z}dz \text{ is the gamma function.} \\ &= \frac{\theta^3}{(\theta+3)} \left[\frac{1}{(\theta-t)^2} + \frac{3\theta}{(\theta-t)^4}\right], \\ &= \frac{\theta^3}{(\theta+3)} \left[(\theta-t)^{-2} + 3\theta(\theta-t)^{-4}\right]; t \in \Re. \end{aligned} \tag{5.18}$$

The cumulant generating function of X is obtained as

$$\begin{aligned}
K_X(t) &= \ln M_X(t), \\
&= \ln \frac{\theta^3}{(\theta+3)} \left[(\theta-t)^{-2} + 3\theta(\theta-t)^{-4}\right], \\
&= \ln \frac{\theta^3}{(\theta+3)(\theta-t)^2} + \ln\left[1+3\theta(\theta-t)^{-2}\right]; t \in \Re. \qquad (5.19)
\end{aligned}$$

5.3 Distributions of order statistics

In this section, we find the distributions of extreme order statistics for $LBXG(\theta)$.

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n drawn from $X \sim LBXG(\theta)$. Denote $X_{j:n}$ as the j^{th} order statistic. Then $X_{1:n}$ and $X_{2:n}$ denote the smallest and largest order statistics for a sample of size n drawn from length-biased xgamma distribution with parameter θ, respectively.
For any $x > 0$, the pdf of $X_{1:n}$ is derived as

$$\begin{aligned}
f_{X_{1:n}}(x) &= n[1-F(x)]^{n-1} f(x), \\
&= \frac{n\theta^3\left(x+\frac{\theta}{2}x^3\right)}{(\theta+3)^n}\left[(3+\theta)+(3+\theta)\theta x+\frac{3}{2}\theta^2x^2+\frac{1}{2}\theta^3x^3\right]^{n-1} e^{-n\theta x}. \qquad (5.20)
\end{aligned}$$

Similarly, for any $x > 0$, the pdf of $X_{n:n}$ is obtained as

$$\begin{aligned}
f_{X_{n:n}}(x) &= n[F(x)]^{n-1} f(x), \\
&= \frac{n\theta^3\left(x+\frac{\theta}{2}x^3\right)}{(\theta+3)}\left[1-\frac{(3+\theta)+(3+\theta)\theta x+\frac{3}{2}\theta^2x^2+\frac{1}{2}\theta^3x^3}{(\theta+3)}e^{-\theta x}\right]^{n-1} e^{-\theta x},
\end{aligned}$$

after simple arrangements,

$$= \frac{n\theta^3\left(x+\frac{\theta}{2}x^3\right)}{(\theta+3)^n}\left[(\theta+3)\{1-e^{-\theta x}(1+\theta x)\}-\frac{1}{2}\theta^2x^2e^{-\theta x}(3+\theta x)\right]^{n-1} e^{-\theta x}. \qquad (5.21)$$

5.4 Entropy measures

We first derive the Rényi entropy measure when $X \sim LBXG(\theta)$. We derive,

$$\begin{aligned}
&\int_0^\infty f^\gamma(x)dx \\
&= \int_0^\infty \left[\frac{\theta^3}{(\theta+3)}\left(x+\frac{\theta}{2}x^3\right)e^{-\theta x}\right]^\gamma dx, \text{ for } \gamma > 0(\neq 1), \\
&= \frac{\theta^{3\gamma}}{(\theta+3)^\gamma}\int_0^\infty \left(x+\frac{\theta}{2}x^3\right)^\gamma e^{-\gamma\theta x}dx, \\
&= \frac{\theta^{3\gamma}}{(\theta+3)^\gamma}\int_0^\infty x^\gamma\left(1+\frac{\theta}{2}x^2\right)^\gamma e^{-\gamma\theta x}dx, \\
&\text{Putting } \left(1+\frac{\theta}{2}x^2\right)^\gamma = \sum_{j=0}^{\gamma}\binom{\gamma}{j}\left(\frac{\theta x^2}{2}\right)^j, \\
&= \frac{\theta^{3\gamma}}{(\theta+3)^\gamma}\sum_{j=0}^{\gamma}\binom{\gamma}{j}\int_0^\infty \left(\frac{\theta}{2}\right)^j x^{2j+\gamma}e^{-\gamma\theta x}dx, \\
&= \frac{\theta^{3\gamma}}{(\theta+3)^\gamma}\sum_{j=0}^{\gamma}\binom{\gamma}{j}\frac{\Gamma(2j+\gamma+1)}{2^j\theta^{j+\gamma+1}\gamma^{2j+\gamma+1}}, \\
&\text{Here } \Gamma(a) = \int_0^\infty z^{a-1}e^{-z}dz \text{ is the gamma function.} \\
&= \frac{\theta^{3\gamma}}{(\theta+3)^\gamma}\sum_{j=0}^{\gamma}\binom{\gamma}{j}\frac{\Gamma(2j+\gamma+1)}{2^j\theta^{j+\gamma+1}\gamma^{2j+\gamma+1}}
\end{aligned}$$

to obtain Rényi entropy as

$$\begin{aligned}
H_R(\gamma) &= \frac{1}{1-\gamma}\ln\left[\int_0^\infty f^\gamma(x)dx\right], \\
&= \frac{1}{1-\gamma}\left[3\gamma\ln\theta - \gamma\ln(\theta+3)\right] + \frac{1}{1-\gamma}\ln\left[\sum_{j=0}^{\gamma}\binom{\gamma}{j}\frac{\Gamma(2j+\gamma+1)}{2^j\theta^{j+\gamma+1}\gamma^{2j+\gamma+1}}\right].
\end{aligned} \tag{5.22}$$

Now, when $X \sim LBXG(\theta)$, to obtain Tallis measure of entropy, defined by

$$S_q(X) = \frac{1}{q-1}\ln\left[1 - \int_0^\infty f^q(x)dx\right] \text{ for } q > 0\,(\neq 1),$$

we calculate,

$$
\begin{aligned}
&\int_0^\infty f^q(x)dx\\
&=\int_0^\infty \left[\frac{\theta^3}{(\theta+3)}\left(x+\frac{\theta}{2}x^3\right)e^{-\theta x}\right]^q dx,\\
&=\frac{\theta^{3q}}{(\theta+3)^q}\int_0^\infty \left(x+\frac{\theta}{2}x^3\right)^q e^{-q\theta x}dx,\\
&=\frac{\theta^{3q}}{(\theta+3)^q}\int_0^\infty x^q\left(1+\frac{\theta}{2}x^2\right)^q e^{-q\theta x}dx,\\
&\text{Putting } \left(1+\frac{\theta}{2}x^2\right)^q=\sum_{j=0}^{q}\binom{q}{j}\left(\frac{\theta x^2}{2}\right)^j,\\
&=\frac{\theta^{3q}}{(\theta+3)^q}\sum_{j=0}^{q}\binom{q}{j}\int_0^\infty \left(\frac{\theta}{2}\right)^j x^{2j+q}e^{-q\theta x}dx,\\
&=\frac{\theta^{3q}}{(\theta+3)^q}\sum_{j=0}^{q}\binom{q}{j}\frac{\Gamma(2j+q+1)}{2^j\theta^{j+q+1}q^{2j+q+1}},\\
&\text{Here } \Gamma(a)=\int_0^\infty z^{a-1}e^{-z}dz \text{ is the gamma function.}\\
&=\frac{\theta^{3q}}{(\theta+3)^q}\sum_{j=0}^{q}\binom{q}{j}\frac{\Gamma(2j+q+1)}{2^j\theta^{j+q+1}q^{2j+q+1}}.
\end{aligned}
$$

Hence, the final form of Tsallis entropy is given by

$$
S_q(x)=\frac{1}{1-q}\left[1-\frac{\theta^{3q}}{(\theta+3)^q}\sum_{j=0}^{q}\binom{q}{j}\frac{\Gamma(2j+q+1)}{2^j\theta^{j+q+1}q^{2j+q+1}}\right]. \tag{5.23}
$$

5.5 Survival properties

In this section we study survival properties of $LBXG(\theta)$.

The survival function of $X\sim LBXG(\theta)$ is obtained as

$$
S(x)=\Pr(X>x)=\frac{\left[(3+\theta)+(3+\theta)\theta x+\frac{3}{2}\theta^2x^2+\frac{1}{2}\theta^3x^3\right]}{(\theta+3)}e^{-\theta x}, x>0. \tag{5.24}
$$

5.5.1 Hazard rate or failure rate function

The hazard rate (or failure rate) function is obtained as

$$h(x) = \frac{f(x)}{S(x)} = \frac{\theta^3 \left(x + \frac{\theta}{2}x^3\right)}{\left[(3+\theta) + (3+\theta)\theta x + \frac{3}{2}\theta^2 x^2 + \frac{1}{2}\theta^3 x^3\right]}, x > 0. \tag{5.25}$$

The hazard rate plots for different values of θ is shown in the Figure 5.3. It is observed that the hazard rate function in (5.25) is increasing in θ and x.

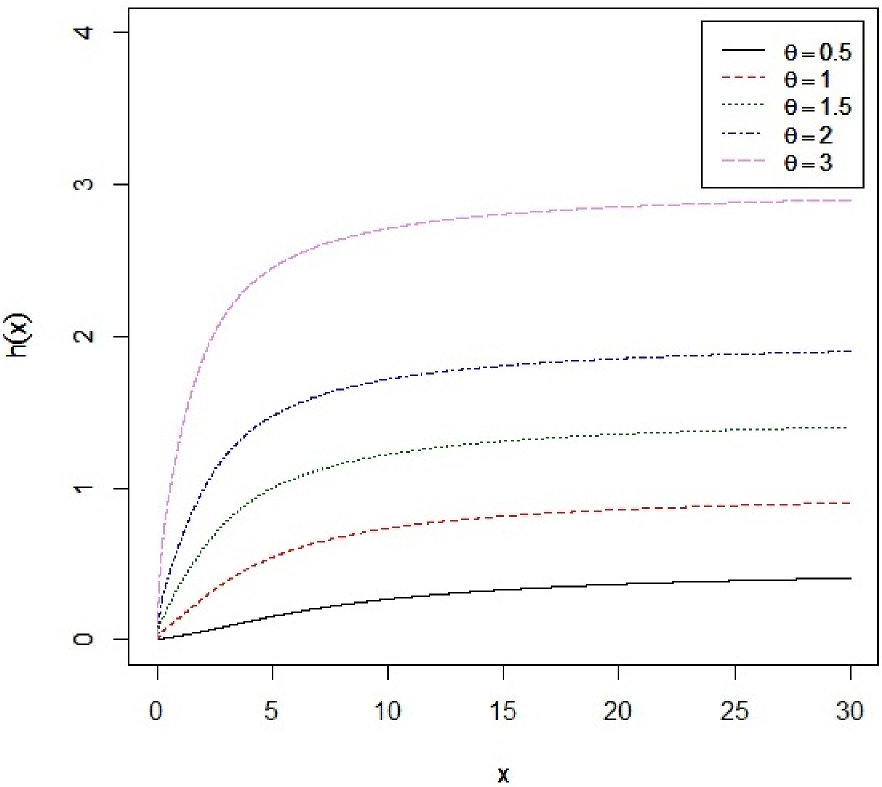

FIGURE 5.3: Hazard rate function of length biased xgamma distribution for different values of θ.

5.5.2 MRL function

When $X \sim LBXG(\theta)$, the MRL function can be derived as below.

$$\begin{aligned} m(x) &= \frac{1}{S(x)} \int_x^\infty S(t)dt, \\ &= \frac{1}{(\theta+3)S(x)} \int_x^\infty \left[(3+\theta) + (3+\theta)\theta t + \frac{3}{2}\theta^2 t^2 + \frac{1}{2}\theta^3 t^3\right] e^{-\theta t} dt. \end{aligned} \quad (5.26)$$

Now,

$$\begin{aligned} &\int_x^\infty \left[(3+\theta) + (3+\theta)\theta t + \frac{3}{2}\theta^2 t^2 + \frac{1}{2}\theta^3 t^3\right] e^{-\theta t} dt \\ &= (3+\theta)\int_x^\infty e^{-\theta t}dt + (3+\theta)\theta \int_x^\infty t e^{-\theta t}dt + \frac{3}{2}\theta^2 \int_x^\infty t^2 e^{-\theta t}dt + \frac{\theta^3}{2}\int_x^\infty t^3 e^{-\theta t}dt, \end{aligned}$$

Using the expressions of integration in (2.3), (2.4), (2.5) and (**??**), we have,

$$\begin{aligned} &= (3+\theta)\frac{e^{-\theta x}}{\theta} + (3+\theta)\theta\left(\frac{xe^{-\theta x}}{\theta} + \frac{e^{-\theta x}}{\theta^2}\right) + \frac{3\theta^2}{2}\left\{\frac{x^2 e^{-\theta x}}{\theta} + \frac{2}{\theta}\left(\frac{xe^{-\theta x}}{\theta} + \frac{e^{-\theta x}}{\theta^2}\right)\right\} \\ &+ \frac{\theta^3}{2}\left(\frac{x^3 e^{-\theta x}}{\theta} + \frac{3x^2 e^{-\theta x}}{\theta^2} + \frac{6xe^{-\theta x}}{\theta^3} + \frac{6e^{-\theta x}}{\theta^4}\right), \\ &= e^{-\theta x}\left[\frac{\theta+3}{\theta} + \frac{(\theta+3)(1+\theta x)}{\theta} + \frac{3(\theta^2x^2 + 2\theta x + 2)}{2\theta} + \frac{(\theta^3x^3 + 3\theta^2x^2 + 6\theta x + 6)}{2\theta}\right], \end{aligned}$$

On simplification,

$$\begin{aligned} &= e^{-\theta x}\left[\frac{4\theta + 24 + 2\theta^2 x + 18\theta x + 6\theta^2x^2 + \theta^3x^3}{2\theta}\right], \\ &= \frac{e^{-\theta x}}{\theta}\left[2\theta + 12 + \theta^2 x + 9\theta x + 3\theta^2x^2 + \frac{1}{2}\theta^3x^3\right]. \end{aligned}$$

Using (5.26), we have then,

$$\begin{aligned} m(x) &= \frac{1}{(\theta+3)S(x)}\frac{e^{-\theta x}}{\theta}\left[2\theta + 12 + \theta^2 x + 9\theta x + 3\theta^2x^2 + \frac{1}{2}\theta^3x^3\right], \\ &= \frac{\left[2\theta + 12 + \theta^2 x + 9\theta x + 3\theta^2x^2 + \frac{1}{2}\theta^3x^3\right]}{\theta\left[(3+\theta) + (3+\theta)\theta x + \frac{3}{2}\theta^2x^2 + \frac{1}{2}\theta^3x^3\right]}. \end{aligned}$$

On adjustment of the numerator, we have,

$$m(x) = \frac{\left[(3+\theta) + (3+\theta)\theta x + \frac{3}{2}\theta^2 x^2 + \frac{1}{2}\theta^3 x^3\right] + \left(\theta + 9 + 6\theta x + \frac{3}{2}\theta^2 x^2\right)}{\theta\left[(3+\theta) + (3+\theta)\theta x + \frac{3}{2}\theta^2 x^2 + \frac{1}{2}\theta^3 x^3\right]},$$

$$= \frac{1}{\theta} + \frac{\theta + 9 + 6\theta x + \frac{3}{2}\theta^2 x^2}{\theta\left[(3+\theta) + (3+\theta)\theta x + \frac{3}{2}\theta^2 x^2 + \frac{1}{2}\theta^3 x^3\right]}.$$

Hence, the MRL function is given by

$$m(x) = \frac{1}{\theta} + \frac{(\theta+3) + 6(1+\theta x) + \frac{3}{2}\theta^2 x^2}{\theta\left[(\theta+3) + (\theta+3)\theta x + \frac{3}{2}\theta^2 x^2 + \frac{1}{2}\theta^3 x^3\right]}. \tag{5.27}$$

The plots of MRL function for different values of θ is shown in the Figure 5.4.

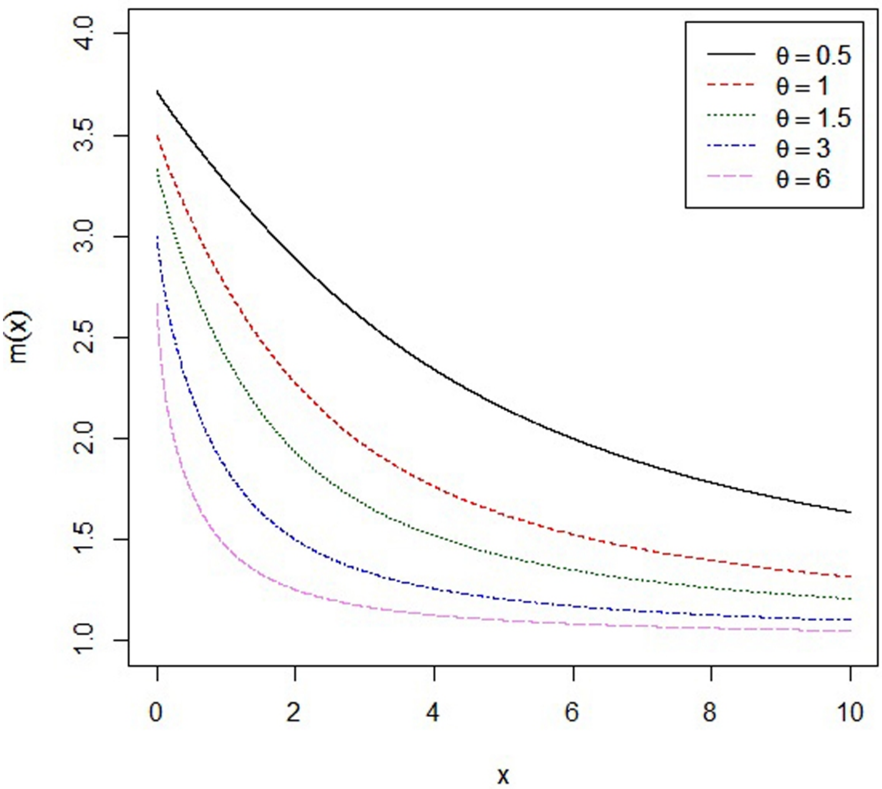

FIGURE 5.4: Mean residual life function of length biased xgamma distribution for different values of θ

The following points are noted.

(i) It is clear that the hazard rate is increasing function in x (> 0). The fact can easily be identified as the length biased distribution given in (5.9) is log-concave.

(ii) The MRL function in (5.27) is bounded below by $1/\theta$ and bounded above by $\frac{2(\theta+6)}{\theta(\theta+3)} = E(X)$ and is decreasing in x.

(iii) Therefore, the distribution posses increasing failure rate (IFR) and decreasing mean residual life (DMRL) property.

5.5.3 Reversed hazard rate function

The reversed hazard rate function of $X \sim LBXG(\theta)$ is given by (see Figure 5.5 for the plots of reversed hazard rate function for selected values of θ)

$$\begin{aligned} r(x) &= \frac{f(x)}{F(x)}, \\ &= \frac{\theta^3\left(x + \frac{\theta}{2}x^3\right)e^{-\theta x}}{(\theta+3) - \{(\theta+3) + (\theta+3)\theta x + \frac{3}{2}\theta^2 x^2 + \frac{1}{2}\theta^3 x^3\}e^{-\theta x}}, \\ &= \frac{\theta^3\left(x + \frac{\theta}{2}x^3\right)e^{-\theta x}}{(\theta+3)\{1-(1+\theta x)e^{-\theta x}\} - \frac{1}{2}\theta^2 x^2(3+\theta x)e^{-\theta x}}, x > 0. \end{aligned} \tag{5.28}$$

5.5.4 Stochastic ordering

In this sub-section, we study stochastic order relationship of length biased xgamma random variables.

The following theorem shows that length biased xgamma random variables possess strong stochastic ordering depending the value of the parameter.

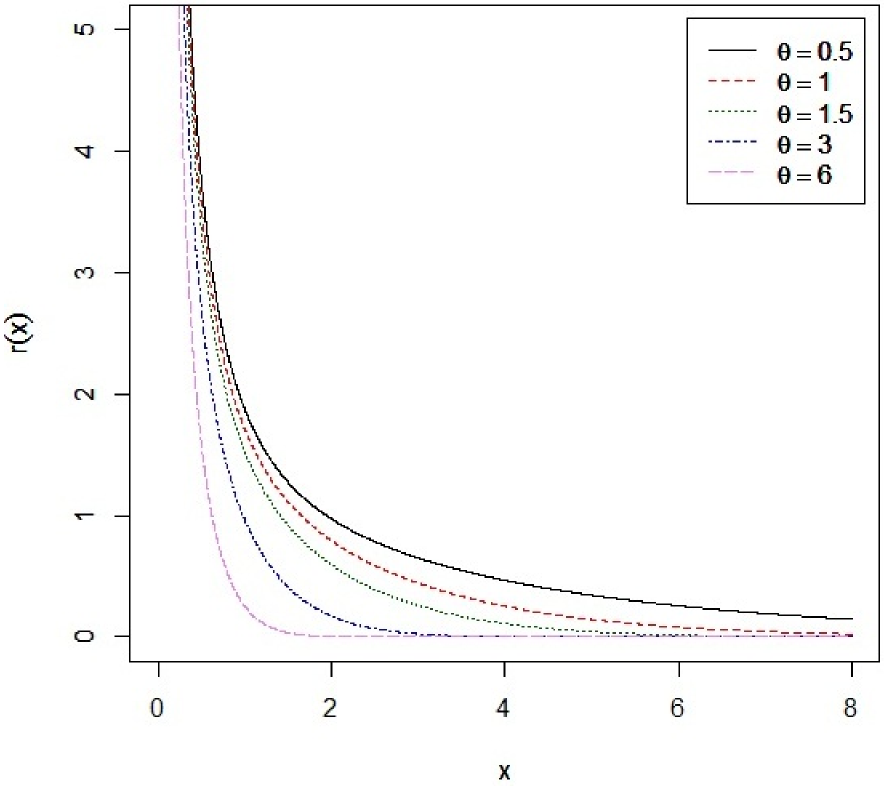

FIGURE 5.5: Reversed hazard rate function of length biased xgamma distribution for different values of θ.

Theorem 5.3. *If $X \sim LBXG(\theta_1)$ and $Y \sim LBXG(\theta_2)$, then for $\theta_1 > \theta_2$, X is smaller than Y in hazard rate order (i.e.,$X \leq_{HR} Y$) and thereby in mean residual life order ($X \leq_{MRL} Y$) and stochastic order ($X \leq_{ST} Y$), respectively.*

Proof. For $t > 0$, we have the ratio of the hazard functions of X and Y as

$$\frac{h_X(t)}{h_Y(t)} = \left(\frac{\theta_1}{\theta_2}\right)^3 \left(\frac{2t + \theta_1 t^3}{2t + \theta_2 t^3}\right) \left[\frac{(3+\theta_2) + (3+\theta_2)\theta_2 t + \frac{3}{2}\theta_2^2 t^2 + \frac{1}{2}\theta_2^3 t^3}{(3+\theta_1) + (3+\theta_1)\theta_1 t + \frac{3}{2}\theta_1^2 t^2 + \frac{1}{2}\theta_1^3 t^3}\right],$$

which is more than unity if $\theta_1 > \theta_2$ (see Figure 5.6 for the plots of $\frac{h_X(t)}{h_Y(t)}$ for selected values of θ_1 and θ_2). Hence, $h_X(t) > h_Y(t)$ for $\theta_1 > \theta_2$ and $t > 0$. So, $X \leq_{HR} Y$. Again by Shaked and Shanthikumar (1994), $X \leq_{HR} Y \Rightarrow X \leq_{MRL} Y$ and $X \leq_{HR} Y \Rightarrow X \leq_{ST} Y$, and hence the proof.

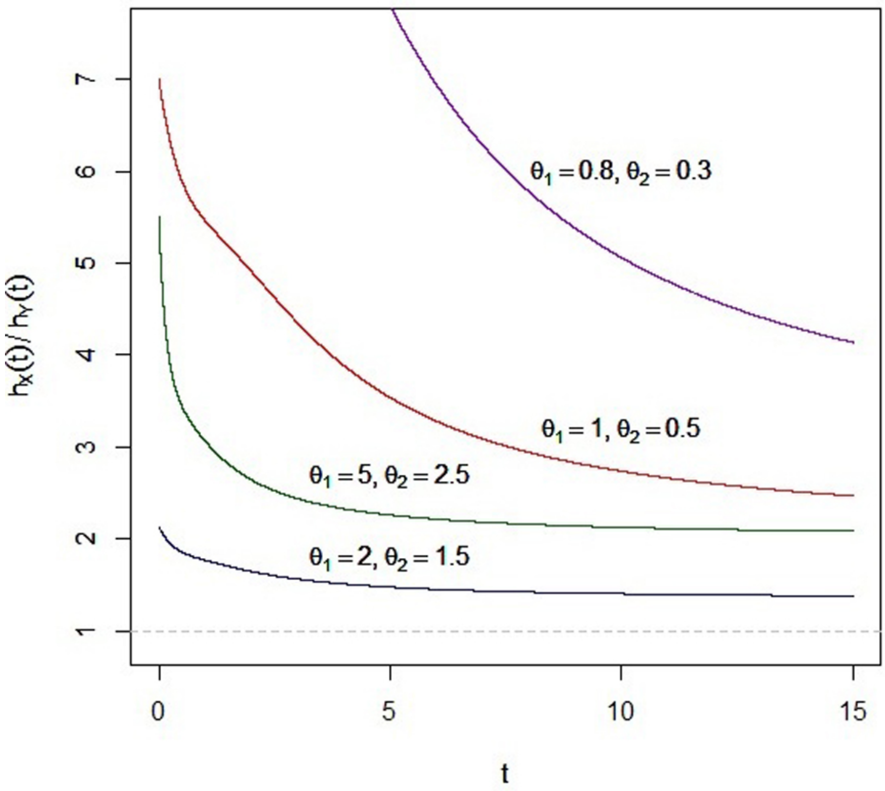

FIGURE 5.6: Plots for $\frac{h_X(t)}{h_Y(t)}$ for selected values of θ_1 and θ_2 ($\theta_1 > \theta_2$).

5.6 Parameter estimation

In this section, method of moments and method maximum likelihood are been proposed for estimating θ when $X \sim LBXG(\theta)$. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n drawn from $LBXG(\theta)$.

5.6.1 Method of moments

If $\bar{X}$ denotes the sample mean, then by applying the method of moments, we have

$$\bar{X} = \frac{2(\theta + 6)}{\theta(\theta + 3)},$$

which gives a quadratic equation in θ as

$$\bar{X}\theta^2 + (3\bar{X} - 2)\theta - 12 = 0. \tag{5.29}$$

Denoting $\hat{\theta}_M$ as the method of moment estimator for θ. $\hat{\theta}_M$ is the solution of (5.29) and is obtained as

$$\hat{\theta}_M = \frac{-(3\bar{X} - 2) + \sqrt{(3\bar{X} - 2)^2 + 48\bar{X}}}{2\bar{X}} \text{ for } \bar{X} > 0. \tag{5.30}$$

5.6.2 Method of maximum likelihood

Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ be a particular realization on $X_1, X_2, \ldots, X_n$. The likelihood function of θ given $\boldsymbol{x}$ is then written as

$$L(\theta|\boldsymbol{x}) = \prod_{i=1}^{n} \frac{\theta^3}{(\theta + 3)} \left(x_i + \frac{\theta}{2} x_i^3 \right) e^{-\theta x_i} = \frac{\theta^{3n}}{(\theta + 3)^n} e^{-\theta \sum_{i=1}^{n} x_i} \prod_{i=1}^{n} \left(x_i + \frac{\theta}{2} x_i^3 \right).$$

The log-likelihood function is given by

$$\ln L(\theta|\boldsymbol{x}) = 3n \ln \theta - n \ln(\theta + 3) - \theta \left(\sum_{i=1}^{n} x_i \right) + \sum_{i=1}^{n} \ln \left(x_i + \frac{\theta}{2} x_i^3 \right). \tag{5.31}$$

Differentiating (5.31) with respect to θ and equating with zero, the log-likelihood equation is

$$\begin{aligned} &\frac{\partial}{\partial \theta} \ln L(\theta|\boldsymbol{x}) = 0 \\ &\Rightarrow \frac{3n}{\theta} - \frac{n}{(\theta + 3)} - \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \frac{x_i^2/2}{(1 + \frac{\theta}{2} x_i^2)} = 0. \end{aligned} \tag{5.32}$$

Differentiating (5.31) twice with respect to θ, we have

$$\frac{\partial^2}{\partial \theta^2} \ln L(\theta|\boldsymbol{x}) = \frac{n}{(\theta + 3)^2} - \frac{3n}{\theta^2} - \sum_{i=1}^{n} \left(\frac{x_i^2/2}{1 + \frac{\theta}{2} x_i^2} \right)^2. \tag{5.33}$$

The equation (5.32) can not be solved analytically, hence for finding the maximum likelihood estimator, say $\hat{\theta}$, of θ numerical method like *Newton-Raphson* is applied.

5.7 Simulation study

The procedure for simulating random sample of specific size from $LBXG(\theta)$ is discussed in this section along with a simulation study.
The fact that length biased xgamma distribution is a special mixture of $gamma(2, \theta)$ and $gamma(4, \theta)$ with mixing proportions $\theta/(3+\theta)$ and $3/(3+\theta)$, respectively, is utilized for constructing the simulation algorithm from the distribution.
If $X \sim LBXG(\theta)$, then for generating a random sample of size n we can have the following algorithm.

1. Generate $U_i \sim uniform(0, 1); i = 1, 2, \ldots, n$.
2. Generate $V_i \sim gamma(2, \theta); i = 1, 2, \ldots, n$.
3. Generate $W_i \sim gamma(4, \theta); i = 1, 2, \ldots, n$.
4. If $U_i \leq \frac{\theta}{\theta+3}$, then set $X_i = Vi$, otherwise set $X_i = Wi$.

A Monte-Carlo simulation study is carried out by considering $N = 10,000$ times for selected values of n and θ. Samples of sizes $20, 40, 60$ and 100 are considered and values of θ are taken as $0.1, 0.5, 1.0, 1.5, 3, 4.5$ and 6. The required numerical evaluations are carried out using R software. The following two measures are been computed.

(i) Average estimate of θ:
$\hat{\theta} = \frac{1}{N}\sum_{i=1}^{N}\hat{\theta}_i$, where $\hat{\theta}_i$'s are simulated estimates.

(ii) Mean Square Error (MSE) of the simulated estimates $\hat{\theta}_i, i = 1, 2, \ldots, N$:
$\frac{1}{N}\sum_{i=1}^{N}(\hat{\theta}_i - \theta)^2$.

The results of the simulation study is presented in Table 5.1. The following observations are made from the simulation study.

1. For a given value of θ, the average mean square error (MSE) decreases as sample size n increases.
2. For a larger given value of θ, MSE gets higher and follow the similar trends as indicated in (i) above.

Table 5.1: Estimate and average MSE for different sample sizes

	$n = 20$		$n = 40$	
θ	Estimate	MSE	Estimate	MSE
0.1	0.09989	0.00013	0.09976	0.00006
0.5	0.48862	0.00326	0.48647	0.00171
1.0	0.95012	0.01445	0.94499	0.00884
1.5	1.39521	0.03706	1.38425	0.02610
3.0	2.64595	0.22346	2.62256	0.18895
4.5	3.80880	0.68941	3.78011	0.61890
6.0	4.91580	1.53158	4.88066	1.42160
	$n = 60$		$n = 100$	
θ	Estimate	MSE	Estimate	MSE
0.1	0.09963	0.00004	0.09935	0.00002
0.5	0.48404	0.00126	0.48376	0.00085
1.0	0.94193	0.00714	0.94072	0.00576
1.5	1.38165	0.02240	1.38010	0.01935
3.0	2.61816	0.17740	2.61517	0.16657
4.5	3.76977	0.59767	3.76117	0.58579
6.0	4.85885	1.41716	4.85344	1.38345

5.8 Application

In this section, a real life data set is analyzed to illustrate the applicability of length biased xgamma distribution.

Fatigue is an important factor in determining the service life of ball bearings. Bearing manufacturers are therefore constantly engaged in fatigue-testing operations in order to obtain information relating fatigue life to load and other factors. The data set of 23 fatigue life for deep-groove ball bearings, compiled by American Standards Association and reported in Lieblein and Zelen (1956) is used to illustrate the applicability of the length biased xgamma model.
The data set (given in Table 5.2) is positively skewed (skewness=0.94 and kurtosis=0.49) with mean value 72.22, median 67.80 and is unimodal (mode at 50).

Table 5.2: Data on fatigue lives of 23 deep-groove ball bearings

17.88	28.92	33.00	41.52	42.12	45.60	48.48	51.84	51.96	54.12
55.56	67.80	68.64	68.64	68.88	84.12	93.12	98.64	105.12	105.84
127.92	128.04	173.40							

For comparison purpose, besides length biased xgamma distribution with parameter θ, five other different life distributions, namely, exponential with rate θ, gamma distribution with shape α and rate θ, Weibull distribution with shape α and scale β, xgamma distribution with parameter θ and length biased weighted exponential distribution with parameters α and λ, i.e., $LBWE(\alpha, \lambda)$ (Das and Kundu, 2016), are considered.
In order to compare lifetime models, criteria like, negative log-likelihood, AIC and BIC are taken. The better fitted distribution corresponds to smaller negative log-likelihood, AIC and BIC values. Maximum likelihood estimates (MLEs) are obtained for the parameters involved in the distributions considered for the purpose. Statistical software R is utilized for computation.
Table 5.3 shows the estimates of the model parameter(s) with standard error(s) of estimates in parenthesis and different model selection criteria. From Table 5.3, it is observed that $LBXG(\theta)$ better fits the data as compared to the other models. Moreover, added flexibility over xgamma distribution is observed in real data application.

TABLE 5.3: MLEs of model parameters and model selection criteria for fatigue lives of ball bearing data.

Distributions	Estimate(Std. Error)	-Log-likelihood	AIC	BIC
Exponential(θ)	$\hat{\theta}$=0.0138 (0.0029)	121.435	244.870	246.005
Gamma(α, θ)	$\hat{\alpha}$=4.0260 (1.1396) $\hat{\theta}$=0.0557 (0.0168)	113.029	230.059	232.330
Weibull(α, β)	$\hat{\alpha}$=2.1021 (0.3286) $\hat{\beta}$=81.8683 (8.5986)	113.691	231.383	233.654
Xgamma(θ)	$\hat{\theta}$=0.0407 (0.0049)	113.966	229.931	231.067
LBWE(α, λ)	$\hat{\alpha}$=0.0251 (0.8960) $\hat{\lambda}$=0.0410 (0.0182)	113.522	231.045	233.326
LBXG(θ)	$\hat{\theta}$=0.0549 (0.0057)	113.086	228.171	229.307

The Figure 5.7 shows the plot of histogram and fitted exponential, gamma, Weibull, xgamma, $LBWE(\alpha, \lambda)$ and $LBXG(\theta)$ curves for fatigue lives data.

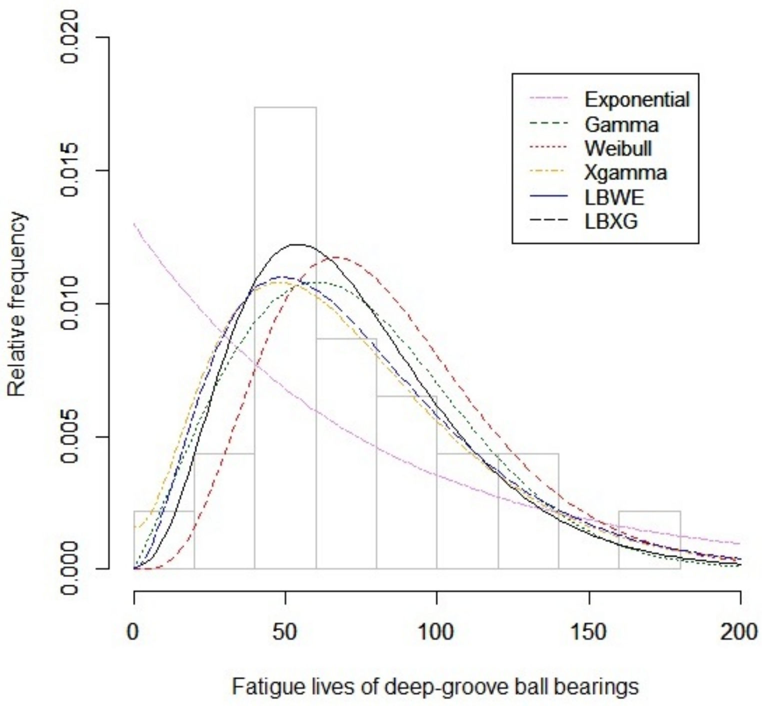

FIGURE 5.7: Plot of histogram and fitted lifetime models for fatigue lives data.

5.9 Conclusion

Owing the importance of weighted distributions in statistical literature, the weighted xgamma distribution, considering a special non-negative weight function, is proposed and studied in this chapter as a generalization of xgamma distribution. As a special case of weighted xgamma distribution, length biased version of xgamma distribution is obtained and its different distributional and survival properties are studied in detail. Method of moments and method of maximum likelihood are proposed for estimating unknown parameter in the length biased xgamma distribution. Real data are analyzed to show the applicability of the proposed model and compared with other life distributions. The following important findings are obtained in this chapter.

1. It is observed that the length biased xgamma is a special case of weighted xgamma distribution and is a special finite mixture of $gamma(2, \theta)$ and $gamma(4, \theta)$.

2. Length biased xgamma distribution is unimodal and holds IFR and DMRL property.

3. Length biased xgamma random variable possesses strong hazard rate, mean residual life and stochastic ordering for certain restriction on parameter.

4. Simulation study shows that the estimator of the unknown parameter in length biased xgamma distribution behaves satisfactorily for larger sample. Real data illustration shows that the length biased xgamma distribution is a potential model in describing real life time-to-event data and can be utilized as a flexible lifetime model against the standard lifetime models available in the literature.

This chapter opens some further scope for future research on the distribution proposed.

Open research problems:

Listed below some future research problems one could be interested in.

- Investigation for a suitable method of discriminating between xgamma distribution and length biased xgamma distribution for a given sample data.
- Bayesian estimation aspects for length biased xgamma distribution for different loss functions and under different censoring schemes could be potential research interest.
- Bivariate and multivariate extensions of length biased xgamma distribution could be interesting generalizations.

Chapter 6

Two extensions of xgamma distribution

So far in Chapter 2 to Chapter 5, we have studied xgamma distribution, its upper truncated version and weighted version for investigating flexibility in data analysis, mainly related to time-to-event set up. In Chapter 4, we have observed that the upper truncated xgamma model although provides added flexibility over xgamma model in analyzing real life data, the form of the distribution and its certain properties are not much user friendly or sometimes difficult to apply in terms of flexibility of their final forms.

However, the length biased xgamma distribution, studied in Chapter 5 as a special case of weighted xgamma distribution, provides better flexibility in terms of application as well as in its different properties. But, the length biased xgamma distribution has only single parameter like xgamma distribution and hence, might not be appropriate in modeling wide range of time-to-event data sets where two-parameter life distributions provide better options. So, we search further for some extensions or generalizations of xgamma distribution with two non-negative parameters involved in the density and intend to study different properties of such forms that could possibly reveal additional flexibility and could provide possible improvements over xgamma and other life distributions in data analysis as well.

Adding an extra parameter to an existing family of distributions is very common in the statistical distribution theory. Often introducing an extra parameter brings additional flexibility to a class of probability distributions, and, in turn, it can be very useful for data analysis purposes. However, adding more parameters to an existing family of distributions may create complications in its basic structural properties and/or in methods of estimating the additional parameters, see for more details Johnson et al. (1994).
Several authors in statistical literature have been proposed excellent methods in adding extra parameter(s) to an existing distribution for added flexibility in terms of distributional properties, computations, statistical inferences and in describing uncertainties behind real world phenomena, see for more survey on methods of adding parameters to standard models Azzalini (1985), Marshall and Olkin (1997), Eugene et al. (2002), Lee et al. (2013), Mudholkar and Srivastava (2013), Alzaatreh et al. (2013) and Jones (2014).

Therefore, introducing new probability distributions and/or extending (or generalizing) existing probability distributions by adding extra parameters into its form has become a time-honored device for obtaining more flexible new families of distributions.

The present chapter, which is the last chapter of this thesis, contemplates on introducing two different extensions, viz. the quasi xgamma and the two-parameter xgamma, of xgamma distribution, studying their essential distributional and survival properties, aspects of estimating unknown parameters for complete sample situation and possible applications with real data illustrations.
The chapter is broadly classified into two major sections, section 6.1 along with its delegate subsections deals with the study of the quasi xgamma distribution and section 6.2 along with its subsections deals with the study of the two-parameter

xgamma distribution. Finally section 6.3 summarizes the chapter with important findings and points out some open research problems for future investigations.

6.1 The quasi xgamma distribution

An extra non-negative parameter is incorporated to the one parameter xgamma distribution in (2.2) for more flexibility in describing data that might follow situations. The family of distributions, thus obtained, is named as *quasi xgamma*. The name *quasi xgamma* is proposed not in view of any technical term, here the term "quasi" stands for "similar form" as xgamma distribution. It is to be noted that the quasi xgamma distribution such obtained includes xgamma distribution as a special case.

For synthesizing the density form of quasi xgamma distribution, we consider $f_1(x)$ to follow an exponential distribution with parameter θ and $f_2(x)$ to follow a gamma distribution with scale parameter θ and shape parameter 3 i.e., $f_1(x) \sim exp(\theta)$ and $f_1(x) \sim gamma(3, \theta)$ with $\pi_1 = \frac{\alpha}{(1+\alpha)}$ and $\pi_2 = 1 - \pi_1$ in (2.1).

We have the following definition for the quasi xgamma distribution.

Definition 6.1. A non-negative continuous random variable, X, is said to follow a quasi xgamma (QXG) distribution with parameters α and θ if its pdf is of the form

$$f(x) = \frac{\theta}{(1+\alpha)} \left(\alpha + \frac{\theta^2}{2} x^2 \right) e^{-\theta x}, x > 0, \theta > 0 \text{ and } \alpha > 0, \qquad (6.1)$$

and is denoted by $X \sim QXG(\alpha, \theta)$.

Special cases:

For particular values of α, from (6.1) the following special cases are observed.

1. When $\alpha = 0$, gamma distribution with shape parameter 3 and scale parameter θ, i.e., $X \sim G(\theta, 3)$ is obtained.

2. When $\alpha = 1$, a new class of distributions can be obtained with pdf

$$f(x) = \frac{\theta}{2}\left(\alpha + \frac{\theta^2}{2}x^2\right)e^{-\theta x}, x > 0 \text{ and } \theta > 0. \tag{6.2}$$

3. When $\alpha = \theta$, the xgamma distribution is obtained with pdf given in (2.2).

The plot of the density functions of quasi xgamma distribution for different values of α and θ is shown in Figure 6.1.

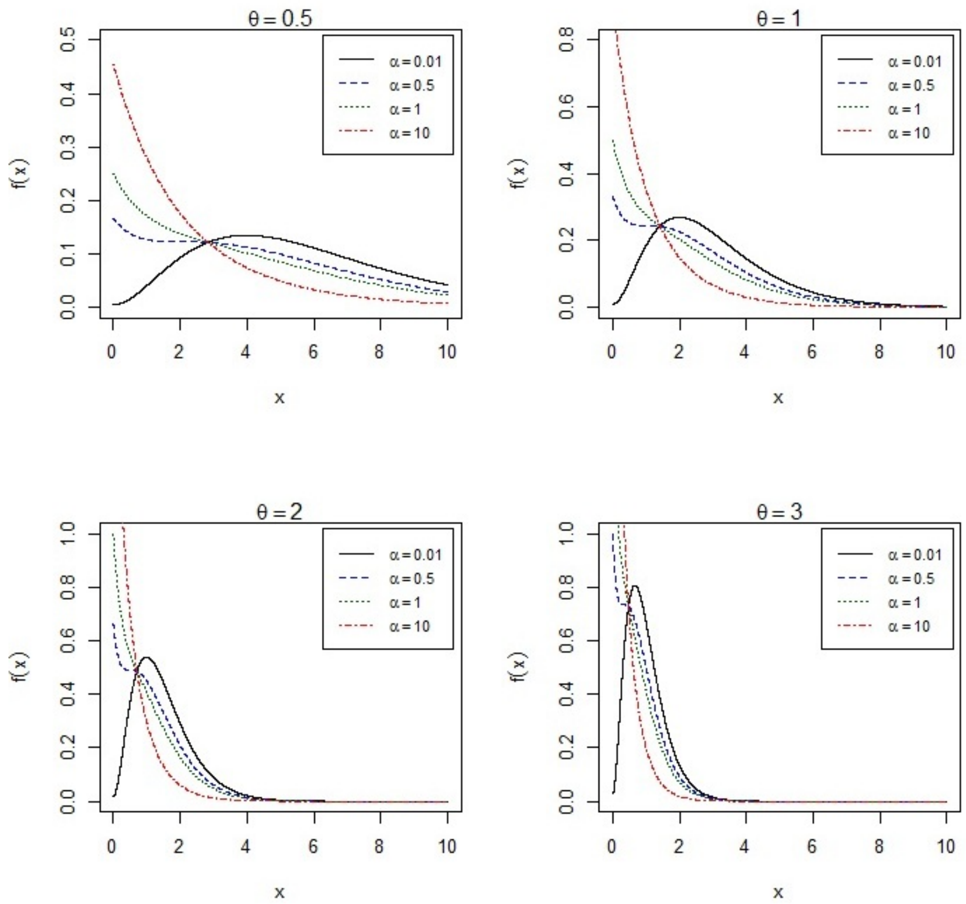

FIGURE 6.1: Probability density curves of $QXG(\alpha, \theta)$ for different values of α and θ.

Now, we derive the cdf of $QXG(\alpha, \theta)$ as below.

We have,

$$F(x) = 1 - \Pr(X > x) = 1 - \int_x^\infty f(t)dt. \tag{6.3}$$

Now,

$$\begin{aligned}
\Pr(X > x) &= \int_x^\infty \frac{\theta}{(1+\alpha)} \left(\alpha + \frac{\theta^2}{2}t^2\right) e^{-\theta t} dt, \\
&= \frac{\theta}{(1+\alpha)} \left[\alpha \int_x^\infty e^{-\theta t} dt + \frac{\theta^2}{2} \int_x^\infty t^2 e^{-\theta t} dt\right].
\end{aligned}$$

Using the expressions of integration in (2.3) and (2.5), we have,

$$\begin{aligned}
\Pr(X > x) &= \frac{\theta}{(1+\alpha)} \left[\frac{\alpha e^{-\theta x}}{\theta} + \frac{\theta^2}{2}\left\{\frac{x^2 e^{-\theta x}}{\theta} + \frac{2}{\theta}\left(\frac{x e^{-\theta x}}{\theta} + \frac{e^{-\theta x}}{\theta^2}\right)\right\}\right], \\
&= \frac{\theta}{(1+\alpha)} \left[\frac{\alpha e^{-\theta x}}{\theta} + \frac{\theta x^2 e^{-\theta x}}{2} + x e^{-\theta x} + \frac{e^{-\theta x}}{\theta}\right], \\
&= \frac{\theta e^{-\theta x}}{(1+\alpha)} \left[\left(\frac{\alpha}{\theta} + \frac{1}{\theta}\right) + x + \frac{\theta x^2}{2}\right], \\
&= \frac{\theta e^{-\theta x}}{(1+\alpha)} \left[\left(\frac{1+\alpha}{\theta}\right) + x + \frac{\theta x^2}{2}\right], \\
&= \frac{e^{-\theta x}}{(1+\alpha)} \left[(1+\alpha) + \theta x + \frac{\theta^2 x^2}{2}\right].
\end{aligned}$$

Hence, the cdf of X is given by

$$F(x) = 1 - \frac{\left(1 + \alpha + \theta x + \frac{\theta^2 x^2}{2}\right)}{(1+\alpha)} e^{-\theta x}, x > 0. \tag{6.4}$$

6.1.1 Moments and related measures

In this section we find moments and some related measures of $QXG(\alpha, \theta)$.

The r^{th} order non-central moment of quasi xgamma distribution can be derived as

$$\begin{aligned}
\mu_r' &= E(X^r), \\
&= \int_0^\infty x^r \frac{\theta}{(1+\alpha)}\left(\alpha + \frac{\theta^2}{2}x^2\right) e^{-\theta x} dx, \\
&= \frac{\theta}{(1+\alpha)}\left[\alpha \int_0^\infty x^r e^{-\theta x} dx + \frac{\theta^2}{2}\int_0^\infty x^{r+2} e^{-\theta x} dx\right], \\
&= \frac{\theta}{(1+\alpha)}\left[\frac{\alpha\Gamma(r+1)}{\theta^{r+1}} + \frac{\theta^2\Gamma(r+3)}{2\theta^{r+2}}\right],
\end{aligned}$$

Here $\Gamma(a) = \int_0^\infty z^{a-1}e^{-z}dz$ is the gamma function.

$$= \frac{\Gamma(r+1)}{(1+\alpha)\theta^r}\left[\alpha + \frac{1}{2}(r+1)(r+2)\right].$$

Hence, The r^{th} order non-central moment is given by

$$\mu_r' = \frac{r!}{\theta^r(1+\alpha)}\left[\alpha + \frac{1}{2}(r+1)(r+2)\right] \text{ for } r = 1, 2, \dots. \tag{6.5}$$

In particular, we have

$$\mu_1' = E(X) = \frac{(3+\alpha)}{\theta(1+\alpha)} = \mu(\text{say}) \tag{6.6}$$

$$\mu_2' = E(X^2) = \frac{2(6+\alpha)}{\theta^2(1+\alpha)} \quad ; \quad \mu_3' = E(X^3) = \frac{6(10+\alpha)}{\theta^3(1+\alpha)} \tag{6.7}$$

$$\mu_4' = E(X^4) = \frac{24(15+\alpha)}{\theta^4(1+\alpha)}. \tag{6.8}$$

The j^{th} order central (about μ) moment can be obtained using the relationship given in (1.3).

In particular, we have

$$\begin{aligned}
\mu_2 = Var(X) &= \mu_2' - \mu^2, \\
&= \frac{2(6+\alpha)}{\theta^2(1+\alpha)} - \left[\frac{(3+\alpha)}{\theta(1+\alpha)}\right]^2.
\end{aligned}$$

On simplification, we have,

$$\mu_2 = Var(X) = \frac{\alpha^2 + 8\alpha + 3}{\theta^2(1+\alpha)^2} = \sigma^2(say). \tag{6.9}$$

Similarly, one can easily calculate

$$\begin{aligned}\mu_3 &= \mu_3' - 3\mu_2'\mu + 2\mu^3, \\ &= \frac{2(\alpha^3 + 15\alpha^2 + 9\alpha + 3)}{\theta^3(1+\alpha)^3}.\end{aligned} \tag{6.10}$$

$$\begin{aligned}\mu_4 &= \mu_4' - 4\mu_3'\mu + 6\mu_2'\mu^2 - 3\mu^4, \\ &= \frac{3(\alpha^4 + 88\alpha^3 + 310\alpha^2 + 288\alpha + 177)}{\theta^4(1+\alpha)^4}.\end{aligned} \tag{6.11}$$

The coefficient of variation (γ), coefficient of skewness ($\sqrt{\beta_1}$) and coefficient of kurtosis (β_2) are obtained by

$$\gamma = \frac{\sqrt{Var(X)}}{E(X)} = \frac{\sqrt{(\alpha^2 + 8\alpha + 3)}}{(3+\alpha)}, \tag{6.12}$$

$$\sqrt{\beta_1} = \frac{\mu_3}{\mu_2^{3/2}} = \frac{2(\alpha^3 + 15\alpha^2 + 9\alpha + 3)}{(\alpha^2 + 8\alpha + 3)^{3/2}} \tag{6.13}$$

and

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{3(\alpha^4 + 88\alpha^3 + 310\alpha^2 + 288\alpha + 177)}{(\alpha^2 + 8\alpha + 3)^2}, \tag{6.14}$$

respectively.

The following theorem shows that the pdf of $QXG(\alpha, \theta)$ is decreasing in x for $\alpha > 1/2$.

Theorem 6.2. *For $\alpha > 1/2$, the pdf in (6.1) is decreasing in x.*

Proof. We have from (6.1) the first derivative of $f(x)$ with respect to x as

$$f'(x) = \frac{\theta^2}{(1+\alpha)}\left(\theta x - \alpha - \frac{1}{2}\theta^2 x^2\right) e^{-\theta x}.$$

$f'(x)$ is negative in x if $\alpha > 1/2$, and hence the proof.

So, from the above Theorem 6.2, for $\alpha \leq 1/2$, $\frac{d}{dx}f(x) = 0$ implies that $(1 + \sqrt{(1-2\alpha)})/\theta$ is the unique critical point at which $f(x)$ is maximized.
Hence, the mode of quasi xgamma distribution is given by

$$Mode(X) = \begin{cases} \frac{1+\sqrt{1-2\alpha}}{\theta}\ , & \text{if } 0 < \alpha \leq 1/2. \\ 0\ , & \text{otherwise.} \end{cases} \tag{6.15}$$

6.1.2 Characteristic and generating functions

In this sub-section, we derive the characteristic, moment generating and cumulant generating functions of $X \sim QXG(\alpha, \theta)$.

The characteristic function of X is derived as

$$\begin{aligned} \phi_X(t) &= E(e^{itX}), \\ &= \int_0^\infty e^{itx} \frac{\theta}{(1+\alpha)}\left(\alpha + \frac{\theta^2}{2}x^2\right) e^{-\theta x} dx, \\ &= \frac{\theta}{(1+\alpha)}\left[\alpha \int_0^\infty e^{(\theta - it)x} dx + \frac{\theta^2}{2}\int_0^\infty x^2 e^{(\theta - it)x} dx\right], \end{aligned}$$

Hence,

$$\begin{aligned}
\phi_X(t) &= \frac{\theta}{(1+\alpha)}\left[\frac{\alpha}{(\theta-it)}+\frac{\theta^2}{2}\frac{\Gamma(3)}{(\theta-it)^3}\right],\\
&\text{Here } \Gamma(a)=\int_0^{\infty} z^{a-1}e^{-z}dz \text{ is the gamma function.}\\
&= \frac{\theta}{(1+\alpha)}\left[\alpha(\theta-it)^{-1}+\theta^2(\theta-it)^{-3}\right],\\
&= \frac{1}{(1+\alpha)}\left[\alpha\left(1-i\frac{t}{\theta}\right)^{-1}+\left(1-i\frac{t}{\theta}\right)^{-3}\right]; t\in\Re, i=\sqrt{-1}. \qquad (6.16)
\end{aligned}$$

Now, to find the moment generating function, we calculate

$$\begin{aligned}
M_X(t) &= E(e^{tX}),\\
&= \int_0^{\infty} e^{tx}\frac{\theta}{(1+\alpha)}\left(\alpha+\frac{\theta^2}{2}x^2\right)e^{-\theta x}dx,\\
&= \frac{\theta}{(1+\alpha)}\left[\alpha\int_0^{\infty} e^{(\theta-t)x}dx+\frac{\theta^2}{2}\int_0^{\infty} x^2e^{(\theta-t)x}dx\right],\\
&= \frac{\theta}{(1+\alpha)}\left[\frac{\alpha}{(\theta-t)}+\frac{\theta^2}{2}\frac{\Gamma(3)}{(\theta-t)^3}\right],\\
&\text{Here } \Gamma(a)=\int_0^{\infty} z^{a-1}e^{-z}dz \text{ is the gamma function.}\\
&= \frac{1}{(1+\alpha)}\left[\alpha\left(1-\frac{t}{\theta}\right)^{-1}+\left(1-\frac{t}{\theta}\right)^{-3}\right]; t\in\Re. \qquad (6.17)
\end{aligned}$$

The cumulant generating function is obtained by taking logarithm of $M_X(x)$ and is given by

$$\begin{aligned}
K_X(t) &= \ln[M_X(t)],\\
&= \ln\frac{\theta}{(1+\alpha)(\theta-t)}+ln\left[\alpha+\frac{\theta^2}{(\theta-t)^2}\right]; t\in\Re, \qquad (6.18)
\end{aligned}$$

6.1.3 Entropy measures

We first find the Rényi entropy measure for $QXG(\alpha, \theta)$.

We derive,

$$\int_0^\infty f^\gamma(x)dx$$
$$= \frac{\theta^\gamma}{(1+\alpha)^\gamma} \int_0^\infty \left(\alpha + \frac{\theta^2}{2}x^2\right)^\gamma e^{-\gamma\theta x} dx,$$
$$= \frac{(\alpha\theta)^\gamma}{(1+\alpha)^\gamma} \int_0^\infty \left(1 + \frac{\theta^2}{2\alpha}x^2\right)^\gamma e^{-\gamma\theta x} dx,$$

Using power series expansion $\left(1 + \frac{\theta^2}{2\alpha}x^2\right)^\gamma = \sum_{j=0}^{\gamma} \binom{\gamma}{j} \left(\frac{\theta^2 x^2}{2\alpha}\right)^j$, we have,

$$= \frac{(\alpha\theta)^\gamma}{(1+\alpha)^\gamma} \int_0^\infty \sum_{j=0}^{\gamma} \binom{\gamma}{j} \left(\frac{\theta^2 x^2}{2\alpha}\right)^j e^{-\gamma\theta x} dx,$$
$$= \frac{(\alpha\theta)^\gamma}{(1+\alpha)^\gamma} \sum_{j=0}^{\gamma} \binom{\gamma}{j} \left(\frac{\theta^2}{2\alpha}\right)^j \int_0^\infty x^{2j} e^{-\gamma\theta x} dx,$$
$$= \frac{(\alpha\theta)^\gamma}{(1+\alpha)^\gamma} \sum_{j=0}^{\gamma} \binom{\gamma}{j} \left(\frac{\theta^2}{2\alpha}\right)^j \frac{\Gamma(2j+1)}{(\gamma\theta)^{2j+1}},$$

Here $\Gamma(a) = \int_0^\infty z^{a-1} e^{-z} dz$ is the gamma function.

$$= \frac{\alpha^\gamma \theta^{\gamma-1}}{(1+\alpha)^\gamma} \sum_{j=0}^{\gamma} \binom{\gamma}{j} \frac{\Gamma(2j+1)}{2^j \alpha^j \gamma^{2j+1}}.$$

Hence, the Rényi entropy is given by

$$H_R(\gamma) = \frac{1}{1-\gamma}[\gamma \ln \alpha + (\gamma - 1)\ln \theta - \gamma \ln(1+\alpha)] + \frac{1}{1-\gamma} \ln \left[\sum_{j=0}^{\gamma} \binom{\gamma}{j} \frac{\Gamma(2j+1)}{2^j \alpha^j \gamma^{2j+1}}\right]. \quad (6.19)$$

Shannon measure of entropy is a special case of Rényi entropy, see (1.11).

If $X \sim QXG(\alpha, \theta)$, the Shannon entropy is derived as

$$\begin{aligned}
H(f) &= E[-\ln f(x)], \\
&= -\int_0^\infty \ln\left\{\frac{\theta}{(1+\alpha)}\left(\alpha+\frac{\theta^2}{2}x^2\right)e^{-\theta x}\right\} f(x)dx, \\
&= -\int_0^\infty \left[\ln\left(\frac{\theta}{1+\alpha}\right)+\ln\left(\alpha+\frac{\theta^2}{2}x^2\right)-\theta x\right] f(x)dx, \\
&= -\left[\int_0^\infty \ln\left(\frac{\theta}{1+\alpha}\right) f(x)dx+\int_0^\infty \ln\left(\alpha+\frac{\theta^2}{2}x^2\right) f(x)dx-\theta\int_0^\infty xf(x)dx\right], \\
&\text{Now, since } \int_0^\infty xf(x)dx = E(X) \text{ and } \int_0^\infty f(x)dx = 1, \text{ we have,} \\
&= -\left[\ln\left(\frac{\theta}{1+\alpha}\right)+\int_0^\infty \ln\left(\alpha+\frac{\theta^2}{2}x^2\right) f(x)dx-\frac{\theta(3+\alpha)}{\theta(1+\alpha)}\right].
\end{aligned} \tag{6.20}$$

Now, we find,

$$\int_0^\infty \ln\left(\alpha+\frac{\theta^2}{2}x^2\right) f(x)dx.$$

$$\begin{aligned}
&\int_0^\infty \ln\left(\alpha+\frac{\theta^2}{2}x^2\right) f(x)dx \\
&= \frac{\theta}{(1+\alpha)}\int_0^\infty \left(\alpha+\frac{\theta^2}{2}x^2\right)\ln\left(\alpha+\frac{\theta^2}{2}x^2\right)e^{-\theta x}dx, \\
&= \frac{\theta}{(1+\alpha)}\left[\alpha\int_0^\infty \ln\left(\alpha+\frac{\theta^2}{2}x^2\right)e^{-\theta x}dx+\frac{\theta^2}{2}x^2\int_0^\infty \ln\left(\alpha+\frac{\theta^2}{2}x^2\right)e^{-\theta x}dx\right], \\
&= \frac{\theta}{(1+\alpha)}\left[\alpha\int_0^\infty \left\{\ln\left(\frac{\alpha+\frac{\theta^2x^2}{2}}{\alpha}\right)+\ln\alpha\right\}e^{-\theta x}dx\right] \\
&+ \frac{\theta^3}{2(1+\alpha)}\int_0^\infty x^2\left\{\ln\left(\frac{\alpha+\frac{\theta^2x^2}{2}}{\alpha}\right)+\ln\alpha\right\}e^{-\theta x}dx.
\end{aligned}$$

So,

$$\int_0^\infty \ln\left(\alpha + \frac{\theta^2}{2}x^2\right) f(x)dx$$
$$= \frac{\theta}{(1+\alpha)}\left[\alpha \int_0^\infty \ln\left(1+\frac{\theta^2 x^2}{2\alpha}\right) e^{-\theta x}dx + \alpha \ln\alpha \int_0^\infty e^{-\theta x}dx\right]$$
$$+ \frac{\theta^3}{2(1+\alpha)}\int_0^\infty x^2 \ln\left(1+\frac{\theta^2 x^2}{2\alpha}\right) e^{-\theta x}dx + \frac{\theta^3 \ln\alpha}{2(1+\alpha)}\int_0^\infty x^2 e^{-\theta x}dx,$$

Using $\ln\left(1+\frac{\theta^2 x^2}{2\alpha}\right) = \sum_{j=1}^{\infty}(-1)^{j+1}\frac{\left(\frac{\theta^2 x^2}{2\alpha}\right)^j}{j}$, we have,

$$= \frac{\theta}{(1+\alpha)}\left[\alpha \sum_{j=1}^{\infty}\frac{(-1)^{j+1}\theta^{2j}}{j(2\alpha)^j}\int_0^\infty x^{2j}e^{-\theta x}dx + \frac{\alpha\ln\alpha}{\theta}\right]$$
$$+ \frac{\theta^3}{2(1+\alpha)}\sum_{j=1}^{\infty}\frac{(-1)^{j+1}\theta^{2j}}{j(2\alpha)^j}\int_0^\infty x^{2j+2}e^{-\theta x}dx + \frac{\theta^3\ln\alpha}{2(1+\alpha)}\frac{\Gamma(3)}{\theta^3},$$
$$= \frac{\theta}{(1+\alpha)}\left[\alpha \sum_{j=1}^{\infty}\frac{(-1)^{j+1}\theta^{2j}}{j(2\alpha)^j}\frac{\Gamma(2j+1)}{\theta^{2j+1}} + \frac{\alpha\ln\alpha}{\theta}\right] + \frac{\ln\alpha}{(1+\alpha)}$$
$$+ \frac{\theta^3}{2(1+\alpha)}\sum_{j=1}^{\infty}\frac{(-1)^{j+1}\theta^{2j}}{j(2\alpha)^j}\frac{\Gamma(2j+3)}{\theta^{2j+3}},$$

Here $\Gamma(a) = \int_0^\infty z^{a-1}e^{-z}dz$ is the gamma function.

$$= \frac{\alpha}{(1+\alpha)}\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j(2\alpha)^j}\Gamma(2j+1) + \frac{\alpha}{(1+\alpha)}\ln\alpha + \frac{1}{(1+\alpha)}\ln\alpha$$
$$+ \frac{1}{2(1+\alpha)}\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j(2\alpha)^j}\Gamma(2j+3),$$
$$= \frac{\alpha}{(1+\alpha)}\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j(2\alpha)^j}\Gamma(2j+1) + \ln\alpha + \frac{1}{2(1+\alpha)}\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j(2\alpha)^j}\Gamma(2j+3),$$
$$= \ln\alpha + \frac{1}{(1+\alpha)}\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j(2\alpha)^j}\left[\alpha\Gamma(2j+1) + \frac{1}{2}\Gamma(2j+3)\right]. \tag{6.21}$$

Using (6.21), from (6.20) we have the final expression for Shannon entropy.

Hence, the Shannon entropy is given by

$$H(f) = \left(\frac{3+\alpha}{1+\alpha}\right) - \ln\left(\frac{\alpha\theta}{1+\alpha}\right) - \frac{1}{(1+\alpha)} \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j 2^j \alpha^j} \left[\alpha\Gamma(2j+1) + \frac{1}{2}\Gamma(2j+3)\right]. \quad (6.22)$$

6.1.4 Distributions of order statistics

In this sub-section, we derive the distributions of extreme order statistics for $QXG(\alpha, \theta)$.

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n drawn from $QXG(\alpha, \theta)$. Denote $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ be n order statistics.

Then for any $x > 0$, the pdf $X_{n:n}$, is obtained as

$$\begin{aligned} f_{X_{n:n}}(x) &= n[F(x)]^{n-1} f(x), \\ &= \frac{n\theta}{(1+\alpha)^n} \left[(1+\alpha)(1-e^{-\theta x}) - \theta x \left(1 + \frac{\theta x}{2}\right) e^{-\theta x}\right]^{n-1} \left(\alpha + \frac{\theta^2}{2} x^2\right) e^{-\theta x}. \end{aligned} \quad (6.23)$$

Similarly, for any $x > 0$, the pdf of the smallest order statistic, $X_{1:n}$, is derived as

$$\begin{aligned} f_{X_{1:n}}(x) &= n[1 - F(x)]^{n-1} f(x), \\ &= \frac{n\theta}{(1+\alpha)^n} \left(1 + \alpha + \theta x + \frac{\theta^2 x^2}{2}\right)^{n-1} \left(\alpha + \frac{\theta^2}{2} x^2\right) e^{-\theta x}. \end{aligned} \quad (6.24)$$

6.1.5 Survival properties

In this sub-section, some properties of quasi xgamma distribution are derived and studied that are useful in the context of survival analysis and/or reliability analysis. If $X \sim QXG(\alpha, \theta)$, the survival function of X is given by

$$S(x) = \Pr(X > x) = \frac{\left(1 + \alpha + \theta x + \frac{\theta^2}{2} x^2\right)}{(1 + \alpha)} e^{-\theta x}, x > 0. \tag{6.25}$$

6.1.5.1 Hazard rate or failure rate function

For quasi xgamma distribution, the failure rate function is obtained as

$$h(x) = \frac{f(x)}{S(x)} = \frac{\theta \left(\alpha + \frac{\theta^2}{2} x^2\right)}{\left(1 + \alpha + \theta x + \frac{\theta^2}{2} x^2\right)}. \tag{6.26}$$

Note. $h(x)$ obtained in (6.26) is bounded, i.e.,

$$\frac{\alpha\theta}{(1 + \alpha)} < h(x) < \theta, \quad \text{moreover,} \quad h(0) = f(0) = \frac{\alpha\theta}{(1 + \alpha)}.$$

6.1.5.2 MRL and reversed hazard rate functions

For quasi xgamma distribution the MRL function is derived as

$$\begin{aligned}
m(x) &= \frac{1}{S(x)} \int_x^\infty S(t)dt, \\
&= \frac{1}{(1 + \alpha)S(x)} \int_x^\infty \left(1 + \alpha + \theta t + \frac{\theta^2}{2} t^2\right) e^{-\theta t} dt, \\
&= \frac{1}{(1 + \alpha)S(x)} \left[(1 + \alpha) \int_x^\infty e^{-\theta t} dt + \theta \int_x^\infty t e^{-\theta t} dt + \frac{\theta^2}{2} \int_x^\infty t^2 e^{-\theta t} dt\right],
\end{aligned}$$

Using the expressions for integration in (2.3), (2.4) and (2.5), we have,

$$= \frac{1}{(1 + \alpha)S(x)} \left[\frac{3e^{-\theta x}}{\theta} + \frac{\alpha e^{-\theta x}}{\theta} + 2x e^{-\theta x} + \frac{\theta x^2 e^{-\theta x}}{2}\right].$$

So, we have,

$$\begin{aligned} m(x) &= \frac{e^{-\theta x}}{(1+\alpha)S(x)}\left[\frac{(3+\alpha)}{\theta}+2x+\frac{\theta x^2}{2}\right], \\ &= \frac{\left(3+\alpha+2\theta x+\frac{\theta^2 x^2}{2}\right)}{\theta\left(1+\alpha+\theta x+\frac{\theta^2}{2}x^2\right)}, \\ &= \frac{\left(1+\alpha+\theta x+\frac{\theta^2}{2}x^2\right)+(2+\theta x)}{\theta\left(1+\alpha+\theta x+\frac{\theta^2}{2}x^2\right)}. \end{aligned}$$

Hence, the MRL function is given by

$$m(x) = \frac{1}{\theta}+\frac{(2+\theta x)}{\theta\left(1+\alpha+\theta x+\frac{\theta^2}{2}x^2\right)}. \tag{6.27}$$

The MRL function of quasi xgamma distribution in (6.27) has the following properties

(i) $m(0) = E(X) = \frac{(3+\alpha)}{\theta(1+\alpha)}$.

(ii) $m(x)$ is decreasing in x with bounds $\frac{1}{\theta} < m(x) < \frac{(3+\alpha)}{\theta(1+\alpha)}$.

The reversed hazard rate function, as defined in (1.19), for $QXG(\alpha,\theta)$ is obtained as

$$r(x) = \frac{f(x)}{F(x)} = \frac{\theta\left(\alpha+\frac{\theta^2}{2}x^2\right)e^{-\theta x}}{(1+\alpha)(1-e^{-\theta x})-\theta x\left(1+\frac{\theta x}{2}\right)e^{-\theta x}}. \tag{6.28}$$

6.1.5.3 Stochastic ordering

Now, stochastic orderings of quasi xgamma random variables are studied. Recall the basic definition described in sub-section1.3.5 of Chapter 1, the following theorem states that quasi xgamma random variables follow strong likelihood ratio and other orderings.

Theorem 6.3. *Let $X \sim QXG(\alpha_1, \beta_1)$ and $Y \sim QXG(\alpha_2, \beta_2)$. If $\alpha_1 = \alpha_2$ and $\theta_1 \geq \theta_2$ (or, if $\theta_1 = \theta_2$ and $\alpha_1 \geq \alpha_2$), then $X \leq_{LR} Y$ and hence $X \leq_{HR} Y$, $X \leq_{MRL} Y$ and $X \leq_{ST} Y$.*

Proof. Let us denote the pdf of X as $f_X(x)$ and that of Y be $f_Y(x)$.
We have then the ratio

$$\frac{f_X(x)}{f_Y(x)} = \frac{\theta_1(1+\alpha_2)\,(2\alpha_1 + \theta_1^2 x^2)}{\theta_2(1+\alpha_1)\,(2\alpha_2 + \theta_2^2 x^2)} e^{-(\theta_1-\theta_2)x}.$$

So,

$$\ln\left[\frac{f_X(x)}{f_Y(x)}\right] = \ln\frac{\theta_1(1+\alpha_2)}{\theta_2(1+\alpha_1)} + \ln\left(2\alpha_1 + \theta_1^2 x^2\right) - \ln\left(2\alpha_2 + \theta_2^2 x^2\right) - (\theta_1 - \theta_2)x.$$

The first derivative with respect to x gives

$$\frac{d}{dx}\ln\left[\frac{f_X(x)}{f_Y(x)}\right] = \frac{4x(\theta_1\alpha_2 - \theta_2\alpha_1)}{(2\alpha_1 + \theta_1^2 x^2)\,(2\alpha_2 + \theta_2^2 x^2)} - (\theta_1 - \theta_2),$$

which is negative when $\alpha_1 = \alpha_2$ and $\theta_1 \geq \theta_2$ (or, when $\theta_1 = \theta_2$ and $\alpha_1 \geq \alpha_2$), so $X \leq_{LR} Y$, rest of the orderings are well justified by Shaked and Shanthikumar (1994). Hence the proof.

6.1.6 Parameter estimation

Two classical methods of estimation, viz. method of moments and method of maximum likelihood, are proposed in this sub-section for estimating the unknown parameters of the quasi xgamma distribution under complete sample situation.
As usual, let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ be n observations or realizations on a random sample $X_1, X_2, \ldots, X_n$ of size n drawn from $QXG(\alpha, \theta)$.

6.1.6.1 Method of moments

To obtain the moment estimators of the parameters, we equate

$$\mu_1^{'} = \frac{(3+\alpha)}{\theta(1+\alpha)} = \text{sample mean} = m_1^{'} = \bar{X}$$
$$\mu_2^{'} = \frac{2(6+\alpha)}{\theta^2(1+\alpha)} = m_2^{'} = \frac{1}{n}\sum_{i=1}^{n} X_i^2$$

Let

$$b = \frac{\mu_2'}{\mu_1'^2} = \frac{2(6+\alpha)(1+\alpha)}{(3+\alpha)^2},$$

which implies

$$(2-b)\alpha^2 + (14-6b)\alpha + (12-9b) = 0. \tag{6.29}$$

The equation (6.29) is a quadratic in α. The moment estimator, say $\hat{\alpha}_M$, of α, can be obtained by solving (6.29). It is to be noted that the value of b can be easily estimated from the sample moments.

The moment estimator of θ, say $\hat{\theta}_M$, is then obtained as

$$\hat{\theta}_M = \frac{(3+\hat{\alpha}_M)}{\bar{X}(1+\hat{\alpha}_M)}. \tag{6.30}$$

6.1.6.2 Method of maximum likelihood

Now, we obtain the maximum Likelihood estimators (MLEs) of the parameters α and θ. The likelihood function is

$$L(\alpha,\theta|\boldsymbol{x}) = \prod_{i=1}^{n} \frac{\theta}{(1+\alpha)}\left(\alpha + \frac{\theta^2}{2}x_i^2\right)e^{-\theta x_i}.$$

The log-likelihood function is given by

$$\ln L(\alpha,\theta|\boldsymbol{x}) = n\ln(\theta) - n\ln(1+\alpha) + \sum_{i=1}^{n}\ln\left(\alpha + \frac{\theta^2}{2}x_i^2\right) - \theta\sum_{i=1}^{n}x_i. \quad (6.31)$$

To find out the MLEs of α and θ, we have two likelihood equations as

$$\frac{\partial \ln L(\alpha,\theta|\mathbf{x})}{\partial\alpha} = \sum_{i=1}^{n}\frac{1}{\left(\alpha+\frac{\theta^2}{2}x_i^2\right)} - \frac{n}{(1+\alpha)} = 0 \quad (6.32)$$

and

$$\frac{\partial \ln L(\alpha,\theta|\mathbf{x})}{\partial\theta} = \frac{n}{\theta} + \sum_{i=1}^{n}\frac{\theta x_i^2}{\left(\alpha+\frac{\theta^2}{2}x_i^2\right)} - \sum_{i=1}^{n}x_i = 0, \quad (6.33)$$

respectively.

Though the values of α and θ cannot be obtained analytically, we can utilize any numerical method, such as *Newton-Raphson*, for solving the non-linear equations (6.32) and (6.33) to obtain those.

Moreover, we can apply Fisher's scoring method for getting the MLEs of α and θ. The second order derivatives are obtained as

$$\frac{\partial^2 \ln L(\alpha,\theta|\mathbf{x})}{\partial\alpha^2} = \frac{n}{(1+\alpha)^2} - \sum_{i=1}^{n}\frac{1}{\left(\alpha+\frac{\theta^2}{2}x_i^2\right)^2} \quad (6.34)$$

$$\frac{\partial^2 \ln L(\alpha,\theta|\mathbf{x})}{\partial\theta^2} = \sum_{i=1}^{n}\frac{\alpha x_i^2 - \frac{\theta^2}{2}x_i^4}{\left(\alpha+\frac{\theta^2}{2}x_i^2\right)^2} - \frac{n}{\theta^2} \quad (6.35)$$

$$\frac{\partial^2 \ln L(\alpha,\theta|\mathbf{x})}{\partial\alpha\partial\theta} = \frac{\partial^2 \ln L(\alpha,\theta|\mathbf{x})}{\partial\theta\partial\alpha} = -\sum_{i=1}^{n}\frac{\theta x_i^2}{\left(\alpha+\frac{\theta^2}{2}x_i^2\right)^2}. \quad (6.36)$$

Letting $\hat{\alpha}$ and $\hat{\theta}$ as the MLEs of α and θ, respectively, the following equations are solved.

$$\begin{bmatrix} \frac{\partial^2 \ln L(\alpha,\theta|\mathbf{x})}{\partial\theta^2} & \frac{\partial^2 \ln L(\alpha,\theta|\mathbf{x})}{\partial\theta\partial\alpha} \\ \frac{\partial^2 \ln L(\alpha,\theta|\mathbf{x})}{\partial\alpha\partial\theta} & \frac{\partial^2 \ln L(\alpha,\theta|\mathbf{x})}{\partial\alpha^2} \end{bmatrix}_{\hat{\theta}=\theta_0,\hat{\alpha}=\alpha_0} \begin{bmatrix} \hat{\theta}-\theta_0 \\ \hat{\alpha}-\alpha_0 \end{bmatrix} = \begin{bmatrix} \frac{\partial \ln L(\alpha,\theta|\mathbf{x})}{\partial\theta} \\ \frac{\partial \ln L(\alpha,\theta|\mathbf{x})}{\partial\alpha} \end{bmatrix} \quad (6.37)$$

Method of successive iteration can be applied for initial values α_0 and θ_0 for α and θ, respectively.

6.1.7 Simulation study

An algorithm for generating random samples of specific sizes from quasi xgamma distribution is proposed in this sub-section. A Monte-Carlo simulation study is also carried out to assess the nature of the estimates of the parameters.
The inversion method for generating random data from the quasi xgamma distribution fails because the equation $F(x) = u$, where u is an observation from the uniform distribution on $(0, 1)$, cannot be explicitly solved in x. However, as already mentioned in a note in the section 6.1, we can make use of the fact that $QXG(\alpha, \theta)$ is a special mixture of $exp(\theta)$ and $gamma(3, \theta)$ distributions with mixing proportions $\alpha/(1 + \alpha)$ and $1/(1 + \alpha)$, respectively, to construct a simulation algorithm.
To generate random data $X_i (i = 1, 2, \ldots, n)$ from quasi xgamma distribution with parameters α and θ, the following algorithm can be followed.

1. Generate $U_i \sim uniform(0, 1), i = 1, 2, \ldots, n$.
2. Generate $V_i \sim exp(\theta), i = 1, 2, \ldots, n$.
3. Generate $W_i \sim gamma(3, \theta), i = 1, 2, \ldots, n$.
4. If $U_i \leq \alpha/(1 + \alpha)$, then set $X_i = V_i$, otherwise, set $X_i = W_i$.

A Monte-Carlo simulation study is carried out by considering $N = 10,000$ times for selected values of n, α and θ. Samples of sizes 20, 30, 50, 80 and 100 are considered and values of (α, θ) are taken as (0.5,0.5), (1.5, 2.0) and (3.0, 4.0). The method of maximum likelihood is applied to obtain the estimates. The following measures are computed.

(i) Average mean square error (MSE) of the simulated estimates $\hat{\alpha}_i, i = 1, 2, \ldots, N$:

$$\frac{1}{N}\sum_{i=1}^{N}(\hat{\alpha}_i - \alpha)^2$$

(ii) Average mean square error (MSE) of the simulated estimates $\hat{\theta}_i, i = 1, 2, \ldots, N$:

$$\frac{1}{N}\sum_{i=1}^{N}(\hat{\theta}_i - \theta)^2$$

The results of the simulation study is shown in Table 6.1. Statistical software R is used for the simulation study.

It is clear from Table 6.1 that the MSEs for the estimates of α decrease as the sample size, n, increases and the estimate gets closer to the given value. The similar trend is observed in case of the estimates of θ and its MSE values for different sample sizes.

Table 6.1: Estimates of the parameters with corresponding MSE values.

		$\alpha = 0.5, \theta = 0.5$		
n	$\hat{\alpha}$	MSE of $\hat{\alpha}$	$\hat{\theta}$	MSE of $\hat{\theta}$
20	0.65563	0.64842	0.52574	0.01480
30	0.61249	0.41696	0.51593	0.00873
50	0.55139	0.24084	0.50952	0.00477
80	0.53788	0.14125	0.50112	0.00297
100	0.50544	0.12975	0.50135	0.00257
		$\alpha = 1.5, \theta = 2.0$		
n	$\hat{\alpha}$	MSE of $\hat{\alpha}$	$\hat{\theta}$	MSE of $\hat{\theta}$
20	1.88951	1.53364	2.07156	0.24302
30	1.73208	1.47211	2.04003	0.18286
50	1.61201	1.18401	2.02723	0.12595
80	1.55745	1.07344	2.00678	0.07996
100	1.51757	0.73089	2.00165	0.06368
		$\alpha = 3.0, \theta = 4.0$		
n	$\hat{\alpha}$	MSE of $\hat{\alpha}$	$\hat{\theta}$	MSE of $\hat{\theta}$
20	3.16671	1.27292	3.83684	0.13292
30	3.14370	1.02392	3.84705	0.12729
50	3.12033	0.96721	3.86446	0.09979
80	3.09858	0.78974	3.88473	0.08507
100	3.02271	0.72481	3.94853	0.07146

6.1.8 Application

In this sub-section, a real data set is analyzed to show that the quasi xgamma distribution can be a better model than some recently developed models where the particular data are utilized. The data set, given in Table 6.2, represents an uncensored data set corresponding to remission times (in months) of a random sample of 128 bladder cancer patients reported in Lee and Wang (2003). The data set is positively skewed (skewness = 3.38) with mean remission time of 8.57 months, standard deviation of 10.56 months and unimodal (mode at 5 months).

We consider here gamma, log-normal among standard lifetime models, in addition, Lindley (Lindley, 1958), power Lindley (PL) (Ghitany et al., 2013), transmuted Lindley (TL) (Merovci, 2013), exponentiated Lindley (EL) (Bakouch et al., 2012), weighted Lindley (WL) (Ghitany et al., 2011) and new generalized power Lindley (NGPL) (Mansour and Hamed, 2015) and xgamma models among recently developed or popularized lifetime models, i.e., altogether nine lifetime models are considered to compare with quasi xgamma model for suitability of fit or goodness of fit for the data.

In order to compare the two lifetime models, we consider criteria like, negative log-likelihood, AIC and BIC, for the data set. The better distribution corresponds to smaller negative log-likelihood, AIC and BIC values. We use maximum likelihood method of estimation for estimating the model parameters and statistical software R is utilized for analysis. Table 6.3 shows the estimates of the model parameters with standard error(Std. error) of estimates in parenthesis and model selection criteria.

It is clear from Table 6.3 that the quasi xgamma distribution provides better fit to the bladder cancer data and, hence, the model acts as a strong competitor among the other models considered here for modeling such lifetime data.

TABLE 6.2: Data on remission times (in months) of 128 bladder cancer patients.

0.08	2.09	3.48	4.87	6.94	8.66	13.11	23.63	0.20	2.23
0.26	0.31	0.73	0.52	4.98	6.97	9.02	13.29	0.40	2.26
3.57	5.06	7.09	11.98	4.51	2.07	0.22	13.8	25.74	0.50
2.46	3.64	5.09	7.26	9.47	14.24	19.13	6.54	3.36	0.82
0.51	2.54	3.70	5.17	7.28	9.74	14.76	26.31	0.81	1.76
8.53	6.93	0.62	3.82	5.32	7.32	10.06	14.77	32.15	2.64
3.88	5.32	3.25	12.03	8.65	0.39	10.34	14.83	34.26	0.90
2.69,	4.18	5.34	7.59	10.66	4.50	20.28	12.63	0.96,	36.66
1.05	2.69	4.23	5.41	7.62	10.75	16.62	43.01	6.25	2.02
22.69	0.19	2.75	4.26	5.41	7.63	17.12	46.12	1.26	2.83
4.33	8.37	3.36	5.49	0.66	11.25	17.14	79.05	1.35	2.87
5.62	7.87	11.64	17.36	12.02	6.76	0.40	3.02	4.34	5.71
7.93	11.79	18.1	1.46	4.40	5.85	2.02	12.07		

TABLE 6.3: Estimates of parameters and model selection criteria for bladder cancer data.

Model	Estimate(Std. Error)	-Log-likelihood	AIC	BIC
Gamma(α, β)	$\hat{\alpha}$=0.9154 (0.0910) $\hat{\beta}$=0.1069 (0.0153)	402.624	809.249	814.953
Log-normal(μ, σ)	$\hat{\mu}$=1.5109 (0.1133) $\hat{\sigma}$=1.2819 (0.0801)	406.803	817.605	823.309
Lindley(θ)	$\hat{\theta}$=0.2129(0.0134)	417.924	837.848	840.610
PL(θ, β)	$\hat{\theta}$=0.2943(0.0371) $\hat{\beta}$=0.8302(0.0472)	413.353	830.707	836.410
TL(λ, θ)	$\hat{\lambda}$=0.6169(0.1688) $\hat{\theta}$=0.1557(0.0150)	415.155	834.310	840.014
EL(α, θ)	$\hat{\alpha}$=0.1648(0.0166) $\hat{\theta}$=0.7330(0.0912)	416.285	836.572	842.274
WL(α, θ)	$\hat{\alpha}$=0.1595(0.0172) $\hat{\theta}$=0.6827(0.1115)	416.442	836.885	842.588
NGPL$(\lambda, \theta, \beta, \delta, \alpha)$	$\hat{\lambda}$=-0.858(0.0938) $\hat{\theta}$=2.5044(1.6547) $\hat{\beta}$=0.3292(0.1341) $\hat{\delta}$=6.6798(2.6466) $\hat{\alpha}$=33.738(15.584)	408.966	827.932	842.192
Xgamma(θ)	$\hat{\theta}$=0.2860(0.0159)	425.169	852.338	855.190
Quasi xgamma(α, θ)	$\hat{\alpha}$=16.827(2.0453) $\hat{\theta}$=0.1298(0.0179)	402.320	808.640	814.344

According to model selection criterion, viz., AIC, the following order of best fit is observed.
The Quasi xgamma distribution comes out to be the best model followed by gamma, lognormal, new generalized power Lindley power Lindley, transmuted Lindley, exponentiated Lindley, weighted Lindley, Lindley and xgamma distributions, respectively.
On the other hand, according to model selection criterion, viz., BIC, the following order of best fit is observed.
The quasi xgamma distribution comes out to be the best model followed by gamma, lognormal, power Lindley, transmuted Lindley, Lindley, exponentiated Lindley, new generalized power Lindley, weighted Lindley and xgamma distributions, respectively.

6.2 A two-parameter xgamma distribution

The objective in this section is to introduce and study an another two-parameter generalization of xgamma distribution by adding an additional parameter α (> 0) to it. We name the distribution as *two-parameter* xgamma distribution. The beauty of this two-parameter extension is that this extension or generalization also, like quasi xgamma distribution, contains xgamma distribution as a special case. Different distributional, survival and/or reliability properties are studied for this two-parameter xgamma distribution and its applicability is demonstrated in modeling lifetime data sets with potential flexibility over existing two-parameter lifetime models.

6.2.1 The two-parameter xgamma distribution

In this subsection the two-parameter xgamma distribution is introduced by adding one extra non-negative parameter α in (2.2).

For synthesizing the density form of the two-parameter xgamma distribution, we consider $f_1(x)$ to follow an exponential distribution with parameter θ and $f_2(x)$ to follow a gamma distribution with scale parameter θ and shape parameter 3 i.e., $f_1(x) \sim exp(\theta)$ and $f_1(x) \sim gamma(3, \theta)$ with $\pi_1 = \frac{\theta}{(\alpha+\theta)}$ and $\pi_2 = 1 - \pi_1$ in (2.1). We have, then, the following definition for two-parameter xgamma distribution.

Definition 6.4. A non-negative continuous random variable, X, is said to follow a two-parameter xgamma (TPXG) distribution with parameters α and θ if its pdf is of the form

$$f(x) = \frac{\theta^2}{(\alpha+\theta)} \left(1 + \frac{\alpha\theta}{2} x^2\right) e^{-\theta x}, x > 0, \theta > 0, \alpha > 0. \tag{6.38}$$

It is denoted by $X \sim TPXG(\alpha, \theta)$.

Special case:

1. Putting $\alpha = 1$ in (6.38), the xgamma distribution with parameter θ can be obtained.

The plot of probability density curves with the form (6.38) for different values of α and θ is shown in the Figure 6.2.

Alternative form:

An alternative form of the two-parameter xgamma distribution can be obtained by putting $\beta = 1/\alpha$ in (6.38) and the form of the pdf can be expressed as

$$f(x) = \frac{\theta^2}{(1+\beta\theta)} \left(\beta + \frac{\theta}{2} x^2\right) e^{-\theta x}, x > 0, \theta > 0, \beta > 0. \tag{6.39}$$

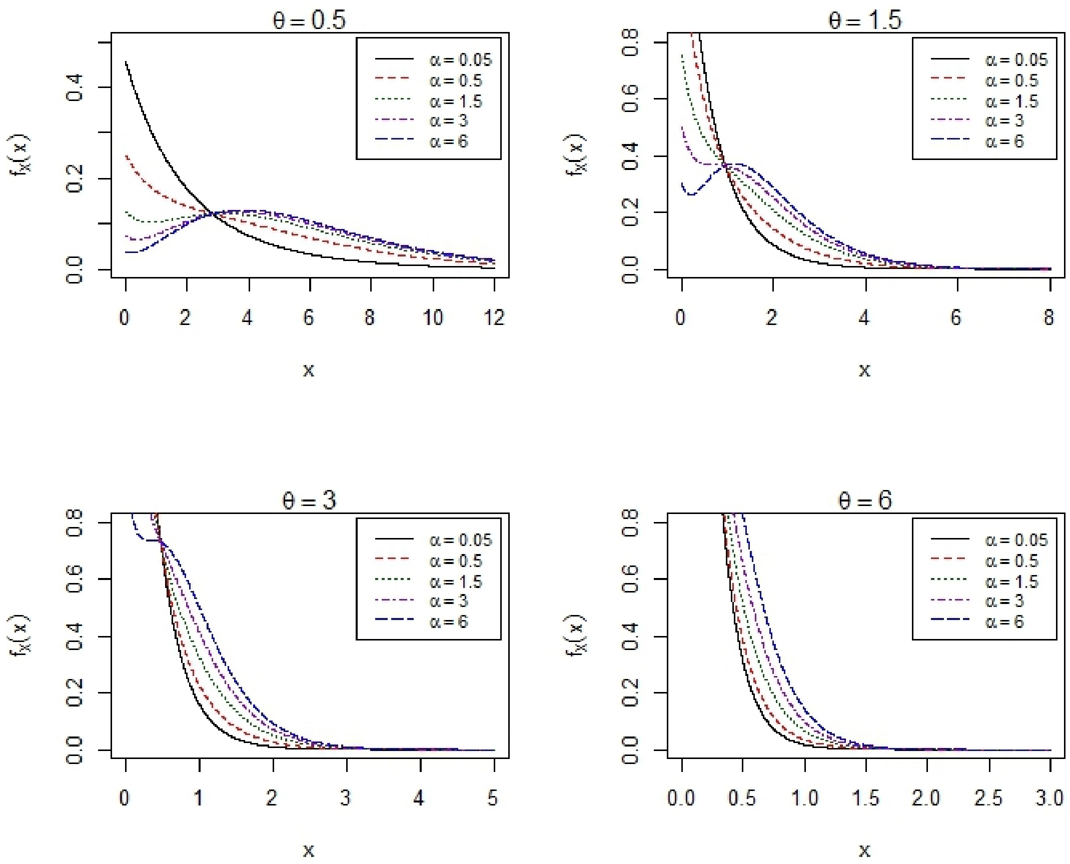

FIGURE 6.2: Probability density curves of $TPXG(\alpha, \theta)$ for different values of α and θ.

Now, for deriving the cdf of $TPXG(\alpha, \theta)$, we calculate,

$$\begin{aligned} \Pr(X > x) &= \int_x^{\infty} \frac{\theta^2}{(\alpha+\theta)} \left(1 + \frac{\alpha\theta}{2} t^2\right) e^{-\theta t} dt, \\ &= \frac{\theta^2}{(\alpha+\theta)} \left[\int_x^{\infty} e^{-\theta t} dt + \frac{\alpha\theta}{2} \int_x^{\infty} t^2 e^{-\theta t} dt\right]. \end{aligned}$$

Using the expressions of integration in (2.3) and (2.5), we have,

$$\begin{aligned}
\Pr(X > x) &= \frac{\theta^2}{(\alpha+\theta)}\left[\frac{e^{-\theta x}}{\theta} + \frac{\alpha\theta}{2}\left\{\frac{x^2 e^{-\theta x}}{\theta} + \frac{2}{\theta}\left(\frac{x e^{-\theta x}}{\theta} + \frac{e^{-\theta x}}{\theta^2}\right)\right\}\right],\\
&= \frac{\theta^2}{(\alpha+\theta)}\left[\frac{e^{-\theta x}}{\theta} + \frac{\alpha x^2 e^{-\theta x}}{2} + \frac{\alpha x e^{-\theta x}}{\theta} + \frac{\alpha e^{-\theta x}}{\theta^2}\right],\\
&= \frac{\theta^2 e^{-\theta x}}{(\alpha+\theta)}\left[\frac{1}{\theta} + \frac{\alpha x^2}{2} + \frac{\alpha x}{\theta} + \frac{\alpha}{\theta^2}\right],\\
&= \frac{\theta^2 e^{-\theta x}}{(\alpha+\theta)}\left[\frac{2\alpha + 2\theta + 2\alpha\theta x + \alpha\theta^2 x^2}{2\theta^2}\right].
\end{aligned}$$

Hence, the cdf corresponding to (6.38) is given by

$$F(x) = 1 - \frac{\left(\alpha + \theta + \alpha\theta x + \frac{1}{2}\alpha\theta^2 x^2\right)}{(\alpha+\theta)} e^{-\theta x}, x > 0. \tag{6.40}$$

6.2.2 Moments and related measures

In this sub-section we study the moments and other related measures for the $TPXG(\alpha, \theta)$.

The r^{th} order raw moment for $X \sim TPXG(\alpha, \theta)$ is obtained as

$$\begin{aligned}
\mu_r' &= E(X^r),\\
&= \int_0^\infty x^r \frac{\theta^2}{(\alpha+\theta)}\left(1 + \frac{\alpha\theta}{2}x^2\right) e^{-\theta x} dx,\\
&= \frac{\theta^2}{(\alpha+\theta)}\left[\int_0^\infty x^r e^{-\theta x} dx + \frac{\alpha\theta}{2}\int_0^\infty x^{r+2} e^{-\theta x} dx\right],\\
&= \frac{\theta^2}{(\alpha+\theta)}\left[\frac{\Gamma(r+1)}{\theta^{r+1}} + \frac{\alpha\theta\Gamma(r+3)}{2\theta^{r+2}}\right],
\end{aligned}$$

Here $\Gamma(a) = \int_0^\infty z^{a-1} e^{-z} dz$ is the gamma function.

$$= \frac{\Gamma(r+1)}{(\alpha+\theta)\theta^{r-1}}\left[1 + \frac{\alpha}{2\theta}(r+1)(r+2)\right].$$

Hence, the r^{th} order non-central moment for $X \sim TPXG(\alpha, \theta)$ is given by

$$\mu'_r = \frac{r!}{2\theta^r(\alpha+\theta)}\left[2\theta + \alpha(1+r)(2+r)\right] \text{ for } r = 1, 2, \ldots. \tag{6.41}$$

In particular, we have

$$\mu'_1 = E(X) = \frac{(\theta+3\alpha)}{\theta(\alpha+\theta)} = \mu(say) \quad ; \quad \mu'_2 = E(X^2) = \frac{2(\theta+6\alpha)}{\theta^2(\alpha+\theta)}. \tag{6.42}$$

Now, using (1.3), the expression for second order central (about mean) moment or the population variance for X can be obtained as

$$\begin{aligned} Var(X) = \mu_2 &= \mu'_2 - \mu^2, \\ &= \frac{2(\theta+6\alpha)}{\theta^2(\alpha+\theta)} - \left[\frac{(\theta+3\alpha)}{\theta(\alpha+\theta)}\right]^2, \\ &= \frac{2(\theta^2+8\alpha\theta+3\alpha^2)}{\theta^2(\alpha+\theta)^2} \text{ (On simplification)}, \end{aligned} \tag{6.43}$$

so that the coefficient of variation becomes

$$\gamma = \frac{\sqrt{2(\theta^2+8\alpha\theta+3\alpha^2)}}{(\theta+3\alpha)}. \tag{6.44}$$

The following theorem shows that $TPXG(\alpha, \theta)$ is unimodal.

Theorem 6.5. *For $\theta > \alpha/2$, the pdf $f(x)$ in (6.38) is decreasing in x.*

Proof. We have from (6.38) the first derivative of $f(x)$ with respect to x as

$$f'(x) = \frac{\theta^2}{(\alpha+\theta)}\left(\alpha\theta x - \theta - \frac{1}{2}\alpha\theta^2 x^2\right)e^{-\theta x}$$

$f'(x)$ is negative in x when $\theta > \alpha/2$, and hence the proof.

So, from the above theorem 6.5, for $\theta \le \alpha/2$, $\frac{d}{dx}f(x) = 0$ implies that $\frac{\left(1+\sqrt{1-2\frac{\theta}{\alpha}}\right)}{\theta}$ is the unique critical point at which $f(x)$ is maximized.

Hence, the mode of $TPXG(\alpha, \theta)$ is given by

$$Mode(X) = \begin{cases} \frac{1+\sqrt{1-\frac{2\theta}{\alpha}}}{\theta}, & \text{if } 0 < \theta \leq \alpha/2. \\ 0, & \text{otherwise.} \end{cases} \tag{6.45}$$

6.2.3 Characteristic and generating functions

In this sub-section, we derive the characteristic, moment generating and cumulant generating functions for $X \sim TPXG(\alpha, \theta)$.
The characteristic function of X is derived as

$$\begin{aligned}\phi_X(t) &= E(e^{itX}),\\ &= \int_0^\infty e^{itx} \frac{\theta^2}{(\alpha+\theta)} \left(1 + \frac{\alpha\theta}{2}x^2\right) e^{-\theta x} dx,\\ &= \frac{\theta^2}{(\alpha+\theta)} \left[\int_0^\infty e^{(\theta - it)x} dx + \frac{\alpha\theta}{2} \int_0^\infty x^2 e^{(\theta-it)x} dx\right],\\ &= \frac{\theta^2}{(\alpha+\theta)} \left[\frac{1}{(\theta - it)} + \frac{\alpha\theta}{2} \frac{\Gamma(3)}{(\theta - it)^3}\right],\end{aligned}$$

Here $\Gamma(a) = \int_0^\infty z^{a-1} e^{-z} dz$ is the gamma function.

$$= \frac{\theta^2}{(\alpha+\theta)} \left[(\theta - it)^{-1} + \alpha\theta(\theta - it)^{-3}\right]; t \in \Re, i = \sqrt{-1}. \tag{6.46}$$

In a very similar way, the moment generating function of X is derived as

$$\begin{aligned}M_X(t) &= E(e^{tX}),\\ &= \int_0^\infty e^{tx} \frac{\theta^2}{(\alpha+\theta)} \left(1 + \frac{\alpha\theta}{2}x^2\right) e^{-\theta x} dx,\\ &= \frac{\theta^2}{(\alpha+\theta)} \left[\int_0^\infty e^{(\theta - t)x} dx + \frac{\alpha\theta}{2} \int_0^\infty x^2 e^{(\theta-t)x} dx\right].\end{aligned}$$

Hence,

$$M_X(t) = \frac{\theta^2}{(\alpha+\theta)}\left[\frac{1}{(\theta-t)} + \frac{\alpha\theta}{2}\frac{\Gamma(3)}{(\theta-t)^3}\right],$$

Here $\Gamma(a) = \int_0^\infty z^{a-1}e^{-z}dz$ is the gamma function.

$$= \frac{\theta^2}{(\alpha+\theta)}\left[(\theta-t)^{-1} + \alpha\theta(\theta-t)^{-3}\right]; t \in \Re. \tag{6.47}$$

The cumulant generating function of X is obtained as

$$K_X(t) = \ln M_X(t),$$
$$= \ln\frac{\theta^2}{(\alpha+\theta)(\theta-t)} + \ln\left[1 + \frac{\alpha\theta}{(\theta-t)^2}\right]; t \in \Re. \tag{6.48}$$

6.2.4 Entropy measures

When $X \sim TPXG(\alpha, \theta)$, we derive the Rényi entropy. For $\gamma > 0 (\neq 1)$, we have,

$$\int_0^\infty f^\gamma(x)dx$$
$$= \frac{\theta^{2\gamma}}{(\alpha+\theta)^\gamma}\int_0^\infty \left(1 + \frac{\alpha\theta}{2}x^2\right)^\gamma e^{-\gamma\theta x}dx,$$
$$= \frac{(\theta)^{2\gamma}}{(\alpha+\theta)^\gamma}\int_0^\infty \left(1 + \frac{\alpha\theta}{2}x^2\right)^\gamma e^{-\gamma\theta x}dx.$$

Using power series expansion

$$\left(1 + \frac{\alpha\theta}{2}x^2\right)^\gamma = \sum_{j=0}^{\gamma}\binom{\gamma}{j}\left(\frac{\alpha\theta x^2}{2}\right)^j,$$

we obtain,

$$\begin{aligned}
\int_0^\infty f^\gamma(x)dx &= \frac{\theta^{2\gamma}}{(\alpha+\theta)^\gamma}\int_0^\infty \sum_{j=0}^{\gamma}\binom{\gamma}{j}\left(\frac{\alpha\theta x^2}{2}\right)^j e^{-\gamma\theta x}dx,\\
&= \frac{\theta^{2\gamma}}{(\alpha+\theta)^\gamma}\sum_{j=0}^{\gamma}\binom{\gamma}{j}\left(\frac{\alpha\theta}{2}\right)^j \int_0^\infty x^{2j}e^{-\gamma\theta x}dx,\\
&= \frac{\theta^{2\gamma}}{(\alpha+\theta)^\gamma}\sum_{j=0}^{\gamma}\binom{\gamma}{j}\left(\frac{\alpha\theta}{2}\right)^j \frac{\Gamma(2j+1)}{(\gamma\theta)^{2j+1}},\\
&\text{Here } \Gamma(a) = \int_0^\infty z^{a-1}e^{-z}dz \text{ is the gamma function.}\\
&= \frac{\theta^{2\gamma}}{(\alpha+\theta)^\gamma}\sum_{j=0}^{\gamma}\binom{\gamma}{j}\left(\frac{\alpha}{2}\right)^j \frac{\Gamma(2j+1)}{\theta^{j+1}\gamma^{2j+1}}. \qquad (6.49)
\end{aligned}$$

Hence, the Rényi entropy is given by

$$\begin{aligned}
H_R(\gamma) &= \frac{1}{1-\gamma}\ln\left[\int_0^\infty f^\gamma(x)dx\right],\\
&= \frac{1}{1-\gamma}\left[2\gamma\ln\theta - \gamma\ln(\alpha+\theta)\right] + \frac{1}{1-\gamma}\ln\left[\sum_{j=0}^{\gamma}\binom{\gamma}{j}\left(\frac{\alpha}{2}\right)^j\frac{\Gamma(2j+1)}{\theta^{j+1}\gamma^{2j+1}}\right].
\end{aligned} \qquad (6.50)$$

When $X \sim TPXG(\alpha,\theta)$, to find Tsallis measure of entropy, we derive $\int_0^\infty f^q(x)dx$ in a very similar fashion by replacing γ with q in (6.49) to obtain

$$\int_0^\infty f^q(x)dx = \frac{\theta^{2q}}{(\alpha+\theta)^q}\sum_{j=0}^{q}\binom{q}{j}\left(\frac{\alpha}{2}\right)^j\frac{\Gamma(2j+1)}{\theta^{j+1}q^{2j+1}} \text{ for } q>0(\neq 1).$$

Hence, Tsallis measure of entropy is given by

$$S_q(X) = \frac{1}{q-1}\ln\left[1 - \frac{\theta^{2q}}{(\alpha+\theta)^q}\sum_{j=0}^{q}\binom{q}{j}\left(\frac{\alpha}{2}\right)^j\frac{\Gamma(2j+1)}{\theta^{j+1}q^{2j+1}}\right]. \qquad (6.51)$$

Next, we find Shannon measure of entropy for $TPXG(\alpha,\theta)$.

We have, by definition of Shannon entropy,

$$
\begin{aligned}
H(f) &= E[-\ln f(x)],\\
&= -\int_0^\infty \ln\left\{\frac{\theta^2}{(\alpha+\theta)}\left(1+\frac{\alpha\theta}{2}x^2\right)e^{-\theta x}\right\}f(x)dx,\\
&= -\int_0^\infty \left[\ln\left(\frac{\theta^2}{\alpha+\theta}\right)+\ln\left(1+\frac{\alpha\theta}{2}x^2\right)-\theta x\right]f(x)dx,\\
&= -\left[\int_0^\infty \ln\left(\frac{\theta^2}{\alpha+\theta}\right)f(x)dx+\int_0^\infty \ln\left(1+\frac{\alpha\theta}{2}x^2\right)f(x)dx-\theta\int_0^\infty xf(x)dx\right],\\
&\text{Since } \int_0^\infty xf(x)dx = E(X) \text{ and } \int_0^\infty f(x)dx = 1, \text{ we have,}\\
&= -\left[\ln\left(\frac{\theta^2}{\alpha+\theta}\right)+\int_0^\infty \ln\left(1+\frac{\alpha\theta}{2}x^2\right)f(x)dx-\frac{\theta(\theta+3\alpha)}{\theta(\alpha+\theta)}\right]. \qquad (6.52)
\end{aligned}
$$

Now, we find

$$\int_0^\infty \ln\left(1+\frac{\alpha\theta}{2}x^2\right)f(x)dx.$$

We have,

$$
\begin{aligned}
&\int_0^\infty \ln\left(1+\frac{\alpha\theta}{2}x^2\right)f(x)dx\\
&= \frac{\theta^2}{(\alpha+\theta)}\int_0^\infty \ln\left(1+\frac{\alpha\theta}{2}x^2\right)\left(1+\frac{\alpha\theta}{2}x^2\right)e^{-\theta x}dx,\\
&= \frac{\theta^2}{(\alpha+\theta)}\left[\int_0^\infty \ln\left(1+\frac{\alpha\theta}{2}x^2\right)e^{-\theta x}dx+\frac{\alpha\theta}{2}\int_0^\infty \ln\left(1+\frac{\alpha\theta}{2}x^2\right)x^2e^{-\theta x}dx\right].
\end{aligned}
$$

Putting

$$\ln\left(1+\frac{\theta}{2}x^2\right) = \sum_{j=1}^{\infty}(-1)^{j+1}\frac{(\frac{\alpha\theta}{2}x^2)^j}{j},$$

we have,

$$\begin{aligned}
&\int_0^\infty \ln\left(1+\frac{\alpha\theta}{2}x^2\right) f(x)dx \\
&= \frac{\theta^2}{(\alpha+\theta)}\sum_{j=1}^{\infty}\frac{(-1)^{j+1}(\alpha\theta)^j}{j2^j}\int_0^\infty x^{2j}e^{-\theta x}dx \\
&+ \frac{\alpha\theta^3}{2(\alpha+\theta)}\sum_{j=1}^{\infty}\frac{(-1)^{j+1}(\alpha\theta)^j}{j2^j}\int_0^\infty x^{2j+2}e^{-\theta x}dx, \\
&= \frac{\theta^2}{(\alpha+\theta)}\sum_{j=1}^{\infty}\frac{(-1)^{j+1}\alpha^j\theta^j}{j2^j}\frac{\Gamma(2j+1)}{\theta^{2j+1}} + \frac{\alpha\theta^3}{2(\alpha+\theta)}\sum_{j=1}^{\infty}\frac{(-1)^{j+1}\alpha^j\theta^j}{j2^j}\frac{\Gamma(2j+3)}{\theta^{2j+3}}.
\end{aligned} \tag{6.53}$$

Here $\Gamma(a) = \int_0^\infty z^{a-1}e^{-z}dz$ is the gamma function.

Using (6.53), from (6.52) the final expression for Shannon entropy is given by

$$\begin{aligned}
H(f) = \left(\frac{3\alpha+\theta}{\alpha+\theta}\right) - \ln\frac{\theta^2}{(\alpha+\theta)} - \frac{\theta^2}{(\alpha+\theta)}\sum_{j=1}^{\infty}(-1)^{j+1}\frac{(\alpha/2)^j}{\theta^{j+1}j} \\
\left[\Gamma(2j+1) + \frac{\alpha}{2\theta}\Gamma(2j+3)\right].
\end{aligned} \tag{6.54}$$

6.2.5 Distributions of order statistics

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n drawn from $TPXG(\alpha, \theta)$. Denote $X_{j:n}$ as the j^{th} order statistic. Then $X_{1:n}$ and $X_{n:n}$ denote the smallest and largest order statistics, respectively.

For any $x > 0$, the pdf of $X_{1:n}$ can be derived as

$$\begin{aligned}
f_{X_{1:n}}(x) &= n[1-F(x)]^{n-1}f(x), \\
&= \frac{n\theta^2}{(\alpha+\theta)^n}\left(1+\frac{\alpha\theta}{2}x^2\right)\left[\alpha+\theta+\alpha\theta x+\frac{1}{2}\alpha\theta^2x^2\right]^{n-1}e^{-n\theta x}.
\end{aligned} \tag{6.55}$$

Similarly, for any $x > 0$, the pdf of $X_{n:n}$ is obtained as

$$f_{X_{n:n}}(x) = n[F(x)]^{n-1}f(x),$$
$$= \frac{n\theta^2}{(\alpha+\theta)^n}\left(1+\frac{\alpha\theta}{2}x^2\right)\left[(\alpha+\theta)(1-e^{-\theta x})-\left(1+\frac{\theta x}{2}\right)\alpha\theta x e^{-\theta x}\right]^{n-1} e^{-\theta x}. \quad (6.56)$$

6.2.6 Survival properties

In this sub-section different properties related to survival and/or reliability for $TPXG(\alpha,\theta)$ are studied.

The survival function of X is given by

$$S(x) = (X > x) = \frac{\left(\alpha+\theta+\alpha\theta x+\frac{1}{2}\alpha\theta^2x^2\right)}{(\alpha+\theta)}e^{-\theta x}, x > 0. \quad (6.57)$$

6.2.6.1 Hazard rate or failure rate function

The hazard rate (or failure rate) function of X is obtained as

$$h(x) = \frac{f(x)}{S(x)} = \frac{\theta^2\left(1+\frac{\alpha\theta}{2}x^2\right)}{\left(\alpha+\theta+\alpha\theta x+\frac{1}{2}\alpha\theta^2x^2\right)}, x > 0. \quad (6.58)$$

Note. The hazard rate function in (6.58) is increasing for $x > \sqrt{\frac{2}{\alpha\theta}}$ with the bounds

$$f(0) = \frac{\theta^2}{(\alpha+\theta)} < h(x) < \theta.$$

Figure 6.3 shows the plot of hazard rate function of $TPXG(\alpha,\theta)$ for different values of α and θ.

The following theorem shows that the failure rate function of $TPXG(\alpha,\theta)$ is sometimes IFR and sometimes DFR depending on values of x.

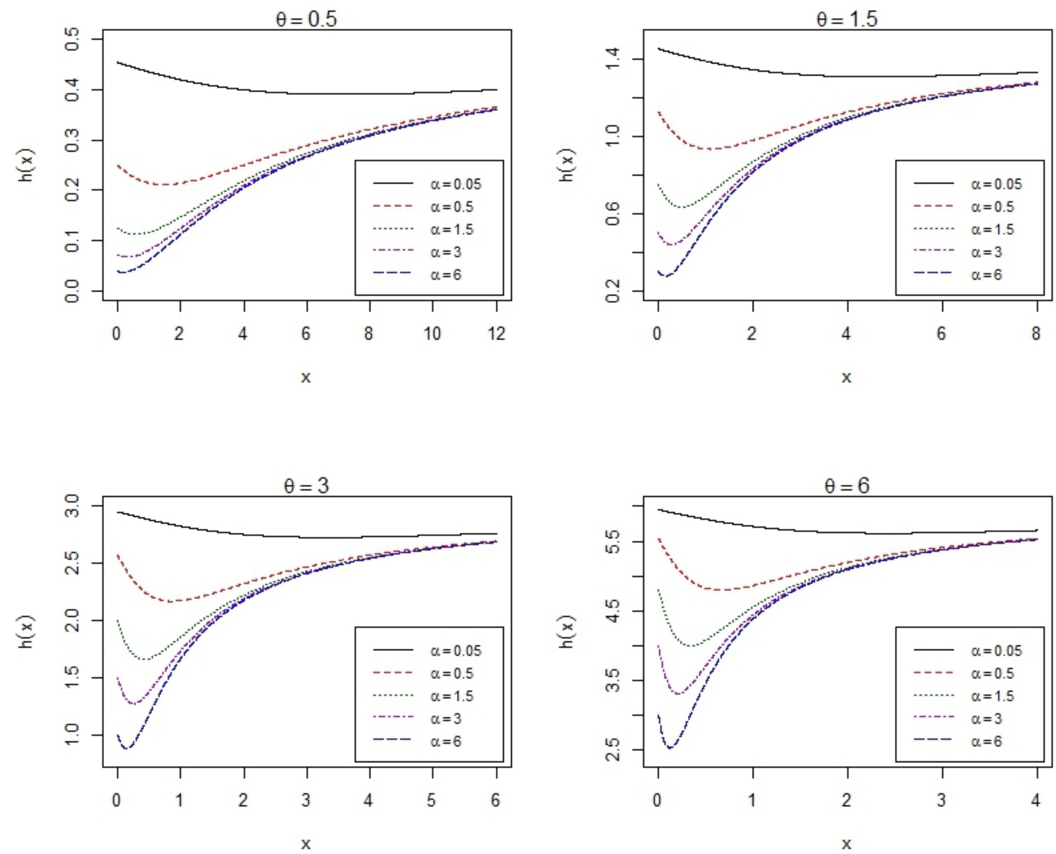

FIGURE 6.3: Plot of hazard rate function of $TPXG(\alpha, \theta)$ for different values of α and θ.

Theorem 6.6. *The failure rate $h(x)$ given in (6.58) is increasing failure rate (IFR) in distribution for $x > \sqrt{\frac{2}{\alpha\theta}}$ and is decreasing failure rate (DFR) in distribution for $x < \sqrt{\frac{2}{\alpha\theta}}$.*

Proof. The proof comes immediately as the pdf given in (6.38) is log-concave for $x > \sqrt{\frac{2}{\alpha\theta}}$ and log-convex for $x < \sqrt{\frac{2}{\alpha\theta}}$.

6.2.6.2 MRL and reversed hazard rate function

The MRL function of $TPXG(\alpha, \theta)$ can be derived as

$$
\begin{aligned}
m(x) &= \frac{1}{S(x)} \int_x^\infty S(t)dt, \\
&= \frac{1}{(\alpha+\theta)S(x)} \int_x^\infty \left(\alpha + \theta + \alpha\theta t + \frac{1}{2}\alpha\theta^2 t^2\right) e^{-\theta t} dt, \\
&= \frac{1}{(\alpha+\theta)S(x)} \left[(\alpha+\theta)\int_x^\infty e^{-\theta t}dt + \alpha\theta \int_x^\infty t e^{-\theta t}dt + \frac{\alpha\theta^2}{2}\int_x^\infty t^2 e^{-\theta t}dt\right],
\end{aligned}
$$

Using the expressions for integration in (2.3), (2.4) and (2.5), we have,

$$
\begin{aligned}
&= \frac{1}{(\alpha+\theta)S(x)} \left[e^{-\theta x} + \frac{3\alpha e^{-\theta x}}{\theta} + 2\alpha x e^{-\theta x} + \frac{\alpha\theta x^2 e^{-\theta x}}{2}\right], \\
&= \frac{e^{-\theta x}}{(\alpha+\theta)S(x)} \left[1 + \frac{3\alpha}{\theta} + 2\alpha x + \frac{\alpha\theta x^2}{2}\right], \\
&= \frac{\left(\theta + 3\alpha + 2\alpha\theta x + \frac{1}{2}\alpha\theta^2 x^2\right)}{\theta\left(\alpha + \theta + \alpha\theta x + \frac{1}{2}\alpha\theta^2 x^2\right)}, \\
&= \frac{\left(\alpha + \theta + \alpha\theta x + \frac{1}{2}\alpha\theta^2 x^2\right) + \alpha(2+\theta x)}{\theta\left(\alpha + \theta + \alpha\theta x + \frac{1}{2}\alpha\theta^2 x^2\right)}.
\end{aligned}
$$

Hence, the MRL function is given by

$$
m(x) = \frac{1}{\theta} + \frac{\alpha(2+\theta x)}{\theta\left(\alpha + \theta + \alpha\theta x + \frac{1}{2}\alpha\theta^2 x^2\right)}. \tag{6.59}
$$

Note. The MRL function in (6.59) is bounded with the following limits.

$$
\frac{1}{\theta} < m(x) < \frac{(\theta + 3\alpha)}{\theta(\alpha+\theta)} = E(X).
$$

The reversed hazard rate function of $TPXG(\alpha, \theta)$ is obtained as

$$
r(x) = \frac{f(x)}{F(x)} = \frac{\theta^2\left(1 + \frac{\alpha\theta}{2}x^2\right)e^{-\theta x}}{(\alpha+\theta)(1-e^{-\theta x}) - \left(1 + \frac{\theta x}{2}\right)\alpha\theta x e^{-\theta x}}, x > 0. \tag{6.60}
$$

6.2.6.3 Stochastic ordering

Here, the stochastic ordering relations for random variables following $TPXG(\alpha, \theta)$ are studied.

The following theorems shows that the two-parameter xgamma distribution is ordered with respect to the strongest likelihood ratio ordering and thereby the other orderings.

Theorem 6.7. *Let $X_1 \sim TPXG(\alpha_1, \theta_1)$ and $X_2 \sim TPXG(\alpha_2, \theta_2)$. If $\alpha_1 = \alpha_2$ and $\theta_1 \geq \theta_2$ (or, if $\theta_1 = \theta_2$ and $\alpha_1 \leq \alpha_2$), then $X_1 \leq_{LR} X_2$ and hence $X_1 \leq_{HR} X_2$, $X_1 \leq_{MRL} X_2$ and $X_1 \leq_{ST} X_2$.*

Proof. Let us denote the pdf of X_1 as $f_{X_1}(x)$ and that of X_2 be $f_{X_2}(x)$ for $x > 0$. We have then the ratio

$$\frac{f_{X_1}(x)}{f_{X_2}(x)} = \frac{\theta_1^2(\alpha_2 + \theta_2)}{\theta_2^2(\alpha_1 + \theta_1)} \left(\frac{1 + \frac{\alpha_1\theta_1}{2}x^2}{1 + \frac{\alpha_2\theta_2}{2}x^2} \right) e^{-(\theta_1 - \theta_2)x}$$

Taking logarithm both sides, we have

$$\ln\left[\frac{f_{X_1}(x)}{f_{X_2}(x)}\right] = 2\ln\left(\frac{\theta_1}{\theta_2}\right) + \ln\left(\frac{\alpha_2 + \theta_2}{\alpha_1 + \theta_1}\right) + \ln\left(\frac{1 + \frac{\alpha_1\theta_1}{2}x^2}{1 + \frac{\alpha_2\theta_2}{2}x^2}\right) - (\theta_1 - \theta_2)x.$$

The first derivative with respect to x gives

$$\frac{d}{dx}\ln\left[\frac{f_{X_1}(x)}{f_{X_2}(x)}\right] = \frac{(\alpha_1\theta_1 - \alpha_2\theta_2)x}{\left(1 + \frac{\alpha_1\theta_1}{2}x^2\right)\left(1 + \frac{\alpha_2\theta_2}{2}x^2\right)} - (\theta_1 - \theta_2),$$

which is negative when $\alpha_1 = \alpha_2$ and $\theta_1 \geq \theta_2$ (or, when $\theta_1 = \theta_2$ and $\alpha_1 \leq \alpha_2$), i.e., $\frac{f_{X_1}(x)}{f_{X_2}(x)}$ decreases in x when $\alpha_1 = \alpha_2$ and $\theta_1 \geq \theta_2$ (or, when $\theta_1 = \theta_2$ and $\alpha_1 \leq \alpha_2$), so $X_1 \leq_{LR} X_2$ and the other orderings follow automatically by (1.21). Hence the proof.

Now, we establish stochastic order relationships between two random variables, X and Y, when $X \sim TPXG(\alpha_1, \theta_1)$ and $Y \sim QXG(\alpha_2, \theta_2)$. We have the following theorem.

Theorem 6.8. *Let $X \sim TPXG(\alpha_1, \theta_1)$ and $Y \sim QXG(\alpha_2, \theta_2)$. If $\alpha_1 = \alpha_2 = \alpha$(say), then $X \leq_{LR} Y$ whenever $\frac{(\theta_1-\theta_2)+\theta_2^2}{\theta_1} \geq \alpha^2$ and $\theta_1 > \theta_2$. Again, if $\theta_1 = \theta_2 = \theta$(say), then $X \leq_{LR} Y$ whenever $\alpha_1 \leq \frac{\theta}{\alpha_2}$.*

Proof. The proof comes immediately following the similar arguments as followed in the proof of the Theorem 6.7. Hence the proof.

6.2.7 Parameter estimation

In this sub-section method of moments and method of maximum likelihood are proposed for estimating α and θ when $X \sim TPXG(\alpha, \theta)$ for complete sample situation.Let $X_1, X_2, \ldots, X_n$ be a random sample of size n drawn from $TPXG(\alpha, \theta)$. Denote $\bar{X}$ as sample mean.

6.2.7.1 Method of moments

Using the first two raw moments given in (6.42), we have

$$\frac{\mu_2'}{\mu_1'^2} = \frac{2(\theta + 6\alpha)(\alpha + \theta)}{(\theta + 3\alpha)^2} = k(\text{say}) \tag{6.61}$$

Taking $\theta = c\alpha$, we have

$$\frac{\mu_2'}{\mu_1'^2} = \frac{2(c+6)(c+1)}{(c+3)^2} = k$$

which gives a quadratic equation in c as

$$(2-k)c^2 + (14-6k)c + (12-9k) = 0. \tag{6.62}$$

An estimate of k is easily obtained by replacing $\mu_1^{'}$ and $\mu_2^{'}$ by sample moments $\bar{X}$ and $m_2^{'}$, respectively, in equation (6.61). This estimate can then be utilized to solve (6.62) to obtain an estimate of c.

Again, from the first moment equation, we have

$$\bar{X} = \frac{(c+3)}{\alpha c(c+1)}$$

and thus moment estimator of α, $\hat{\alpha}_M$ (say), is given by

$$\hat{\alpha}_M = \left[\frac{(c+3)}{c(c+1)}\right]\frac{1}{\bar{X}}. \tag{6.63}$$

Finally, the moment estimator, $\hat{\theta}_M$ (say), of θ is obtained as

$$\hat{\theta}_M = \left(\frac{c+3}{c+1}\right)\frac{1}{\bar{X}}. \tag{6.64}$$

6.2.7.2 Method of maximum likelihood

Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ be n observations or realizations on a random sample $X_1, X_2, \ldots, X_n$ of size n drawn from $X \sim TPXG(\alpha, \theta)$. We have the likelihood function as

$$L(\alpha, \theta|\boldsymbol{x}) = \prod_{i=1}^{n} \frac{\theta^2}{(\alpha+\theta)}\left(1+\frac{\alpha\theta}{2}x_i^2\right)e^{-\theta x_i} = \frac{\theta^{2n}}{(\alpha+\theta)^n}e^{-\theta\sum_{i=1}^{n}x_i}\prod_{i=1}^{n}\left(1+\frac{\alpha\theta}{2}x_i^2\right). \tag{6.65}$$

The log-likelihood function is given by

$$\ln L(\alpha, \theta|\boldsymbol{x}) = 2n\ln\theta - n\ln(\alpha+\theta) - \theta\left(\sum_{i=1}^{n}x_i\right) + \sum_{i=1}^{n}\ln\left(1+\frac{\alpha\theta}{2}x_i^2\right). \tag{6.66}$$

To find out the maximum likelihood estimators (MLEs), $\hat{\alpha}$ and $\hat{\theta}$, of α and θ, we have two log-likelihood equations as

$$\frac{\partial \ln L(\alpha, \theta|\boldsymbol{x})}{\partial \alpha} = \sum_{i=1}^{n} \left(\frac{\frac{\theta}{2} x_i^2}{1 + \frac{\alpha\theta}{2} x_i^2} \right) - \frac{n}{(\alpha + \theta)} = 0 \tag{6.67}$$

and

$$\frac{\partial \ln L(\alpha, \theta|\boldsymbol{x})}{\partial \theta} = \frac{2n}{\theta} - \frac{n}{(\alpha + \theta)} + \sum_{i=1}^{n} \left(\frac{\frac{\alpha}{2} x_i^2}{1 + \frac{\alpha\theta}{2} x_i^2} \right) - \sum_{i=1}^{n} x_i = 0 \tag{6.68}$$

respectively.

Though the log-likelihood equations cannot be solved analytically, one can utilize numerical method, like, *Newton-Raphson*, for solving (6.67) and (6.68) to obtain the maximum likelihood estimators, $\hat{\alpha}$ and $\hat{\theta}$, respectively.

6.2.8 Simulation study

This sub-section deals with the random sample generation algorithm for generating random samples of specific size from the $TPXG(\alpha, \theta)$ distribution supported by a Monte-Carlo simulation study to observe the behaviour of the estimates of unknown parameters α and θ.

We make use of the fact that the distribution given in (6.38) is a special finite mixture of $exp(\theta)$ and $gamma(3, \theta)$ for describing sample generation algorithm.

To generate a random sample of size n from $TPXG(\alpha, \theta)$, we have the following simulation algorithm.

(i) Generate $U_i \sim uniform(0, 1), i = 1, 2, \ldots, n$.

(ii) Generate $V_i \sim exp(\theta), i = 1, 2, \ldots, n$.

(iii) Generate $W_i \sim gamma(3, \theta), i = 1, 2, \ldots, n$.

(iv) If $U_i \leq \frac{\theta}{\alpha+\theta}$, then set $X_i = V_i$, otherwise, set $X_i = W_i$.

A Monte-Carlo simulation study is carried out by considering $N = 10,000$ times for selected values of n, α and θ. Samples of sizes 20, 30, 50, 80 and 100 are considered and values of (α, θ) are taken as $(0.5, 0.5)$, $(1.5, 2.0)$ and $(3.0, 4.0)$.

TABLE 6.4: Estimates and average MSEs of α and θ for different sample sizes.

		$\alpha = 0.1, \theta = 0.5$		
n	$\hat{\alpha}$	MSE of $\hat{\alpha}$	$\hat{\theta}$	MSE of $\hat{\theta}$
20	0.3621	1.3402	0.6597	0.8742
50	0.2106	1.2201	0.5892	0.6420
80	0.1976	1.1046	0.5108	0.5602
100	0.1691	1.0042	0.5032	0.4763
		$\alpha = 0.1, \theta = 1.5$		
n	$\hat{\alpha}$	MSE of $\hat{\alpha}$	$\hat{\theta}$	MSE of $\hat{\theta}$
20	0.3986	1.8756	1.6942	0.8966
50	0.2654	1.4320	1.5730	0.7021
80	0.1976	1.2205	1.5107	0.4503
100	0.1430	0.9986	1.5002	0.3064
		$\alpha = 1.5, \theta = 0.5$		
n	$\hat{\alpha}$	MSE of $\hat{\alpha}$	$\hat{\theta}$	MSE of $\hat{\theta}$
20	2.0166	2.3106	0.6879	0.9845
50	1.9822	1.9658	0.5983	0.6650
80	1.7043	1.4576	0.5127	0.4501
100	1.6503	1.1212	0.5026	0.3326
		$\alpha = 1.5, \theta = 2.5$		
n	$\hat{\alpha}$	MSE of $\hat{\alpha}$	$\hat{\theta}$	MSE of $\hat{\theta}$
20	2.1551	3.2249	2.6158	0.5344
50	1.9256	1.8867	2.5310	0.2776
80	1.8282	1.4404	2.5100	0.2047
100	1.7675	1.2444	2.5004	0.1753
		$\alpha = 3.0, \theta = 5.0$		
n	$\hat{\alpha}$	MSE of $\hat{\alpha}$	$\hat{\theta}$	MSE of $\hat{\theta}$
20	4.6542	2.4328	5.7643	1.2376
50	4.1035	2.0122	5.3066	1.0544
80	3.6479	1.8768	5.1006	0.8790
100	3.4509	1.0256	5.0016	0.6504

The following measures are computed in simulation study.

(a) Average mean square error (MSE) of the simulated estimates $\hat{\alpha}_i$, $i = 1, 2, \ldots, N$:

$$\hat{\alpha} = \frac{1}{N}\sum_{i=1}^{N}(\hat{\alpha}_i - \alpha)^2.$$

(b) Average mean square error (MSE) of the simulated estimates $\hat{\theta}_i$, $i = 1, 2, \ldots, N$:

$$\hat{\theta} = \frac{1}{N}\sum_{i=1}^{N}(\hat{\theta}_i - \theta)^2.$$

The results of the simulation study is shown in Table 6.4. Statistical software R is utilized for computation. The following observations are made from the simulation study.

(i) The estimates of α and θ get closer to the corresponding true values as the sample size, n, increases.

(ii) The average mean square errors for estimates of α and estimates θ decrease with increasing sample size.

6.2.9 Application

In this section two different time-to-event data sets are analyzed for illustrating the applicability of two-parameter xgamma distribution. For comparison purpose, besides two-parameter xgamma distribution, we consider five other two parameter lifetime distributions, viz., gamma distribution with shape α and rate θ, Weibull distribution with shape α and scale β, log-normal distribution with parameters μ and σ, two-parameter Lindley distribution (TPLD) with parameters α and λ (Shanker et al., 2013) and quasi xgamma distribution with parameters α and θ.

In order to compare the models, criteria like, negative log-likelihood, AIC and BIC are considered. The better distribution corresponds to smaller negative log-likelihood, AIC and BIC values. Method of maximum likelihood is used for estimating the model parameters for both the data sets. Statistical software R is utilized for analysis.

Illustration I: As a first illustration we consider a data set on the failure times of an electronic device reported in Wang (2000). Table 6.5 represents the data of 18 failure times of an electronic device.

Illustration II: As a second illustration a data set on the lifetimes of a device reported in Aarset (1987) is considered. Table 6.6 represents the data of 50 lifetimes of a device.

Table 6.5: Data on time to failure of 18 electronic devices.

5	11	21	31	46	75	98	122	145	165	196	224
245	293	321	330	350	420						

Table 6.6: Data on lifetimes of 50 devices.

0.1	0.2	1.0	1.0	1.0	1.0	1.0	2.0	3.0	6.0	7.0	11
12	18	18	18	18	18	21	32	36	40	45	46
47	50	55	60	63	63	67	67	67	67	72	75
79	82	82	83	84	84	84	85	85	85	85	85
86	86										

Table 6.7 shows the estimates of the model parameter(s) with standard error(s) of estimates in parenthesis and model selection criteria for the first data set in Table 6.5. Table 6.8 shows the estimates of the model parameter(s) with standard error(s) of estimates in parenthesis and model selection criteria for the data set represented in Table 6.6.

In each of the above illustrations, $TPXG(\alpha, \theta)$ provides better fit (in view of -log-likelihood, AIC and BIC values) as compared to the well-known lifetime models for the considered data sets. Hence, the two-parameter extension of xgamma distribution provides flexibility in modeling real life data sets in comparison with other two-parameter life distributions in the literature.

Table 6.7: Estimates of model parameters and model selection criteria for failure times data of 18 electronic devices.

Distributions	Estimate(Std. Error)	-Log-likelihood	AIC	BIC
Gamma(α,θ)	$\hat{\alpha}$=1.1131 (0.3206) $\hat{\theta}$=0.0064 (0.0022)	110.60	225.21	226.99
Weibull(α,β)	$\hat{\alpha}$=1.1458 (0.2287) $\hat{\beta}$=179.69 (38.6837)	110.45	224.89	226.67
Log-normal(μ,σ)	$\hat{\mu}$=4.6358 (0.2952) $\hat{\sigma}$=1.2523 (0.2087)	113.03	230.07	231.85
TPLD(α,λ)	$\hat{\alpha}$=0.0090 (0.0134) $\hat{\lambda}$=0.0087 (0.0024)	110.30	224.59	226.37
QXG(α,θ)	$\hat{\alpha}$=0.7251 (0.5740) $\hat{\theta}$=0.0125 (0.0027)	110.24	224.48	226.26
TPXG(α,θ)	$\hat{\alpha}$=0.0173 (0.0158) $\hat{\theta}$=0.0125 (0.0027)	109.62	223.25	225.03

Table 6.8: Estimates of model parameters and model selection criteria for data on lifetimes of 50 devices.

Distributions	Estimate(Std. Error)	-Log-likelihood	AIC	BIC
Gamma(α,θ)	$\hat{\alpha}$=0.7990 (0.1375) $\hat{\theta}$=0.0175 (0.0041)	240.19	484.38	488.20
Weibull(α,β)	$\hat{\alpha}$=0.9492 (0.1196) $\hat{\beta}$=44.9194 (6.9458)	241.00	486.00	489.83
Log-normal(μ,σ)	$\hat{\mu}$=3.0790 (0.2472) $\hat{\sigma}$=1.7481 (0.1748)	252.82	509.65	513.47
TPLD(α,λ)	$\hat{\alpha}$=0.0256 (0.0224) $\hat{\lambda}$=0.0317 (0.0053)	240.16	484.33	488.15
QXG(α,θ)	$\hat{\alpha}$=0.7022 (0.2984) $\hat{\theta}$=0.0476 (0.0056)	237.12	478.24	482.06
TPXG(α,θ)	$\hat{\alpha}$=0.0677 (0.0330) $\hat{\theta}$=0.0476 (0.0056)	236.73	477.47	481.29

6.3 Conclusion

To facilitate better modeling of survival data sets there has been a great interest among statisticians and applied researchers in constructing flexible lifetime models. As a consequence, a significant progress has been made towards the generalization and/or extension of some well-known lifetime models and their successful application to data coming from diverse areas.

In this chapter, two extensions (or generalizations) of one parameter xgamma distribution, viz. quasi xgamma and two-parameter xgamma, are proposed, different distributional and survival properties are studied, methods of estimating unknown parameters are addressed for complete sample situations and their application in the area of survival and/or reliability studies are accomplished with real data illustrations and comparison with other life distributions. The following important findings are made in this chapter.

(i) Both the proposed distributions are special finite mixtures of $exp(\theta)$ and $gamma(3, \theta)$ distributions with different mixing proportions.

(ii) Both the distributions proposed provide additional flexibility over xgamma distribution in view of their distributional and survival properties.

(iii) Both the distributions possess strong likelihood ratio ordering. Moreover, Two-parameter xgamma random variables are stochastically smaller than those of quasi xgamma in likelihood ratio and other orderings.

(iv) Real data analyses revealed that both the proposed distributions are quite competent in modeling time-to-event data sets.

Open research problems:

The present chapter opens the following scopes for future research.

- Besides the investigation for applications in other potential areas apart from survival and reliability, the aspects of Bayesian estimations of the parameters for both the distributions proposed in this chapter under different loss functions and censoring schemes could be important investigation.

- Problem of discriminating between the quasi xgamma and the two-parameter xgamma distributions for a given sample could be an important model selection methodology building.

- Bivariate and multivatiate extensions of both the distributions could be interesting generalizations.

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