Contributions to the Study of Transient Analysis of Bulk Arrival Non Markovian Queueing System with Vacation

A Thesis submitted to *Pondicherry University*

in partial fulfilment for the award of the degree of

Doctor of Philosophy

in Mathematics

By

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Under the guidance of

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CERTIFICATE

This is to certify that the thesis entitled "**Contributions to the Study of Transient Analysis of Bulk Arrival Non Markovian Queueing System with Vacation**" is a bonafide record of research work carried out by **Mrs. K. Sathiya** under my guidance and supervision. This thesis has reached the standard fulfilling the requirements of the regulations relating to the degree. The candidate is hereby permitted to submit the thesis to Pondicherry University, for the award of **Doctor of Philosophy** in Mathematics. This thesis has not previously formed the basis for the award of any degree, diploma, associateship, fellowship or other similar titles of any other university.

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DECLARATION

I hereby declare that this thesis entitled "**Contributions to the Study of Transient Analysis of Bulk Arrival Non Markovian Queueing System with Vacation"** submitted to the Department of Mathematics, Pondicherry Engineering College, Pondicherry, India for the award of the degree of **DOCTOR OF PHILOSOPHY IN MATHEMATICS** is a record of bonafide research work carried out by me under the guidance and supervision of **Dr. G. Ayyappan**, Professor, Department of Mathematics, Pondicherry Engineering College, Puducherry and this has not formed the basis for the award of any other degree by any University / Institution before.

K. SATHIYA

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TABLE OF CONTENTS

CHAPTER ONE

INTRODUCTION

CHAPTER

ONE

INTRODUCTION

In the last seven decades the theory of Stochastic Processes has developed very rapidly and enormously and it has wide range of applications in a large number of fields. Doob (1953) has defined the Stochastic Process as 'The mathematical abstraction of an emprical process whose development is governed by probabilistic laws'. The mathematicians faced with real life problems, conceptualised them as mathematical models and in solving such problems they used stochastic processes. In recent years the theory has been applied with greater advantage to solve complex problems in diversified fields like Nuclear Physics, Statistical Mechanics, Communication Engineering, Computer Science, Chemistry, Astronomy, Astrophysics, Operation Research, Psychology, Sociology, Economics, Acturial Science and Biological Sciences. In this thesis, we present some interesting mathematical models of Queueing theory, an important area of Stochastic Processes.

1.1 Queueing Theory

Congestion is a natural phenomenon in human activities. All of us have experienced the annoyance of having to wait for service. A service facility gets congested if there are more people than the server can possibly handle. Queueing theory attempts to answer questions such as how long must a customer wait and how many people will form the queue through detailed Mathematical Analysis.

It is no surprise that the study of queueing systems began in the fields of telephony when, during the first two decades of last century, Erlang in 1909 developed the basic foundations of the theory long before probability theory was popularized or even well developed. He published his work 'The theory of probabilities and Telephone conversations' in 1909 (see Brockmayer et al. (1948)). The 1920's were basically devoted to the application of his results and until the mid 1930's, when Feller introduced the concept of the birth-death process, the queueing theory was not recognized by the world of Mathematics as an interesting area of research work. During and following the world war II, this theory played an important role in the development of the new field of operation research, which seemed to hold so much promise in the early postwar years. As the enchantment with operations research diminished in the face of the real world's complicated models, the mathematicians proceeded to advance the field of queueing theory rapidly and elegantly. The frontiers of this research proceeded into far-reaches of deep and complex mathematics. It was soon found the really interesting models did not yield to solution and the field quietened down considerably. In early 1930's Pollaczek did some further work. Additonal work was done at that time by Khintchine (1932) and by Palm (1938). Recent contributions are those of Lindley (1952) using integral equations, Bailey (1956), Lendermann and Reuter (1956) on time dependent solutions, Takacs (1962) considering waiting time, Cox (1955) with the concept of supplementary variables, Kendall (1951, 1953, 1964) employing the technique of imbedded markov chains, Champernowe (1956) considering the concept of random walks and Neuts (1978) with the use of Markov chains in queueing theory which have a matrix geometric invariant probability vector.

It is mainly with the advent of digital computers that once again the tools of queueing theory are brought to bear on a class of practical problems, but this time with great success. The fact is that at present, one of the few tools we have for analysing the performance of computer system is that of queueing theory and this explains its popularity among Engineers and Scientists to-day. A good number of new problems are being formulated in terms of this theory and new tools and methods are being developed to meet the challenge posed by these problems. Also the applications of digital computers in solving the equations of queueing theory has produced enormous interest in the field.

The theory of queues has been applied to a large number of problems viz. (1) the telephone traffic (2) the landing of air craft (3) the loading and unloading of ships (4) machine break-down and repair (5) the scheduling of patients in clinics (6) the timing of traffic lights (7) restaurant service (8) checkout stand in supermarkets (9) Inventory control and (10) the theory of dams and provisions.

1.2 Description of the Queueing System

A queueing system can be described by a customer arriving for service, customer waiting for service if it is not immediate and customer leaving the system after being served. The common characteristics of such systems are the following:

1.2.1 Input Process

If the arrivals and services are strictly according to schedule, a queue can be avoided, but in practice this is not in case. In most of the situations arrivals are controlled by external factors. Therefore, the best that can be done is to represent the input process in terms of random variables. Some factors needed for the complete specification of an input process are the source of arrivals, the type of arrivals and the inter-arrival times. It is also necessary to know the reaction of a customer upon entering the system. If a customer decides not to enter the queue upon arrivals, he is said to have balked. On the otherhand, customer may enter the queue, but after sometime lose patience and decide to leave. In this case he is said to have reneged. In the event that there are two or more parallel waiting lines, customers may switch from one to another, that is, jockeying for position.

1.2.2 Service Mechanism

The uncertainties involved in the service mechanism are the number of servers, the number of customers getting served at any time and the duration of service. The situation in which service depends on the number of customers waiting is referred to as state-dependent service.

1.2.3 Queue Discipline

All other factors regarding the rules of conduct of the queue can be pooled under this heading. One of these is the rule followed by the server in taking the customers into service. The most common discipline that can be observed in every day life is First Come First Served (FCFS). Some others in common usages are Last Come First Served (LCFS) which is applicable to many inventory system, selections for service in random order independent of the time of arrival to the queue and a variety of priority schemes where customers are given priorities upon entering the system.

1.2.4 Number of Queues

If in a queueing system there is only one server, then the first three factors can completely define the system. However, in many cases one has to deal with more than one queue in service and/or in parallel.

In is convenient to specify the description of queueing system by a notational representation. Kendall (1953) introduced a shorthand notation to represent queueing system, which is now rather standard throughout the literature. A queueing process is described by the symbol A /B /C /X /Y where 'A' denotes the inter-arrival time distribution, 'B' denotes service time distribution, 'C' denotes the number of servers, 'X' for the system capacity and 'Y' specifies the queue discipline.

During the early periods, queueing models with single arrival and individual service have been studied in depth. But in real life we come across queueing situations with (i) bulk arrival and single service, (ii) single arrival and bulk service and (iii) both bulk arrival and bulk service. Such situations, in general are knows as Bulk queueing systems. Queues with both bulk arrival and bulk service are difficult to analyse. This is because of the fact that added difficulties arise due to the necessity of breaking up arrival batches in order to form service groups. Also the methods used for such systems involve combination of two or more techniques for a single model. To study the recent developments in bulk queues, one may refer Chaudhry and Templeton (1983) and Medhi (1984). Mail bags arriving at a central sorting station and people going to a theater or restaurant are examples of the bulk arrival queueing system. In this system the size of an arriving group may be a random variable or a fixed number.

The theory of bulk queues originated with the pioneering work by Bailey (1954). Gaver (1959) seems to be the first to take up bulk arrival queues for study. Gorsky (1983) used bulk arrival queue for modelling the effect of food intake on the activity pattern of an individual. The transportation problems involving buses, trains, airplanes etc. are bulk service queueing systems in which the arrival occur singly or in bulk, but the service is in bulk.

1.3 Objective

The main objective of this research work is to analyze the behavior of transient analysis of batch arrival queueing system with vacation. Much of the results found in the literature are confined to steady state solutions only. However there are areas in computer and communication system which require time dependent analyse. Eg., adaptive isolated routing and load balancing in computer communication system (incoming customers are directed to an appropriate server based on the estimated current queue length or waiting time), effects of flow congestion control policies in packet switching networks. These are the areas where one needs to know how the system will operate at an instant of time t. There are many systems which are operated only for a specified period of time t. Business establishments, service operations such as banks, doctors clinics, reservation counters etc., which are open and closed, but never operate under steady state (time independent) conditions. The assumptions required to derive the steady state solutions for queueing systems are not always satisfied in the design and analysis of real systems. The situations often occur in communication networks where the load on the network usually depends on time. Thus the investigation of the transient behaviour of the queueing system is very much important, not only from a theoretical view point but also from the point of view of its tremendous use in engineering applications.

The study of transient behavior of queueing models have been increased recently. Takagi (1990) studied the time-dependent analysis of on M/G/1 vacation models with exhaustive service. The theory of continued fractions can be found in Jones and Thron (1980). Parthasarathy and Selvaraju (2001), Thangaraj and Vanitha (2010c) have applied continued fraction technique to study the transient behaviour of the queueing systems. Madan (1992), Thangaraj and Vanitha (2010a), Srinivasan and Maragatha Sundari (2012b) has obtained the time dependent solution of M/G/1 model with compulsory vacation. Khalaf et al. (2010, 2011) studied $M^{[X]}/G/1$ queue with general vacation times. Badamchi Zadeh (2012) have studied a batch arrival queue system with Coxian-2 server vacations and admissibility restricted.

We have used the probability generating functions in terms of Laplace transforms to obtain the transient solution of the $M^{[X]}/G/1$ queue and the following performance measures were derived.

1. Mean number of customer in the queue.
2. Mean number of customer in the system.
3. Mean waiting time in the queue.
4. Mean waiting time in the system.

In this research work, we are analyzing transient and steady state behaviours of our queueing models by implementing various concepts like Breakdown and repair, Second optional service, Feedback service, Vacation policies, Restricted admissibility, Balking, Retrial policy and Staring failure.

Breakdown and Repair

In real life situations, a queueing system might suddenly breakdown and hence the server will not be able to continue providing service unless the system is repaired.

In many waiting line systems, the role of a server is played by mechanical, electrical and electronic devices like robots, computers, traffic lights etc., which can be subject to accidental failures and they need to be repaired to resume service. For instance, in computer network service the servers are deactivated because of virus infected files, the packets (messages) are lost at a node (processor) in communication channel due to the lack of buffer size. Therefore, queueing models which cater to server breakdowns are more realistic in an emerging technological world. Takine and Sengupta (1997), Aissani and Artalejo (1998), Vinck and Bruneel (2006), Thangaraj and Vanitha (2010a), Khalaf et al. (2010) studied different queueing systems subject to breakdowns.

Second Optional Service

Second optional service plays a vital role in queueing systems. The server provides two phases of services namely essential service and second optional service. The essential service will be given first to all arriving customers. The second optional service will be extended to the customers if they demand.

In many applications such as hospital services, production systems, bank services, computer and communication networks. Choudhury (2003) studied some aspects of M/G/1 queueing system with second optional service and derived the steady state queue size distribution at the stationary point of time for general second optional service.

Feedback Service

After the completion of service, if the customer is dissatisfied with his service, he can immediately join the tail of the queue for re-service with some probability. Formulation of queues with feedback mechanism was first introduced by Takacs (1963). Choi and Kulkarni (1992) have studied M/G/1 retrial queue with feedback. The queueing systems which include the possibility for a customer to return to the server for re-service are called feedback queues.

Feedback queues play a vital role in production systems subject to rework, hospital management, super markets and banking businesses, etc. In this area the contributions of eminent authors, like Finch (1959), Scharge (1967), Glenbe and Pujolle (1987), Madan and Al-Rawwash (2005) and Badamchi Zadeh and Shankar (2008).

Vacation Policies

Vacation queueing theory was developed in the 1970's as an extension of the classical queueing theory. In a queueing system with vacations, other than serving randomly arriving customers, the server is allowed to take vacations. The vacations may represent server's working on some supplementary jobs, performing server maintenance inspection and repairs, simply taking a break. The period of temporary server absence for the primary customers is considered as a server vacation. Therefore, queues with vacations or simply called vacation models attracted great attentions of queueing researchers and became an active research area.

In recent years, vacation models had been the subject of interest to queueing theorists because of their applicability and theoretical structures in real life congestion situations such as manufacturing and production, computer and communication systems, service and distribution system, etc. There are different vacation polices such as single, multiple and Bernoulli vacation policy, introduced and applied on queueing models by different researchers. The most remarkable works done in recent past by some researchers on vacation models include Fuhrmann and Cooper (1985), Baba (1986), Lee (1989), Doshi (1986), Takagi (1992), Madan and Anabosi (2003) and Maraghi et al. (2009).

Resrticted Admissibility

In some queueing systems with batch arrival there is a restriction such that not all batches are allowed to join the system at all time. This policy is named restricted admissibility. Earliar, Madan and Abu-Dayyeh (2002) and Madan and Choudhury (2004) studied this type of model with batch arrivals Bernoulli vacation and restricted admissibility, where all arriving batches are not allowed into the system at all time.

Balking

In real practice, it often happens that arrivals become discouraged when the queue is long and do not wish to wait. This type of customer behavior called balking. The remarkable attention has been given on many queueing models with customer impatience. The concept of customer impatience has been studied in 1950's. Haight (1957) has first studied about queueing with balking. Jau-Chuan Ke (2007) analyzed the steady state batch arrival queueing system with balking and a variant vacation policy.

Retrial Policy

At the arrival epoch, if the server is busy the whole batch joins the orbit. Whereas if the server is free, then one of the arriving customer starts its service immediately and the rest joins the orbit. For bibliographies on retrial queues refer Artalejo (1999) and Artalejo and Gomez-Corral (2008).

Starting Failure

One of the most important characteristic in the service facility of a queueing system is its starting failures. An arriving customer who finds the server idle must turn on the server. If the server is started successfully the customer gets the service immediately. Otherwise the repair for the server begins and the customer must join the orbit. The server is assumed to be reliable during service. Such systems with starting failures have been studied in retrial queueing models by Yang and Li (1994), Krishna Kumar et al. (2002b) and Mokaddis et al. (2007).

1.4 Author's Work

This thesis is divided into ten chapters as detailed in the sequel.

The basic definitions of the queue and motivation are given in the first chapter. Also a brief survey of related literature, the objective of the research and a chapter wise structure are given.

Chapter 2 deals with an $M^{[X]}/G/1$ queue with second optional service and second optional vacation. A single server provides two phases of service. The first phase of service is essential for all customers, as soon as the first service of a customer is completed, then with probability θ he may opt for the second service or else with probability $(1-\theta)$, he leaves the system. At each service completion, the server will take compulsory vacation. The server has two heterogeneous phases of vacation. Phase one is compulsory and phase two follows the phase one vacation in such a way that the server may take phase two vacation with probability p or return back to the system with probability $(1-p)$. Customers arrive at the system according to Poisson process with rate λ. The service and vacation periods are assumed to be general (arbitrary) distribution. Stability condition of this model is derived and various system performance measures have been calculated. Numerical results are also presented for some values of parameters.

The research paper related to this model have been published in refereed Journal as given below:

"Time dependent solution of Non-Markovian queue with two phases of service and general vacation time distribution" - *Malaya Journal of Matematik*, Vol. 4, No. 1, 2013, pp. 20-29.

Chapter 3 - This chapter consist of two models.

Model 1 deals with an $M^{[X]}/G/1$ queue with second optional service, optional re-service and Bernoulli vacations, where the arrivals are Poisson. Each customer undergoes first phase of service, after completion of service the customer has the option to repeat or not to repeat the first phase of service and leave the system without taking the second phase or take the second phase service. Similarly after the second phase service he has yet another option to repeat or not to repeat the second phase service. As soon as each service is over, the server may take a vacation with probability θ or may continue to stay in the system with probability $1-\theta$. The service and vacation periods are assumed to be general (arbitrary) distribution. The stability condition for this model have been derived and various system performance measures have been calculated. Numerical results have been done for various values of arrival rate, service rate and vacation rate.

The research paper related to this model have been published in refereed Journal as given below.

“Time Dependent Solution of Batch Arrival Queue with Second Optional Service, Optional Re-Service and Bernoulli Vacation” *Mathematical Theory and Modelling*, Vol. 3, No. 1, 2014, pp. 1-8.

Model II deals with single server queue with Poisson arrivals where customers arrive in batches of variable size, the server provides two types of heterogeneous service. Customer has the option of choosing either type 1 service with probability p_1 or type 2 service with probability p_2 with the service times follow general distribution. After completion type 1 or type 2 service a customer has the option to repeat or not to repeat the same type of service. After every service completion the server has the option to leave for vacation of random length with probability θ or to continue to stay in the system with probability $1 - \theta$. The service period and vacation period are assumed to be general (arbitrary) distribution. Stability condition of this model is derived and various system performance measures have been calculated. Numerical results are also presented for some values of parameters.

The research paper related to this model is *accepted for publication in the Proceedings of International Conference on Applied Mathematical Models*, 2014, PSG Tech, Coimbatore.

Chapter 4 deals with the analysis of a single server batch arrival feedback queue with server vacation and balking. The customers arrive according to Poisson process with rate λ. An arriving batch may join the system with probability b or balks (refuses to join) the system with probability $(1 - b)$ during the period of server's busy or vacation times. As soon as the completion of service, if the customer is dissatisfied with his service, he can immediately join the tail of the original queue as a feedback customer. At each service completion epoch, the server may opt to take vacation with probability p or else with probability $(1-p)$ to stay in the system for the next service. The service and vacation periods are assumed to be general (arbitrary) distributions. The stability condition for this model have been derived and various system performance measures have been calculated. Numerical results are also presented for some values of parameters.

The research paper related to this model have been published in the Proceedings as given below:

"Transient behaviour of batch arrival feedback queue with server vacation and balking"- *Proceedings of National Conference on Recent Advances in Mathematical Analysis and Applications*, 2013, pp. 86-96, Bonfring Publications, India.

Chapter 5 deals with the study of batch arrival queueing system with service interruption and extended server vacation based on Bernoulli schedule. A single server provides essential service to all arriving customers with service time follows general (arbitrary) distribution. After every service completion the server may take vacation or stay in the system. The vacation period has three heterogeneous phase. The server has the option for phase one vacation of random length with probability p or to continue to stay in the system with probability $(1-p)$. The server has the option to go on phase two extended vacation after the phase one vacation completion with probability r or rejoins the system to provide service with probability $(1-r)$. As soon as the completion of phase two vacation, the server undergoes phase three vacation with probability θ or rejoins the system to provide service with probability $(1-\theta)$. The vacation times are assumed to be general. The server is interrupted at random and the duration of attending interruption follows exponential distribution. Also we assume, the customer whose service is interrupted goes back to the head of the queue where the arrivals are Poisson. The stability condition for this model have been derived and various system performance measures have been calculated. Numerical results are also presented for some values of parameters.

Chapter 6 deals with a Poisson arrival queue with two types of service subject to random breakdowns having multiple vacation, where the customers arrive to the system in batches of variable size. A single server provides two types of service, type 1 service with probability p_1 and type 2 service with probability p_2 with the service times follow general (arbitrary) distribution and each arriving customer may choose either type of service. The server takes vacation only if the system becomes empty and the vacation period is assumed to be general. On returning from vacation if the server finds no customer waiting in the system, then the server again goes for vacation until he finds at least one customer in the system. The system may breakdown at random and repair time follow exponential distribution. In addition we assume restricted admissibility of arriving batches in which not all batches are allowed to join the system at all times. Stability condition of this model is derived. We obtained some performance measures of the system. Numerical results are carried out for various values of parameters.

The research paper related to this model has been published in refereed Journal as detailed below:

"Transient solution of batch arrival queue with two types of service, multiple vacation, random breakdown and restricted admissibility"- ***International Journal of Management and Information Technology*****, Vol. 3, No. 3, 2013, pp. 16-25.**

In chapter 7 we studies a single server queue with batch arrival queueing system, two stages of heterogeneous service subject to random breakdowns, delayed repair with Bernoulli schedule server vacations. The customers arrive according to Poisson process with rate λ. After first-stage service the server must provide the second stage service. The repair process does not start immediately after a breakdown and there is a delay waiting time for repairs to start. However, after the completion of each second stage service, the server has the option to leave for a phase one vacation with probability p or continue to serve customers with probability $1 - p$. The server has the option to go on extended vacation after the original vacation completion with probability r or rejoins the system to provide service with probability $1 - r$. The service times, vacation times, extended vacation times, delay times and repair times are all assumed to follow general (arbitrary) distributions, while the breakdown time is exponentially distributed. The stability condition for this model have been derived and various system performance measures have been calculated. Some numerical results were presented to demonstrate how the various parameters of the model influence the behavior of the system.

The research paper related to this model have been published in refereed Journal as detailed below:

" Two stage heterogeneous service, random breakdowns, delayed repairs and extended server vacations with Bernoulli schedule" - *International Journal of Statistical and System*, Vol. 8, No. 3, 2013, pp. 183-201.

Chapter 8 deals with batch arrival queueing system with three stages of heterogeneous service provided by a single server with different general (arbitrary) service time distributions subject to random interruption. Each customer undergoes three stages of heterogeneous service. However at the completion of each third stage of service, the server will take compulsory vacation. After a completion of compulsory vacation the server may take optional vacation with probability p or stay in the system with probability $(1 - p)$ for next service. The server is interrupted at random and the duration of attending interruption follows exponential distribution. Also we assume, the customer whose service is interrupted goes back to the head of the queue where the arrivals are Poisson. The vacation times are assumed to be general (arbitrary) distributions. The stability condition for this model have been derived and various system performance measures have been calculated. Numerical results have been carried out for various parameters.

The research paper related to this model have been published in refereed Journal as detailed below.

"Batch arrival queue with three stages of service having server vacations and service interruptions"- *Advances and Applications in Statistics*, Vol. 3, No. 1, pp. 111-126, 2013.

Chapter 9 deals with batch arrival queueing system with three stage heterogeneous service provided by a single server with different (arbitrary) service time distributions. Customers arrive to the system according to a Poisson process with rate λ. Each customer undergoes three stages of heterogeneous service. As soon as the completion of third stage of service, if the customer is dissatisfied with his service, he can immediately join the tail of the original queue. After service completion of a customer the server may take a vacation or stay in the system. The vacation period has two heterogeneous phases. Phase one is Bernoulli vacation. As soon as the completion of Bernoulli vacation, the server undergoes optional vacation. The vacation times are assumed to be general (arbitrary) distributions. In addition we assume restricted admissibility of arriving batches in which not all batches are allowed to join the system at all times. Stability condition of this model is derived. Numerical results are also presented for some values of parameters.

The research paper related to this model have been published in refereed Journal as detailed below.

"$M^{[X]}/G/1$ **feedback queue with three stage heterogeneous service and server vacations having restricted admissibility"-** ***Journal of Computations and Modelling*****, Vol. 3, No. 2, 2013, pp. 203-225.**

Chapter 10 deals with $M^{[X]}/G/1$ feedback retrial queue, subject to starting failures and Bernoulli vacation. The customers arrive to the system in batches of variable size, but served one by one on a first come - first served basis. We assume that there is no waiting space and therefore if an arriving customer finds the server busy or down, the customer leaves the service area and enters a group of blocked customers called orbit in accordance with an FCFS discipline. That is, only the customer at the head of the orbit queue is allowed for access to the server where the arrival follows Poisson. As soon as the completion of service, if the customer is dissatisfied with his service, he can immediately join the retrial group as a feedback customer for receiving the same service with probability p or to leave the system forever with probability $q(= 1 - p)$. The successful commencement of service for a new customer who finds the server idle and sees no other customer in the orbit with probability δ and is α for all other new and returning customers. After the completion of each service, the server either goes for a vacation with probability β or may wait for serving the next customer with probability $1 - \beta$. Repair times, service times and vacation times are assumed to be general (arbitrary) distributed.

The stability condition for this model have been derived and various system performance measures have been calculated. Numerical results have been carried out for various values of arrival rate, service rate and vacation rate.

The research paper related to this model have been published in refereed Journal as detailed below.

"Transient analysis of batch arrival feedback retrial queue with starting failure and Bernoulli vacation"- *Mathematical Theory and Modelling*, Vol. 3, No. 8, 2013, pp. 60-67.

CHAPTER TWO

$M^{[X]}/G/1$ Queue with Second Optional Service and Second Optional Vacation

CHAPTER

TWO

$M^{[X]}/G/1$ QUEUE WITH SECOND OPTIONAL SERVICE AND SECOND OPTIONAL VACATION

2.1 Introduction

Vacation queues have been the subject of deep study in recent years because of their theoretical structure as well as their applicability in various real life situations. Recently the $M^{[X]}/G/1$ queue with vacation has drawn the attention of various researchers notable among them are Baba (1986, 1987), Lee (1989), Choudhury and Madan (2005) and Badamchi Zadeh (2009).

There is extensive literature on the M/G/1 queue, which has been studied in various forms by numerous authors including Cox (1955), Keilson and Kooharian (1960), Kleinrock (1975), Medhi (1982), Jacob and Madhusoodanan (1988), Choi and Park (1990), Madan (1992) and Singh et al. (2012). Madan (2000b) have studied the time-dependent as well as the steady state behavior of an M/G/1 queue with second optional service, using the supplementary variable technique.

Krishnakumar and Arivudainambi (2001), Choudhury (2003), Artalejo

A part of this chapter is published with entitled:
Time dependent solution of Non-Markovian queue with two phases of service and general vacation time distribution – *Malaya Journal of Matematik*, 4(1):20–29, 2013.

and Choudhury (2004), Wang (2004) and Choudhury and Paul (2005) and Kasturi and Kalidass (2010) have studied queueing system with optional second service. Madan and Al-Rawwash (2005) have studied the $M^{[X]}/G/1$ queue with feedback and optional server vacations based on a single vacation policy.

In this chapter, we consider an $M^{[X]}/G/1$ queue with second optional service, with different service time and second optional vacation. A single server provides two phases of service. The first phase of service is essential for all customers, as soon as the first service of a customer is completed, then with probability θ, he may opt for the second optional service or else with probability $(1-\theta)$, he leaves the system. At each service completion, the server will take vacation. The vacation period of the server has two heterogeneous phases. Phase one is compulsory and phase two is optional. After completion of phase one vacation the server may take phase two vacation with probability p or return back to the system with probability $(1-p)$ where the arrival follows Poisson. The service and vacation periods follow general (arbitrary) distribution.

Here we derive time dependent probability generating functions in terms of Laplace transforms. We also derive the average queue size and average waiting time in the queue and the system. Some particular cases and numerical results are also discussed.

The rest of the chapter is organized as follows. The model description is given in section 2.2. Definitions and equations governing the system are given in section 2.3. The time dependent solution have been obtained in section 2.4 and corresponding steady state results have been derived explicitly in section 2.5. Average queue size and average waiting time are computed in section 2.6. Some particular cases and numerical results are discussed in section 2.7 and 2.8. respectively.

2.2 Model description

We assume the following to describe the queueing model of our study.

a) Customers arrive at the system in batches of variable size in a compound Poisson process and they are provided one by one service on a first come - first served basis. Let $\lambda c_i dt$ $(i = 1, 2, \ldots)$ be the first order probability that a batch of i customers arrives at the system during a short interval of time $(t, t + dt]$, where $0 \leq c_i \leq 1$, $\sum_{i=1}^{\infty} c_i = 1$ and $\lambda > 0$ is the arrival rate of batches.

b) There is a single server who provides two phases of service. The first phase is compulsory and second phase is optional. The first phase of service is essential for all customers, as soon as the essential service of a customer is completed, then with probability θ, he may opt for the second optional service or else with probability $(1 - \theta)$, he leave the system.

c) The service time follows a general (arbitrary) distribution with distribution function $B_i(s)$ and density function $b_i(s)$. Let $\mu_i(x)dx$ be the conditional probability density of service completion during the interval $(x, x + dx]$, given that the elapsed service time is x, so that

$$\mu_i(x) = \frac{b_i(x)}{1 - B_i(x)}, \quad i = 1, 2,$$

and therefore,

$$b_i(s) = \mu_i(s) e^{-\int_0^s \mu_i(x)dx}, \quad i = 1, 2.$$

d) After completion of each service, the server will take vacation of random length. The vacation time has two phases with phase one is compulsory and phase two is second optional vacation. However, after phase one vacation, the server take phase two optional vacation with probability p or may return back to the system with probability $(1-p)$.

e) The server's vacation time follows a general (arbitrary) distribution with distribution function $V_i(t)$ and density function $v_i(t)$. Let $\gamma_i(x)dx$ be the conditional probability density of vacation completion during the interval $(x, x+dx]$, given that the elapsed vacation time is x, so that

$$\gamma_i(x) = \frac{v_i(x)}{1-V_i(x)}, \quad i = 1, 2,$$

and therefore,

$$v_i(t) = \gamma_i(t) e^{-\int_0^t \gamma_i(x)dx}, \ i = 1, 2.$$

f) Various stochastic processes involved in the system are assumed to be independent of each other.

2.3 Definitions and equations governing the system

We define

$P_n^{(1)}(x,t)$ = Probability that at time t, the server is active providing first essential service and there are n $(n \geq 0)$ customers in the queue excluding the one being served and the elapsed service time is x. Consequently $P_n^{(1)}(t) = \int_0^\infty P_n^{(1)}(x,t)dx$ denotes the probability that at time t there are n customers in the queue excluding one customer in the first essential service irrespective of the value of x.

$P_n^{(2)}(x,t)$ = Probability that at time t, the server is active providing second optional service and there are n $(n \geq 0)$ customers in the queue excluding the one being served and the elapsed service time is x. Consequently $P_n^{(2)}(t) = \int\limits_0^\infty P_n^{(2)}(x,t)dx$ denotes the probability that at time t there are n customers in the queue excluding one customer in the second optional service irrespective of the value of x.

$V_n^{(1)}(x,t)$ = Probability that at time t, the server is under phase one compulsory vacation with elapsed vacation time is x and there are n $(n \geq 0)$ customers in the queue. Consequently $V_n^{(1)}(t)= \int\limits_0^\infty V_n^{(1)}(x,t)dx$ denotes the probability that at time t there are n customers in the queue and the server is under phase one compulsory vacation irrespective of the value of x.

$V_n^{(2)}(x,t)$ = Probability that at time t, the server is under second optional vacation with elapsed vacation time is x and there are n $(n \geq 0)$ customers in the queue. Consequently $V_n^{(2)}(t)=\int\limits_0^\infty V_n^{(2)}(x,t)dx$ denotes the probability that at time t there are n customers in the queue and the server is under second optional vacation irrespective of the value of x.

$Q(t)$= Probability that at time t, there are no customers in the system and the server is idle but available in the system.

The system is then governed by the following set of differential-difference equations:

$$\frac{\partial}{\partial x}P_0^{(1)}(x,t) + \frac{\partial}{\partial t}P_0^{(1)}(x,t) + [\lambda + \mu_1(x)]P_0^{(1)}(x,t) = 0 \tag{2.1}$$

$$\frac{\partial}{\partial x}P_n^{(1)}(x,t) + \frac{\partial}{\partial t}P_n^{(1)}(x,t) + [\lambda + \mu_1(x)]P_n^{(1)}(x,t) = \lambda \sum_{k=1}^{n} c_k P_{n-k}^{(1)}(x,t), \quad n \geq 1 \tag{2.2}$$

$$\frac{\partial}{\partial x}P_0^{(2)}(x,t) + \frac{\partial}{\partial t}P_0^{(2)}(x,t) + [\lambda + \mu_2(x)]P_0^{(2)}(x,t) = 0 \tag{2.3}$$

$$\frac{\partial}{\partial x}P_n^{(2)}(x,t) + \frac{\partial}{\partial t}P_n^{(2)}(x,t) + [\lambda + \mu_2(x)]P_n^{(2)}(x,t) = \lambda \sum_{k=1}^{n} c_k P_{n-k}^{(2)}(x,t), \quad n \geq 1 \tag{2.4}$$

$$\frac{\partial}{\partial x}V_0^{(1)}(x,t) + \frac{\partial}{\partial t}V_0^{(1)}(x,t) + [\lambda + \gamma_1(x)]V_0^{(1)}(x,t) = 0 \tag{2.5}$$

$$\frac{\partial}{\partial x}V_n^{(1)}(x,t)+\frac{\partial}{\partial t}V_n^{(1)}(x,t)+[\lambda+\gamma_1(x)]V_n^{(1)}(x,t)=\lambda\sum_{k=1}^{n}c_kV_{n-k}^{(1)}(x,t),\quad n\geq 1 \tag{2.6}$$

$$\frac{\partial}{\partial x}V_0^{(2)}(x,t)+\frac{\partial}{\partial t}V_0^{(2)}(x,t)+[\lambda+\gamma_2(x)]V_0^{(2)}(x,t)=0 \tag{2.7}$$

$$\frac{\partial}{\partial x}V_n^{(2)}(x,t)+\frac{\partial}{\partial t}V_n^{(2)}(x,t)+[\lambda+\gamma_2(x)]V_n^{(2)}(x,t)=\lambda\sum_{k=1}^{n}c_kV_{n-k}^{(2)}(x,t),\quad n\geq 1 \tag{2.8}$$

$$\frac{d}{dt}Q(t)+\lambda Q(t)=(1-p)\int_0^\infty\gamma_1(x)V_0^{(1)}(x,t)dx+\int_0^\infty\gamma_2(x)V_0^{(2)}(x,t)dx \tag{2.9}$$

The above set of equations are to be solved subject to the following boundary conditions:

$$P_n^{(1)}(0,t)=\lambda c_{n+1}Q(t)+(1-p)\int_0^\infty\gamma_1(x)V_{n+1}^{(1)}(x,t)dx+\int_0^\infty\gamma_2(x)V_{n+1}^{(2)}(x,t)dx,\quad n\geq 0 \tag{2.10}$$

$$P_n^{(2)}(0,t)=\theta\int_0^\infty\mu_1(x)P_n^{(1)}(x,t)dx,\quad n\geq 0 \tag{2.11}$$

$$V_n^{(1)}(0,t)=(1-\theta)\int_0^\infty\mu_1(x)P_n^{(1)}(x,t)dx+\int_0^\infty\mu_2(x)P_n^{(2)}(x,t)dx,\quad n\geq 0 \tag{2.12}$$

$$V_n^{(2)}(0,t)=p\int_0^\infty\gamma_1(x)V_n^{(1)}(x,t)dx,\quad n\geq 0 \tag{2.13}$$

We assume that initially there are no customers in the system and the server is idle. So the initial conditions are

$$P_n^{(i)}(0)=V_n^{(i)}(0)=0\ \text{ and }Q(0)=1\quad\text{for }\ n=0,1,2,...,\quad i=1,2. \tag{2.14}$$

2.4 Generating functions of the queue length: The time-dependent solution

In this section, we obtain the transient solution for the above set of differential-difference equations.

Theorem: *The system of differential difference equations to describe an* $M^{[X]}/G/1$ *queue with second optional service and second optional vacation are given by equations (2.1) to (2.13) with initial conditions (2.14) and the generating functions of transient solution are given by equation (2.64) to (2.67).*

Proof: We define the probability generating functions for i=1, 2.

$$P^{(i)}(x,z,t)=\sum_{n=0}^{\infty}z^nP_n^{(i)}(x,t);\quad P^{(i)}(z,t)=\sum_{n=0}^{\infty}z^nP_n^{(i)}(t); \tag{2.15}$$

$$V^{(i)}(x,z,t)=\sum_{n=0}^{\infty}z^nV_n^{(i)}(x,t);\ V^{(i)}(z,t)=\sum_{n=0}^{\infty}z^nV_n^{(i)}(t);C(z)=\sum_{n=1}^{\infty}c_nz^n; \tag{2.16}$$

which are convergent inside the circle given by $|\ z\ |\leq 1$ and define the Laplace transform of a function $f(t)$ as

$$\bar{f}(s)=\int_0^{\infty}e^{-st}f(t)dt,\quad \Re(s)>0. \tag{2.17}$$

Taking the Laplace transform of equations (2.1) to (2.13) and using (2.14), we obtain

$$\frac{\partial}{\partial x}\bar{P}_0^{(1)}(x,s)+(s+\lambda+\mu_1(x))\bar{P}_0^{(1)}(x,s)=0 \tag{2.18}$$

$$\frac{\partial}{\partial x}\bar{P}_n^{(1)}(x,s)+(s+\lambda+\mu_1(x))\bar{P}_n^{(1)}(x,s)=\lambda\sum_{k=1}^{n}c_k\bar{P}_{n-k}^{(1)}(x,s),n\geq 1 \tag{2.19}$$

$$\frac{\partial}{\partial x}\bar{P}_0^{(2)}(x,s)+(s+\lambda+\mu_2(x))\bar{P}_0^{(2)}(x,s)=0 \tag{2.20}$$

$$\frac{\partial}{\partial x}\bar{P}_n^{(2)}(x,s)+(s+\lambda+\mu_2(x))\bar{P}_n^{(2)}(x,s)=\lambda\sum_{k=1}^{n}c_k\bar{P}_{n-k}^{(2)}(x,s), n\geq 1 \tag{2.21}$$

$$\frac{\partial}{\partial x}\bar{V}_0^{(1)}(x,s)+(s+\lambda+\gamma_1(x))\bar{V}_0^{(1)}(x,s)=0 \tag{2.22}$$

$$\frac{\partial}{\partial x}\bar{V}_n^{(1)}(x,s)+(s+\lambda+\gamma_1(x))\bar{V}_n^{(1)}(x,s)=\lambda\sum_{k=1}^{n}c_k\bar{V}_{n-k}^{(1)}(x,s), n\geq 1 \tag{2.23}$$

$$\frac{\partial}{\partial x}\bar{V}_0^{(2)}(x,s)+(s+\lambda+\gamma_2(x))\bar{V}_0^{(2)}(x,s)=0 \tag{2.24}$$

$$\frac{\partial}{\partial x}\bar{V}_n^{(2)}(x,s)+(s+\lambda+\gamma_2(x))\bar{V}_n^{(2)}(x,s)=\lambda\sum_{k=1}^{n}c_k\bar{V}_{n-k}^{(2)}(x,s), n\geq 1 \tag{2.25}$$

$$\begin{aligned}(s+\lambda)\bar{Q}(s)&=1+(1-p)\int_0^\infty\gamma_1(x)\bar{V}_0^{(1)}(x,s)dx\\&+\int_0^\infty\gamma_2(x)\bar{V}_0^{(2)}(x,s)dx\end{aligned} \tag{2.26}$$

$$\begin{aligned}\bar{P}_n^{(1)}(0,s)&=\lambda c_{n+1}\bar{Q}(s)+(1-p)\int_0^\infty\gamma_1(x)\bar{V}_{n+1}^{(1)}(x,s)dx\\&+\int_0^\infty\gamma_2(x)\bar{V}_{n+1}^{(2)}(x,s)dx,\quad n\geq 0\end{aligned} \tag{2.27}$$

$$\bar{P}_n^{(2)}(0,s)=\theta\int_0^\infty\mu_1(x)\bar{P}_n^{(1)}(x,s)dx,\quad n\geq 0 \tag{2.28}$$

$$\begin{aligned}\bar{V}_n^{(1)}(0,s)&=(1-\theta)\int_0^\infty\mu_1(x)\bar{P}_n^{(1)}(x,s)dx\\&+\int_0^\infty\mu_2(x)\bar{P}_n^{(2)}(x,s)dx,\ n\geq 0\end{aligned} \tag{2.29}$$

$$\bar{V}_n^{(2)}(0,s)=p\int_0^\infty\gamma_1(x)\bar{V}_n^{(1)}(x,s)dx,\quad n\geq 0 \tag{2.30}$$

Now multiplying equations (2.19), (2.21), (2.23) and (2.25) by z^n and summing over n from 1 to ∞, adding to equations (2.18), (2.20), (2.22) and (2.24) and using the generating functions defined in (2.15) and (2.16), we get

$$\frac{\partial}{\partial x}\bar{P}^{(1)}(x,z,s)+[s+\lambda-\lambda C(z)+\mu_1(x)]\bar{P}^{(1)}(x,z,s)=0 \tag{2.31}$$

$$\frac{\partial}{\partial x}\bar{P}^{(2)}(x,z,s)+[s+\lambda-\lambda C(z)+\mu_2(x)]\bar{P}^{(2)}(x,z,s)=0 \tag{2.32}$$

$$\frac{\partial}{\partial x}\bar{V}^{(1)}(x,z,s)+[s+\lambda-\lambda C(z)+\gamma_1(x)]\bar{V}^{(1)}(x,z,s)=0 \tag{2.33}$$

$$\frac{\partial}{\partial x}\bar{V}^{(2)}(x,z,s)+[s+\lambda-\lambda C(z)+\gamma_2(x)]\bar{V}^{(2)}(x,z,s)=0 \tag{2.34}$$

For the boundary conditions, we multiply both sides of equation (2.27) by z^n summing over n from 0 to ∞, and use the equation (2.26), we get

$$\begin{aligned} z\bar{P}^{(1)}(0,z,s) &= [1-s\bar{Q}(s)]+\lambda[C(z)-1]\bar{Q}(s) \\ &+(1-p)\int_0^\infty \gamma_1(x)\bar{V}^{(1)}(x,z,s)dx+\int_0^\infty \gamma_2(x)\bar{V}^{(2)}(x,z,s)dx \end{aligned} \tag{2.35}$$

Performing similar operation on equations (2.28) to (2.30), we get

$$\bar{P}^{(2)}(0,z,s)=\theta\int_0^\infty \mu_1(x)\bar{P}^{(1)}(x,z,s)dx \tag{2.36}$$

$$\begin{aligned} \bar{V}^{(1)}(0,z,s) &= (1-\theta)\int_0^\infty \mu_1(x)\bar{P}^{(1)}(x,z,s)dx \\ &+\int_0^\infty \mu_2(x)\bar{P}^{(2)}(x,z,s)dx \end{aligned} \tag{2.37}$$

$$\bar{V}^{(2)}(0,z,s)=p\int_0^\infty \gamma_1(x)\bar{V}^{(1)}(x,z,s)dx \tag{2.38}$$

Integrating equation (2.31) between 0 and x, we get

$$\bar{P}^{(1)}(x,z,s)=\bar{P}^{(1)}(0,z,s)e^{-[s+\lambda-\lambda C(z)]x-\int\limits_0^x \mu_1(t)dt} \tag{2.39}$$

where $P^{(1)}(0,z,s)$ is given by equation (2.35).

Again integrating equation (2.39) by parts with respect to x, yields

$$\bar{P}^{(1)}(z,s)=\bar{P}^{(1)}(0,z,s)\left[\frac{1-\bar{B}_1(s+\lambda-\lambda C(z))}{s+\lambda-\lambda C(z)}\right] \tag{2.40}$$

where

$$\bar{B}_1(s+\lambda-\lambda C(z)) = \int\limits_0^\infty e^{-[s+\lambda-\lambda C(z)]x} dB_1(x) \tag{2.41}$$

is the Laplace-Stieltjes transform of the first phase of service time $B_1(x)$. Now multiplying both sides of equation (2.39) by $\mu_1(x)$ and integrating over x, we obtain

$$\int\limits_0^\infty \bar{P}^{(1)}(x,z,s)\mu_1(x)dx = \bar{P}^{(1)}(0,z,s)\bar{B}_1[s+\lambda(1-C(z))] \tag{2.42}$$

Similarly, on integrating equations (2.32) to (2.34) from 0 to x, we get

$$\bar{P}^{(2)}(x,z,s) = \bar{P}^{(2)}(0,z,s)e^{-[s+\lambda-\lambda C(z)]x-\int\limits_0^x \mu_2(t)dt} \tag{2.43}$$

$$\bar{V}^{(1)}(x,z,s) = \bar{V}^{(1)}(0,z,s)e^{-[s+\lambda-\lambda C(z)]x-\int\limits_0^x \gamma_1(t)dt} \tag{2.44}$$

$$\bar{V}^{(2)}(x,z,s) = \bar{V}^{(2)}(0,z,s)e^{-[s+\lambda-\lambda C(z)]x-\int\limits_0^x \gamma_2(t)dt} \tag{2.45}$$

where $\bar{P}^{(2)}(0,z,s)$, $\bar{V}^{(1)}(0,z,s)$ and $\bar{V}^{(2)}(0,z,s)$ are given by equations (2.36) to (2.38). Again integrating equations (2.43) to (2.45) by parts with respect to x, yields

$$\bar{P}^{(2)}(z,s) = \bar{P}^{(2)}(0,z,s)\left[\frac{1-\bar{B}_2(s+\lambda-\lambda C(z))}{s+\lambda-\lambda c(z)}\right] \tag{2.46}$$

$$\bar{V}^{(1)}(z,s) = \bar{V}^{(1)}(0,z,s)\left[\frac{1-\bar{V}_1(s+\lambda-\lambda C(z))}{s+\lambda-\lambda C(z)}\right] \tag{2.47}$$

$$\bar{V}^{(2)}(z,s) = \bar{V}^{(2)}(0,z,s)\left[\frac{1-\bar{V}_2(s+\lambda-\lambda C(z))}{s+\lambda-\lambda C(z)}\right] \tag{2.48}$$

where

$$\bar{B}_2(s+\lambda-\lambda C(z)) = \int_0^\infty e^{-[s+\lambda-\lambda C(z)]x} dB_2(x) \tag{2.49}$$

$$\bar{V}_1(s+\lambda-\lambda C(z)) = \int_0^\infty e^{-[s+\lambda-\lambda C(z)]x} dV_1(x) \tag{2.50}$$

$$\bar{V}_2(s+\lambda-\lambda C(z)) = \int_0^\infty e^{-[s+\lambda-\lambda C(z)]x} dV_2(x) \tag{2.51}$$

are the Laplace-Stieltjes transform of the second optional service time $B_2(x)$, phase one compulsory vacation time $V_1(x)$ and second optional vacation $V_2(x)$ respectively.

Now multiplying both sides of equations (2.43) to (2.45) by $\mu_2(x)$, $\gamma_1(x)$ and $\gamma_2(x)$ and integrating over x, we obtain

$$\int_0^\infty \bar{P}^{(2)}(x,z,s)\mu_2(x)dx = \bar{P}^{(2)}(0,z,s)\bar{B}_2[s+\lambda-\lambda C(z)] \tag{2.52}$$

$$\int_0^\infty \bar{V}^{(1)}(x,z,s)\gamma_1(x)dx = \bar{V}^{(1)}(0,z,s)\bar{V}_1[s+\lambda-\lambda C(z)] \tag{2.53}$$

$$\int_0^\infty \bar{V}^{(2)}(x,z,s)\gamma_2(x)dx = \bar{V}^{(2)}(0,z,s)\bar{V}_2[s+\lambda-\lambda C(z)] \tag{2.54}$$

Using equations (2.42) and (2.52), we can write equation (2.37) as

$$\bar{V}^{(1)}(0,z,s) = (1-\theta)\bar{B}_1(a)\bar{P}^{(1)}(0,z,s) + \bar{B}_2(a)\bar{P}^{(2)}(0,z,s) \tag{2.55}$$

Using equation (2.42) in (2.36), we get

$$\bar{P}^{(2)}(0,z,s) = \theta\bar{B}_1(a)\bar{P}^{(1)}(0,z,s) \tag{2.56}$$

By using equation (2.56) in (2.55), we get

$$\bar{V}^{(1)}(0,z,s) = \bar{B}_1(a)[1-\theta+\theta\bar{B}_2(a)]\bar{P}^{(1)}(0,z,s) \tag{2.57}$$

Using equations (2.53) and (2.57) in (2.38), we get

$$\bar{V}^{(2)}(0,z,s) = p\bar{B}_1(a)\bar{V}_1(a)[1-\theta+\theta\bar{B}_2(a)]\bar{P}^{(1)}(0,z,s) \tag{2.58}$$

Similarly using equations (2.53), (2.54), (2.57) and (2.58) in (2.35), we get

$$\bar{P}^{(1)}(0,z,s) = \frac{[1-s\bar{Q}(s)]+\lambda[(C(z)-1]\bar{Q}(s)}{Dr} \tag{2.59}$$

where $$Dr = z-\bar{B}_1(a)\bar{V}_1(a)[1-\theta+\theta\bar{B}_2(a)](1-p+p\bar{V}_2(a)), \tag{2.60}$$

and $a = s+\lambda-\lambda C(z)$.
Substituting the value of $\bar{P}^{(1)}(0,z,s)$ from equation (2.59) into equations (2.56), (2.57) and (2.58), we get

$$\bar{P}^{(2)}(0,z,s) = \frac{\theta\bar{B}_1(a)[(1-s\bar{Q}(s))+\lambda(C(z)-1)\bar{Q}(s)]}{Dr} \tag{2.61}$$

$$\bar{V}^{(1)}(0,z,s) = \bar{B}_1(a)(1-\theta+\theta\bar{B}_2(a)) \times \frac{[(1-s\bar{Q}(s))+\lambda(C(z)-1)\bar{Q}(s)]}{Dr} \tag{2.62}$$

$$\bar{V}^{(2)}(0,z,s) = p\bar{B}_1(a)(1-\theta+\theta\bar{B}_2(a))\bar{V}_1(a) \times \frac{[(1-s\bar{Q}(s))+\lambda(C(z)-1)\bar{Q}(s)]}{Dr} \tag{2.63}$$

Using equations (2.59), (2.61), (2.62) and (2.63) in (2.40), (2.46), (2.47) and (2.48), we get

$$\bar{P}^{(1)}(z,s) = \frac{[(1-s\bar{Q}(s))+\lambda(C(z)-1)\bar{Q}(s)]}{Dr}\frac{[1-\bar{B}_1(a)]}{a} \tag{2.64}$$

$$\bar{P}^{(2)}(z,s) = \frac{\theta\bar{B}_1(a)[(1-s\bar{Q}(s))+\lambda(C(z)-1)\bar{Q}(s)]}{Dr}\frac{[1-\bar{B}_2(a)]}{a} \tag{2.65}$$

$$\bar{V}^{(1)}(z,s) = \frac{[1-\theta+\theta\bar{B}_2(a)]\bar{B}_1(a)}{Dr} \times [(1-s\bar{Q}(s))+(\lambda C(z)-\lambda)\bar{Q}(s)]\frac{[1-\bar{V}_1(a)]}{a} \tag{2.66}$$

$$\bar{V}^{(2)}(z,s) = p\bar{B}_1(a)\bar{V}_1(a)\frac{[1-\theta+\theta\bar{B}_2(a)]}{Dr} \times [(1-s\bar{Q}(s)) + (\lambda C(z)-\lambda)\bar{Q}(s)]\frac{[1-\bar{V}_2(a)]}{a} \quad (2.67)$$

where Dr is given by equation (2.60). Thus $\bar{P}^{(1)}(z,s)$, $\bar{P}^{(2)}(z,s)$, $\bar{V}^{(1)}(z,s)$ and $\bar{V}^{(2)}(z,s)$ are completely determined from equations (2.64) to (2.67) which completes the proof of the theorem.

2.5 The steady state results

In this section, we shall derive the steady state probability distribution for our queueing model. To define the steady state probabilities, we suppress the argument t wherever it appears in the time-dependent analysis. This can be obtained by applying the Tauberian property

$$\lim_{s\to 0} s\bar{f}(s) = \lim_{t\to\infty} f(t) \quad (2.68)$$

In order to determine $\bar{P}^{(1)}(z,s)$, $\bar{P}^{(2)}(z,s)$, $\bar{V}^{(1)}(z,s)$ and $\bar{V}^{(2)}(z,s)$ completely, we have yet to determine the unknown Q which appears in the numerators of the right hand sides of equations (2.64) to (2.67).
For that purpose, we shall use the normalizing condition

$$P^{(1)}(1) + P^{(2)}(1) + V^{(1)}(1) + V^{(2)}(1) + Q = 1 \quad (2.69)$$

The steady state probabilities for an $M^{[X]}/G/1$ queue with second optional service and second optional vacation are given by

$$P^{(1)}(1) = \frac{\lambda E(I)E(B_1)Q}{dr} \quad (2.70)$$

$$P^{(2)}(1) = \frac{\theta\lambda E(I)E(B_2)Q}{dr} \tag{2.71}$$

$$V^{(1)}(1) = \frac{\lambda E(I)E(V_1)Q}{dr} \tag{2.72}$$

$$V^{(2)}(1) = \frac{p\lambda E(I)E(V_2)Q}{dr} \tag{2.73}$$

where

$$dr = 1 - \lambda E(I)[E(B_1) + \theta E(B_2) + E(V_1) + pE(V_2)]. \tag{2.74}$$

$P^{(1)}(1)$, $P^{(2)}(1)$, $V^{(1)}(1)$, $V^{(2)}(1)$ and Q are the steady state probabilities that the server is providing first essential service, second optional service, server under phase one compulsory vacation, second optional vacation and server under idle respectively without regard to the number of customers in the queue.

Multiplying both sides of equations (2.64) to (2.67) by s, taking limit as $s \to 0$, applying property (2.68) and simplifying, we obtain

$$P^{(1)}(z) = \frac{[\bar{B}_1(b) - 1]Q}{D(z)} \tag{2.75}$$

$$P^{(2)}(z) = \frac{\theta\bar{B}_1(b)[\bar{B}_2(b) - 1]Q}{D(z)} \tag{2.76}$$

$$V^{(1)}(z) = \frac{\bar{B}_1(b)[1 - \theta + \theta\bar{B}_2(b)][\bar{V}_1(b) - 1]Q}{D(z)} \tag{2.77}$$

$$V^{(2)}(z) = \frac{p\bar{B}_1(b)[1 - \theta + \theta\bar{B}_2(b)]\bar{V}_1(b)[\bar{V}_2(b) - 1]Q}{D(z)} \tag{2.78}$$

where

$$D(z) = z - \bar{V}_1(b)\bar{B}_1(b)[1 - \theta + \theta\bar{B}_2(b)][1 - p + p\bar{V}_2(b)], \tag{2.79}$$

and $b = \lambda - \lambda C(z)$.

Let $W_q(z)$ denote the probability generating function of the queue size irrespective of the state of the system. Then adding equations (2.75) to (2.78), we obtain

$$
\begin{aligned}
W_q(z) =& P^{(1)}(z) + P^{(2)}(z) + V^{(1)}(z) + V^{(2)}(z) \\
W_q(z) =& \frac{[\bar{B}_1(b) - 1]Q}{D(z)} + \frac{\theta \bar{B}_1(b)[\bar{B}_2(b) - 1]Q}{D(z)} \\
&+ \frac{\bar{B}_1(b)[1 - \theta + \theta \bar{B}_2(b)][\bar{V}_1(b) - 1]Q}{D(z)} \\
&+ \frac{p\bar{B}_1(b)[1 - \theta + \theta \bar{B}_2(b)]\bar{V}_1(b)[\bar{V}_2(b) - 1]Q}{D(z)}
\end{aligned} \tag{2.80}
$$

In order to find Q, we use the normalization condition $W_q(1) + Q = 1$. We see that for z=1, $W_q(1)$ is indeterminate of the form 0/0. Therefore, we apply L'Hopital's rule and on simplifying, we get

$$
W_q(1) = \frac{\lambda E(I)[E(B_1) + \theta E(B_2) + E(V_1) + pE(V_2)]}{1 - \lambda E(I)[E(B_1) + \theta E(B_2) + E(V_1) + pE(V_2)]} Q \tag{2.81}
$$

where $C(1)$= 1, $C'(1) = E(I)$ is mean batch size of the arriving customers, $E(B_i) = -\bar{B}_i'(0)$, $E(V_i) = -\bar{V}_i'(0)$ for $i = 1, 2$.

Therefore adding Q to equation (2.81), equating to 1 and simplifying, we get

$$
Q = 1 - \rho \tag{2.82}
$$

and hence the utilization factor ρ of the system is given by

$$
\rho = \lambda E(I)[E(B_1) + \theta E(B_2) + E(V_1) + pE(V_2)] \tag{2.83}
$$

where $\rho < 1$ is the stability condition under which the steady state exists. Equation (2.82) gives the probability that the server is idle. Substituting Q from (2.82) into (2.80), we have completely and explicitly determined $W_q(z)$, the probability generating function of the queue size.

2.6 The average queue size and average waiting time

Let L_q denote the mean number of customers in the queue under the steady state. Then

$$L_q = \frac{d}{dz} W_q(z) \ \text{ at } z = 1$$

since this formula gives 0/0 form, then we write $W_q(z)$ given in (2.80) as $W_q(z) = \dfrac{N(z)}{D(z)} Q$ where

$$N(z) = \bar{B}_1(b)\bar{V}_1(b)(1 - \theta + \theta\bar{B}_2(b))(1 - p + p\bar{V}_2(b)) - 1$$

and $D(z)$ is given by equation (2.79).

$$\begin{aligned}
N'(z) =& \bar{B}_1'(b)(b')\bar{V}_1(b)(1 - \theta + \theta\bar{B}_2(b))(1 - p + p\bar{V}_2(b)) \\
&+ \bar{B}_1(b)\bar{V}_1'(b)(b')(1 - \theta + \theta\bar{B}_2(b))(1 - p + p\bar{V}_2(b)) \\
&+ \bar{B}_1(b)\bar{V}_1(b)\theta\bar{B}_2'(b)(b')(1 - p + p\bar{V}_2(b)) \\
&+ \bar{B}_1(b)\bar{V}_1(b)(1 - \theta + \theta\bar{B}_2(b))p\bar{V}_2'(b)(b') \\
D'(z) =& 1 - [\bar{V}_1'(b)b'\bar{B}_1(b) + \bar{V}_1(b)\bar{B}_1'(b)b'](1 - \theta + \theta\bar{B}_2(b)) \\
&\times (1 - p + p\bar{V}_2(b)) - \bar{B}_1(b)\bar{V}_1(b)[\theta\bar{B}_2'(b)b'(1 - p + p\bar{V}_2(b)) \\
&+ (1 - \theta + \theta\bar{B}_2(b))p\bar{V}_2'(b)b'] \\
N''(z) =& (\bar{B}_1''(b)(b')^2\bar{V}_1(b) + b''\bar{B}_1'(b)\bar{V}_1(b) \\
&+ 2\bar{B}_1'(b)\bar{V}_1'(b)(b')^2 + \bar{B}_1(b)\bar{V}_1''(b)(b')^2 \\
&+ \bar{B}_1(b)\bar{V}_1'(b)b'')(1 - \theta + \theta\bar{B}_2(b))(1 - p + p\bar{V}_2(b)) \\
&+ 2(\bar{B}_1'(b)(b')\bar{V}_1(b) + \bar{B}_1(b)\bar{V}_1'(b)b') \\
&\times (\theta\bar{B}_2'(b)b'(1 - p + p\bar{V}_2(b)) + (1 - \theta + \theta\bar{B}_2(b))p\bar{V}_2'(b)b') \\
&+ \bar{B}_1(b)\bar{B}_2(b)(\theta\bar{B}_2''(b)(b')^2(1 - p + p\bar{V}_2(b))
\end{aligned}$$

$$
\begin{aligned}
&+ \theta b'' \bar{B}_2'(b)(1-p+p\bar{V}_2'(b)) \\
&+ 2\theta \bar{B}_2'(b)(b')p\bar{V}_2'(b)(b') \\
&+ (1-\theta+\theta\bar{B}_2(b))p\bar{V}_2''(b)(b')^2 \\
&+ (1-\theta+\theta\bar{B}_2(b))p\bar{V}_2'(b)(b''))
\end{aligned}
$$

$$
\begin{aligned}
D''(z) =& - [(\bar{V}_1''(b)\bar{B}_1(b) + 2\bar{V}_1'(b)\bar{B}_1'(b) + \bar{V}_1(b)\bar{B}_1''(b)) \\
&\times (b')^2(1-\theta+\theta\bar{B}_2(b))(1-p+p\bar{V}_2(b)) \\
&+ 2b'(\bar{V}_1'(b)\bar{B}_1(b) + \bar{V}_1(b)\bar{B}_1'(b)) \\
&\times [\theta\bar{B}_2'(b)b'(1-p+p\bar{V}_2(b)) + (1-\theta+\theta\bar{B}_2(b))p\bar{V}_2'(b)b'] \\
&+ \bar{V}_1(b)\bar{B}_1(b)[(\theta\bar{B}_2''(b)b'^2 + \theta\bar{B}_2'(b)b'')(1-p+p\bar{V}_2(b)) \\
&+ 2\theta\bar{B}_2'(b)b'p\bar{V}_2'(b)b' + (1-\theta+\theta\bar{B}_2(b))(p\bar{V}_2''(b)b'^2 + p\bar{V}_2'(b)b'')]]
\end{aligned}
$$

$$
L_q = \frac{d}{dz}W_q(z) = \left[\frac{D'(1)N''(1) - N'(1)D''(1)}{2(D'(1))^2}\right] Q \tag{2.84}
$$

where primes and double primes in (2.84) denote first and second derivative at $z = 1$ respectively. Carrying out the derivative at $z = 1$, we have

$$
N'(1) = \lambda E(I)[E(B_1) + \theta E(B_2) + E(V_1) + pE(V_2)] \tag{2.85}
$$

$$
\begin{aligned}
N''(1) =& \lambda^2(E(I))^2[E(B_1^2) + \theta E(B_2^2) + E(V_1^2) + pE(V_2^2)] \\
&+ \lambda E(I(I-1))[E(B_1) + \theta E(B_2) + E(V_1) + pE(V_2)] \\
&+ 2\lambda^2(E(I))^2[E(B_1)E(V_1) + p\theta E(B_2)E(V_2)] \\
&+ 2\lambda^2(E(I))^2[E(B_1) + E(V_1)][\theta E(B_2) + pE(V_2)]
\end{aligned} \tag{2.86}
$$

$$
D'(1) = 1 - \lambda E(I)[E(B_1) + \theta E(B_2) + E(V_1) + pE(V_2)] \tag{2.87}
$$

$$
\begin{aligned}
D''(1) =& - \lambda^2(E(I))^2[E(B_1^2) + \theta E(B_2^2) + E(V_1^2) + pE(V_2^2)] \\
&- 2\lambda^2(E(I))^2[E(B_1)E(V_1) + p\theta E(B_2)E(V_2)] \\
&- 2\lambda^2(E(I))^2[E(B_1) + E(V_1)][\theta E(B_2) + pE(V_2)] \\
&- \lambda E(I(I-1))[E(B_1) + \theta E(B_2) + E(V_1) + pE(V_2)]
\end{aligned} \tag{2.88}
$$

where $E(B_1^2)$, $E(B_2^2)$, $E(V_1^2)$ and $E(V_2^2)$ are the second moment of the service times and vacation times respectively. $E(I(I-1))$ is the second factorial moment of the batch size of arriving customers.

Then if we substitute the values $N'(1), N''(1), D'(1), D''(1)$ from equations (2.85) to (2.88) into equation (2.84), we obtain L_q in the closed form.

Further, we find the mean system size L using Little's formula. Thus we have

$$L = L_q + \rho \tag{2.89}$$

where L_q has been found by equation (2.84) and ρ is obtained from equation (2.83).

Let W_q and W denote the mean waiting time in the queue and in the system respectively. Then by using Little's formula, we obtain

$$W_q = \frac{L_q}{\lambda}$$

and

$$W = \frac{L}{\lambda}$$

where L_q and L have been found in equations (2.84) and (2.89).

2.7 Particular cases

Case 1: If there is no optional service, i.e, $\theta = 0$. Then our model reduces to the $M^{[X]}/G/1$ queue with compulsory vacation and optional vacation.

Using this in the main result of (2.82), (2.83) and (2.84), we can find the idle probability Q, utilization factor ρ, and the mean queue size L_q can be simplified to the following expressions.

$$Q = 1 - \lambda E(I)[E(B_1) + E(V_1) + pE(V_2)]$$

$$\rho = \lambda E(I)[E(B_1) + E(V_1) + pE(V_2)]$$

$$L_q = \left[\frac{D'(1)N''(1) - N'(1)D''(1)}{2(D''(1))^2}\right] Q$$

where

$$\begin{aligned}
N'(1) =& \lambda E(I)[E(B_1) + E(V_1) + pE(V_2)] \\
N''(1) =& \lambda^2 (E(I))^2 [E(B_1^2) + E(V_1^2) + pE(V_2^2)] \\
& + \lambda E(I(I-1))[E(B_1) + E(V_1) + pE(V_2)] \\
& + 2\lambda^2 (E(I))^2 E(B_1) E(V_1) \\
& + 2\lambda^2 (E(I))^2 [E(B_1) + E(V_1)] pE(V_2) \\
D'(1) =& 1 - \lambda E(I)[E(B_1) + E(V_1) + pE(V_2)] \\
D''(1) =& -\lambda^2 (E(I))^2 [E(B_1^2) + E(V_1^2) + pE(V_2^2)] \\
& - 2\lambda^2 (E(I))^2 E(B_1) E(V_1) \\
& - 2p\lambda^2 (E(I))^2 [E(B_1) + E(V_1)] E(V_2) \\
& - \lambda E(I(I-1))[E(B_1) + E(V_1) + pE(V_2)]
\end{aligned}$$

Case 2: If the server has no vacations and $C(z) = z$. i.e, $p = 0$, $E(I) = 1, E(I(I-1)) = 0$ then our model reduces to the $M/G/1$ queue with second optional service.

Using this in the main result of (2.82), (2.83) and (2.84), we can find the idle probability Q, utilization factor ρ, and the mean queue size L_q can be simplified to the following expressions.

$$\begin{aligned}
Q =& 1 - \rho \\
\rho =& \lambda[E(B_1) + \theta E(B_2)] \\
L_q =& \left[\frac{D'(1)N''(1) - N'(1)D''(1)}{2(D''(1))^2}\right] Q
\end{aligned}$$

where

$$
\begin{aligned}
N'(1) =&\lambda[E(B_1)+\theta E(B_2)]\\
N''(1) =&\lambda^2[E(B_1^2)+\theta E(B_2^2)]+2\lambda^2 E(B_1)\theta E(B_2)\\
D'(1) =&1-\lambda[E(B_1)+\theta E(B_2)]\\
D''(1) =&-\lambda^2[E(B_1^2)+\theta E(B_2^2)]-2\lambda^2 E(B_1)\theta E(B_2)
\end{aligned}
$$

The above equations coincide with result given by Jehad Al-Jararha and Madan (2003).

Case 3: If the second service follows exponential distribution for case 2, then the result coincide with Madan (2000b).

Case 4: When the server has no optional service, no vacation and $C(z) = z$ i.e, $\theta= 0$, $p = 0$, $E(V_1) = 0$, $E(I)= 1$ and $E(I(I-1)) = 0$ then our model reduces to the $M/G/1$ queueing system.

Using this in the main result of (2.82), (2.83) and (2.84), we can find the idle probability Q, utilization factor ρ, and the mean queue size L_q can be simplified to the following expressions.

$$
\begin{aligned}
Q =&1-\lambda E(B_1)\\
\rho =&\lambda E(B_1)\\
L_q =&\left[\frac{D'(1)N''(1)-N'(1)D''(1)}{2(D'(1))^2}\right]Q
\end{aligned}
$$

where

$$
\begin{aligned}
N'(1) =&\lambda E(B_1)\\
N''(1) =&\lambda^2 E(B_1^2)\\
D'(1) =&1-\lambda E(B_1)\\
D''(1) =&-\lambda^2 E(B_1^2)
\end{aligned}
$$

The above equations coincide with result given by Medhi (1982).

2.8 Numerical results

To numerically illustrate the results obtained in this work, we consider that the service times and vacation times are exponentially distributed with rates μ_1, μ_2, γ_1 and γ_2.

In order to see the effect of various parameters on server's idle time Q, utilization factor ρ and various other queue characteristics such as L, W, L_q, W_q. We base our numerical example on the result found in case 1.

For this purpose in Table 2.1, we choose the following arbitrary values: $E(I) = 0.3,\ E(I(I-1)) = 0.05,\ \mu = 3,\ \gamma_1 = 3,\ \gamma_2 = 2$, and $p = 0.25$ while λ varies from 0.1 to 1.0 such that the stability condition is satisfied.

The Table 2.1 gives computed values of the idle time, the utilization factor, the average queue size, system size and average waiting time in the queue and the system of our queueing model. It clearly shows as long as increasing

Table 2.1: Computed values of various queue characteristics

λ	Q	ρ	L_q	L	W_q	W
0.1	0.968333	0.038500	0.002836	0.034502	0.028356	0.345023
0.2	0.936667	0.077000	0.007500	0.070833	0.037500	0.354167
0.3	0.905000	0.115500	0.014185	0.109185	0.047284	0.363950
0.4	0.873333	0.154000	0.023111	0.149777	0.057777	0.374443
0.5	0.841667	0.192500	0.034530	0.192863	0.069059	0.385726
0.6	0.778333	0.231000	0.048735	0.238735	0.081224	0.397891
0.7	0.746667	0.269500	0.066065	0.287732	0.094379	0.411046
0.8	0.715000	0.308000	0.086920	0.340253	0.108650	0.425316
0.9	0.683333	0.346500	0.111766	0.396766	0.124184	0.440851
1.0	0.615000	0.385000	0.141159	0.457825	0.141159	0.457825

the arrival rate, the server's idle time decreases while the utilization factor, the average queue size, system size and average waiting time of our queueing model are all increases.

In Table 2.2, we choose the following arbitrary values: $\mu_1 = 2,\ \gamma_1 = 4,$

Table 2.2: Computed values of various queue characteristics

γ_2	Q	ρ	L_q	L	W_q	W
1	0.250000	0.750000	1.900000	2.650000	0.950000	1.325000
2	0.325000	0.675000	1.107692	1.782692	0.553846	0.891350
3	0.350000	0.650000	0.947619	1.597619	0.473810	0.798810
4	0.362500	0.637500	0.881034	1.518534	0.440517	0.759270
5	0.370000	0.630000	0.844865	1.474865	0.422432	0.737430
6	0.375000	0.625000	0.822222	1.447222	0.411111	0.723610
7	0.378571	0.621429	0.806739	1.428167	0.403369	0.714080
8	0.381250	0.618750	0.795492	1.414242	0.397746	0.707120
9	0.383333	0.616667	0.786695	1.403623	0.393478	0.701810
10	0.385000	0.615000	0.780260	1.395260	0.390130	0.697630

$\lambda = 2$, $E(I) = 0.3$, $E(I(I-1)) = 0.02$ and $p = 0.25$ while γ_2 varies from 1 to 10 such that the stability condition is satisfied.

The Table 2.2 gives computed values of the idle time, the utilization factor, the average queue size, system size and average waiting time in the queue and the system of our queueing model.

It clearly shows as long as increasing the vacation rate, the server's idle time increases while the utilization factor, the average queue size, system size and average waiting time of our queueing model are all decreases.

CHAPTER THREE

$M^{[X]}/G/1$ Queue with Two Phases of Service, Optional Re-Service and Bernoulli Vacation

CHAPTER

THREE

$M^{[X]}/G/1$ QUEUE WITH TWO PHASES OF SERVICE, OPTIONAL RE-SERVICE AND BERNOULLI VACATION

3.1 Introduction

Vacation queues have been studied extensively by numerous authors including Levy and Yechiali (1976), Borthakur and Chaudhury (1997), Fuhrmann and Cooper (1985), Doshi (1986), Madan (1991), Chaudhury (2000) and Chae et al. (2001) due to their various applications in Communication systems, Computer network etc. For the first time the concept of Bernoulli vacation were studied by Keilson and Servi (1986).

A two phase queueing system with vacation have studied by Doshi (1991), Krishna Kumar et al. (2002a), Artalejo and Choudhury (2004), Choudhury and Paul (2005), Badamchi Zadeh and Shankar (2008), Choudhury and Tadj

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(2009), and Gautam Choudhury and Mitali Deka (2012), Arivudainambi and Godhandaraman (2012). Madan and Ayman Baklizi (2002) have studied an M/G/1 queue with additional second stage service and optional re-service. $M^{[X]}/G_1, G_2/1$ queue with optional re-service have studied by Madan et al. (2004). Madan and Anabosi (2003) have studied a single server queue with two types of service, Bernoulli schedule server vacation and a single vacation policy.

This chapter consists of two models. In Model I, we consider $M^{[X]}/G/1$ queues with second optional service, optional re-service and Bernoulli vacation and in Model II, we consider $M^{[X]}/G/1$ queues with two types of service, optional re-service and Bernoulli vacation.

Model 1: $M^{[X]}/G/1$ queue with second optional service, optional re-service and Bernoulli vacation

In Model I, we assume that the customers arrive to the system in batches of variable size, but are served one by one where the arrival follows Poisson. A single server provides two phases of service. Each customer undergoes first phase of essential service whereas second phase of service is optional. After completion of first phase of service, customer has the option to repeat or not to repeat the first phase of service and leave the system without taking the second phase or take the second phase service. Similarly after the second phase service he has yet another option to repeat or not to repeat the second phase service. After each service completion, the server may take a vacation with probability θ or may continue to stay in the system with probability $1-\theta$. The service and vacation periods follow general (arbitrary) distribution. Further, we assume that this option of repeating the first phase or the second phase service can be availed only once.

Here we derive time dependent probability generating functions in terms of

Laplace transforms. We also derive the average queue size and average waiting time in the queue and the system. Some particular cases and numerical results are also discussed.

The Model I is organised as follows. The model description is given in section 3.2. Definitions and equations governing the system are given in section 3.3. The time dependent solution have been obtained in section 3.4 and corresponding steady state results have been derived explicitly in section 3.5. Average queue size and average waiting time in the queue and system are computed in section 3.6. Some particular cases and numerical results are discussed in section 3.7 and 3.8 respectively.

3.2 Model description

We assume the following to describe the queueing model of our study.

a) Customers arrive at the system in batches of variable size in a compound Poisson process and they are provided one by one service on a first come - first served basis. Let $\lambda c_i dt$ ($i \geq 1$) be the first order probability that a batch of i customers arrives at the system during a short interval of time $(t, t+dt]$, where $0 \leq c_i \leq 1$ and $\sum_{i=1}^{\infty} c_i = 1$, $\lambda > 0$ is the arrival rate of batches.

b) There is a single server who provides the first phase of service for all customers, as soon as the first phase of service of a customer is completed, he may opt to repeat the first phase of service with probability r_1 or may not repeat with probability $1 - r_1$. After completing the first phase of service, the customer may opt to take the second phase of service with probability p or may leave the system without taking the second phase of service with probability $1 - p$. Similarly after taking the second phase of service he may demand repetation of second phase of service with probability r_2 or may leave the system without repeating the second phase of service with probability 1- r_2. Further, we assume that this option of repeating the first phase or the second phase of service can be availed only once.

c) The service time follows a general (arbitrary) distribution with distribution function $B_i(s)$ and density function $b_i(s)$. Let $\mu_i(x)dx$ be the conditional probability density of service completion during the interval $(x, x + dx]$, given that the elapsed service time is x, so that

$$\mu_i(x) = \frac{b_i(x)}{1 - B_i(x)}, \quad i = 1, 2,$$

and therefore,

$$b_i(s) = \mu_i(s)e^{-\int_0^s \mu_i(x)dx}, \quad i = 1, 2.$$

d) As soon as each service is over, the server may take a vacation with probability θ or may continue to stay in the system with probability $1 - \theta$.

e) The server's vacation time follows a general (arbitrary) distribution with distribution function $V(t)$ and density function $v(t)$. Let $\gamma(x)dx$ be the conditional probability density of vacation completion during the interval $(x, x + dx]$, given that the elapsed vacation time is x, so that

$$\gamma(x) = \frac{v(x)}{1 - V(x)},$$

and therefore,

$$v(t) = \gamma(t)e^{-\int_0^t \gamma(x)dx}$$

f) Various stochastic processes involved in the system are assumed to be independent of each other.

3.3 Definitions and equations governing the system

We define

$P_n^{(1)}(x,t)$ = Probability that at time t, the server is active providing first phase of service and there are n $(n \geq 0)$ customers in the queue excluding the one being served and the elapsed service time is x. Consequently $P_n^{(1)}(t) = \int_0^\infty P_n^{(1)}(x,t)dx$ denotes the probability that at time t there are n customers in the queue excluding the one customer in the first phase of service irrespective of the value of x.

$P_n^{(2)}(x,t)$ = Probability that at time t, the server is active providing second phase of optional service and there are n $(n \geq 0)$ customers in the queue excluding the one being served and the elapsed service time is x. Consequently $P_n^{(2)}(t) = \int_0^\infty P_n^{(2)}(x,t)dx$ denotes the probability that at time t there are n customers in the queue excluding the one customer in the second phase of optional service irrespective of the value of x.

$R_n^{(1)}(x,t)$ = Probability that at time t, the server is active providing first phase of re-service and there are n $(n \geq 0)$ customers in the queue excluding the one customer who is repeating first phase service and the elapsed service time is x. Consequently $R_n^{(1)}(t) = \int_0^\infty R_n^{(1)}(x,t)dx$ denotes the probability that at time t there are n customers in the queue excluding the one customer who is repeating first phase of service irrespective of the value of x.

$R_n^{(2)}(x,t)$ = Probability that at time t, the server is active providing second phase of re-service and there are n $(n \geq 0)$ customers in the queue excluding the one customer who is repeating second optional service and the elapsed service time is x. Consequently $R_n^{(2)}(t) = \int_0^\infty R_n^{(2)}(x,t)dx$ denotes the probability that at time t there are n customers in the queue excluding the one customer who is repeating second phase of service irrespective of the value of x.

$V_n(x,t)$ = Probability that at time t, the server is under vacation with elapsed vacation time is x and there are n $(n \geq 0)$ customers in the queue. Accordingly $V_n(t) = \int\limits_0^\infty V_n(x,t)dx$ denotes the probability that at time t there are n customers in the queue and the server is under vacation irrespective of the value of x.

$Q(t)$ = Probability that at time t, there are no customers in the system and the server is idle but available in the system.

The system is then governed by the following set of differential - difference equations:

$$\frac{\partial}{\partial x}P_0^{(1)}(x,t) + \frac{\partial}{\partial t}P_0^{(1)}(x,t) + [\lambda + \mu_1(x)]P_0^{(1)}(x,t) = 0 \tag{3.1}$$

$$\frac{\partial}{\partial x}P_n^{(1)}(x,t) + \frac{\partial}{\partial t}P_n^{(1)}(x,t) + [\lambda + \mu_1(x)]P_n^{(1)}(x,t) = \lambda\sum_{k=1}^{n} c_k P_{n-k}^{(1)}(x,t), \quad n \geq 1 \tag{3.2}$$

$$\frac{\partial}{\partial x}P_0^{(2)}(x,t) + \frac{\partial}{\partial t}P_0^{(2)}(x,t) + [\lambda + \mu_2(x)]P_0^{(2)}(x,t) = 0 \tag{3.3}$$

$$\frac{\partial}{\partial x}P_n^{(2)}(x,t) + \frac{\partial}{\partial t}P_n^{(2)}(x,t) + [\lambda + \mu_2(x)]P_n^{(2)}(x,t) = \lambda\sum_{k=1}^{n} c_k P_{n-k}^{(2)}(x,t), \quad n \geq 1 \tag{3.4}$$

$$\frac{\partial}{\partial x}R_0^{(1)}(x,t) + \frac{\partial}{\partial t}R_0^{(1)}(x,t) + [\lambda + \mu_1(x)]R_0^{(1)}(x,t) = 0 \tag{3.5}$$

$$\frac{\partial}{\partial x}R_n^{(1)}(x,t) + \frac{\partial}{\partial t}R_n^{(1)}(x,t) + [\lambda + \mu_1(x)]R_n^{(1)}(x,t) = \lambda\sum_{k=1}^{n} c_k R_{n-k}^{(1)}(x,t), \quad n \geq 1 \tag{3.6}$$

$$\frac{\partial}{\partial x}R_0^{(2)}(x,t) + \frac{\partial}{\partial t}R_0^{(2)}(x,t) + [\lambda + \mu_2(x)]R_0^{(2)}(x,t) = 0 \tag{3.7}$$

$$\frac{\partial}{\partial x}R_n^{(2)}(x,t) + \frac{\partial}{\partial t}R_n^{(2)}(x,t) + [\lambda + \mu_2(x)]R_n^{(2)}(x,t) = \lambda\sum_{k=1}^{n} c_k R_{n-k}^{(2)}(x,t), \quad n \geq 1 \tag{3.8}$$

$$\frac{\partial}{\partial x}V_0(x,t)+\frac{\partial}{\partial t}V_0(x,t)+[\lambda+\gamma(x)]V_0(x,t)=0 \tag{3.9}$$

$$\frac{\partial}{\partial x}V_n(x,t)+\frac{\partial}{\partial t}V_n(x,t)+[\lambda+\gamma(x)]V_n(x,t)=\lambda\sum_{k=1}^{n}c_kV_{n-k}(x,t),\quad n\geq 1 \tag{3.10}$$

$$\begin{aligned}\frac{d}{dt}Q(t)= & -\lambda Q(t)+(1-\theta)(1-p)(1-r_1)\int_0^\infty P_0^{(1)}(x,t)\mu_1(x)dx\\ & +(1-\theta)(1-r_2)\int_0^\infty P_0^{(2)}(x,t)\mu_2(x)dx\\ & +(1-\theta)(1-p)\int_0^\infty R_0^{(1)}(x,t)\mu_1(x)dx\\ & +(1-\theta)\int_0^\infty R_0^{(2)}(x,t)\mu_2(x)dx+\int_0^\infty V_0(x,t)\gamma(x)dx\end{aligned} \tag{3.11}$$

The above set of equations are to be solved subject to the following boundary conditions

$$\begin{aligned}P_n^{(1)}(0,t)= & \lambda c_{n+1}Q(t)+(1-\theta)(1-p)(1-r_1)\int_0^\infty P_{n+1}^{(1)}(x,t)\mu_1(x)dx\\ & +(1-\theta)(1-r_2)\int_0^\infty P_{n+1}^{(2)}(x,t)\mu_2(x)dx\\ & +(1-\theta)(1-p)\int_0^\infty R_{n+1}^{(1)}(x,t)\mu_1(x)dx\\ & +(1-\theta)\int_0^\infty R_{n+1}^{(2)}(x,t)\mu_2(x)dx\\ & +\int_0^\infty V_{n+1}(x,t)\gamma(x)dx,\quad n\geq 0\end{aligned} \tag{3.12}$$

$$\begin{aligned}P_n^{(2)}(0,t)= & p(1-r_1)\int_0^\infty P_n^{(1)}(x,t)\mu_1(x)dx\\ & +p\int_0^\infty R_n^{(1)}(x,t)\mu_1(x)dx,\ n\geq 0\end{aligned} \tag{3.13}$$

$$R_n^{(1)}(0,t)=r_1\int_0^\infty P_n^{(1)}(x,t)\mu_1(x)dx,\quad n\geq 0 \tag{3.14}$$

$$R_n^{(2)}(0,t)=r_2\int_0^\infty P_n^{(2)}(x,t)\mu_2(x)dx,\quad n\geq 0 \tag{3.15}$$

$$V_n(0,t)=(1-p)\theta(1-r_1)\int_0^\infty P_n^{(1)}(x,t)\mu_1(x)dx$$

$$+ \theta(1-r_2)\int_0^\infty P_n^{(2)}(x,t)\mu_2(x)dx$$
$$+ (1-p)\theta\int_0^\infty R_n^{(1)}(x,t)\mu_1(x)dx$$
$$+ \theta\int_0^\infty R_n^{(2)}(x,t)\mu_2(x)dx, \quad n \geq 0 \tag{3.16}$$

We assume that initially there are no customers in the system and the server is idle. So the initial conditions are

$$P_n^{(i)}(0) = R_n^{(i)}(0) = V_n(0) = 0 \text{ for } i = 1,2, \quad n \geq 0 \text{ and } Q(0) = 1. \tag{3.17}$$

3.4 Generating functions of the queue length: The time - dependent solution

In this section, we obtain the transient solution for the above set of differential - difference equations.

Theorem: *The system of differential difference equations to describe an $M^{[X]}/G/1$ queue with second optional service, optional re-services and Bernoulli vacation are given by equations (3.1) to (3.16) with initial conditions (3.17) and the generating functions of transient solution are given by equations (3.72) to (3.76).*

Proof : We define the probability generating functions, for i =1, 2.

$$P^{(i)}(x,z,t) = \sum_{n=0}^{\infty} z^n P_n^{(i)}(x,t); \quad P^{(i)}(z,t) = \sum_{n=0}^{\infty} z^n P_n^{(i)}(t); C(z) = \sum_{n=1}^{\infty} c_n z^n;$$

$$R^{(i)}(x,z,t) = \sum_{n=0}^{\infty} z^n R_n^{(i)}(x,t); \quad R^{(i)}(z,t) = \sum_{n=0}^{\infty} z^n R_n^{(i)}(t); \tag{3.18}$$

$$V(x,z,t) = \sum_{n=0}^{\infty} z^n V_n(x,t); \quad V^{(i)}(z,t) = \sum_{n=0}^{\infty} z^n V_n(t); \tag{3.19}$$

which are convergent inside the circle given by $\mid z \mid \leq 1$ and define the Laplace transform of a function $f(t)$ as

$$\bar{f}(s) = \int_0^\infty e^{-st} f(t)dt, \quad \Re(s) > 0. \tag{3.20}$$

We take the Laplace transform of equations (3.1) to (3.16) and using (3.17), we obtain

$$\frac{\partial}{\partial x}\bar{P}_0^{(1)}(x,s) + (s+\lambda+\mu_1(x))\bar{P}_0^{(1)}(x,s) = 0 \tag{3.21}$$

$$\frac{\partial}{\partial x}\bar{P}_n^{(1)}(x,s) + (s+\lambda+\mu_1(x))\bar{P}_n^{(1)}(x,s) = \lambda\sum_{k=1}^{n} c_k\bar{P}_{n-k}^{(1)}(x,s), n \geq 1 \tag{3.22}$$

$$\frac{\partial}{\partial x}\bar{P}_0^{(2)}(x,s) + (s+\lambda+\mu_2(x))\bar{P}_0^{(2)}(x,s) = 0 \tag{3.23}$$

$$\frac{\partial}{\partial x}\bar{P}_n^{(2)}(x,s) + (s+\lambda+\mu_2(x))\bar{P}_n^{(2)}(x,s) = \lambda\sum_{k=1}^{n} c_k\bar{P}_{n-k}^{(2)}(x,s), n \geq 1 \tag{3.24}$$

$$\frac{\partial}{\partial x}\bar{R}_0^{(1)}(x,s) + (s+\lambda+\mu_1(x))\bar{R}_0^{(1)}(x,s) = 0 \tag{3.25}$$

$$\frac{\partial}{\partial x}\bar{R}_n^{(1)}(x,s) + (s+\lambda+\mu_1(x))\bar{R}_n^{(1)}(x,s) = \lambda\sum_{k=1}^{n} c_k\bar{R}_{n-k}^{(1)}(x,s), n \geq 1 \tag{3.26}$$

$$\frac{\partial}{\partial x}\bar{R}_0^{(2)}(x,s) + (s+\lambda+\mu_2(x))\bar{R}_0^{(2)}(x,s) = 0 \tag{3.27}$$

$$\frac{\partial}{\partial x}\bar{R}_n^{(2)}(x,s) + (s+\lambda+\mu_2(x))\bar{R}_n^{(2)}(x,s) = \lambda\sum_{k=1}^{n} c_k\bar{R}_{n-k}^{(2)}(x,s), n \geq 1 \tag{3.28}$$

$$\frac{\partial}{\partial x}\bar{V}_0(x,s) + [s+\lambda+\gamma(x)]\bar{V}_0(x,s) = 0 \tag{3.29}$$

$$\frac{\partial}{\partial x}\bar{V}_n(x,s) + [s+\lambda+\gamma(x)]\bar{V}_n(x,s) = \lambda\sum_{k=1}^{n} c_k\bar{V}_{n-k}(x,s), \; n \geq 1 \tag{3.30}$$

$$\begin{aligned}(s+\lambda)\bar{Q}(s) = 1 &+ (1-p)(1-r_1)(1-\theta)\int_0^\infty \bar{P}_0^{(1)}(x,s)\mu_1(x)dx \\ &+ (1-r_2)(1-\theta)\int_0^\infty \bar{P}_0^{2}(x,s)\mu_2(x)dx \\ &+ (1-\theta)(1-p)\int_0^\infty \bar{R}_0^{(1)}(x,s)\mu_1(x)dx \\ &+ (1-\theta)\int_0^\infty \bar{R}_0^{(2)}(x,s)\mu_2(x)dx + \int_0^\infty \bar{V}_0(x,s)\gamma(x)dx\end{aligned} \tag{3.31}$$

$$\bar{P}_n^{(1)}(0,s) = (1-\theta)(1-p)(1-r_1)\int_0^\infty \bar{P}_{n+1}^{(1)}(x,s)\mu_1(x)dx$$
$$+ (1-\theta)(1-r_2)\int_0^\infty \bar{P}_{n+1}^{(2)}(x,s)\mu_2(x)dx$$
$$+ (1-\theta)(1-p)\int_0^\infty \bar{R}_{n+1}^{(1)}(x,s)\mu_1(x)dx$$
$$+ (1-\theta)\int_0^\infty \bar{R}_{n+1}^{(2)}(x,s)\mu_2(x)dx + \lambda c_{n+1}\bar{Q}(s)$$
$$+ \int_0^\infty \bar{V}_{n+1}(x,s)\gamma(x)dx, \quad n \geq 0 \tag{3.32}$$

$$\bar{P}_n^{(2)}(0,s) = p(1-r_1)\int_0^\infty \bar{P}_n^{(1)}(x,s)\mu_1(x)dx$$
$$+ p\int_0^\infty \bar{R}_n^{(1)}(x,s)\mu_1(x)dx, \quad n \geq 0 \tag{3.33}$$

$$\bar{R}_n^{(1)}(0,s) = r_1\int_0^\infty \bar{P}_n^{(1)}(x,s)\mu_1(x)dx, \quad n \geq 0 \tag{3.34}$$

$$\bar{R}_n^{(2)}(0,s) = r_2\int_0^\infty \bar{P}_n^{(2)}(x,s)\mu_2(x)dx, \quad n \geq 0 \tag{3.35}$$

$$\bar{V}_n(0,s) = \theta(1-r_1)(1-p)\int_0^\infty \bar{P}_n^{(1)}(x,s)\mu_1(x)dx$$
$$+ \theta(1-r_2)\int_0^\infty \bar{P}_n^{(2)}(x,s)\mu_2(x)dx$$
$$+ \theta(1-p)\int_0^\infty \bar{R}_n^{(1)}(x,s)\mu_1(x)dx$$
$$+ \theta\int_0^\infty \bar{R}_n^{(2)}(x,s)\mu_2(x)dx, \quad n \geq 0 \tag{3.36}$$

Now multiplying equations (3.22), (3.24), (3.26), (3.28) and (3.30) by z^n and summing over n from 1 to ∞, adding to equations (3.21), (3.23), (3.25) (3.27) and (3.29) and using the generating functions defined in (3.18) and (3.19), we get

$$\frac{\partial}{\partial x}\bar{P}^{(1)}(x,z,s) + [s+\lambda-\lambda C(z)+\mu_1(x)]\bar{P}^{(1)}(x,z,s) = 0 \tag{3.37}$$

$$\frac{\partial}{\partial x}\bar{P}^{(2)}(x,z,s) + [s+\lambda-\lambda C(z))+\mu_2(x)]\bar{P}^{(2)}(x,z,s) = 0 \tag{3.38}$$

$$\frac{\partial}{\partial x}\bar{R}^{(1)}(x,z,s) + [s+\lambda-\lambda C(z)+\mu_1(x)]\bar{R}^{(1)}(x,z,s) = 0 \tag{3.39}$$

$$\frac{\partial}{\partial x}\bar{R}^{(2)}(x,z,s)+[s+\lambda-\lambda C(z)+\mu_2(x)]\bar{R}^{(2)}(x,z,s)=0 \tag{3.40}$$

$$\frac{\partial}{\partial x}\bar{V}(x,z,s)+[s+\lambda-\lambda C(z)+\gamma(x)]\bar{V}(x,z,s)=0 \tag{3.41}$$

For the boundary conditions, we multiply both sides of equation (3.32) by z^n summing over n from 0 to ∞, and use the equation (3.31), we get

$$\begin{aligned}
z\bar{P}^{(1)}(0,z,s)=&[1-(s+\lambda)\bar{Q}(s)]+\lambda C(z)\bar{Q}(s)\\
&+(1-\theta)(1-r_1)(1-p)\int_0^\infty \bar{P}^{(1)}(x,z,s)\mu_1(x)dx\\
&+(1-\theta)(1-p)\int_0^\infty \bar{R}^{(1)}(x,z,s)\mu_1(x)dx\\
&+(1-\theta)\int_0^\infty \bar{R}^{(2)}(x,z,s)\mu_2(x)dx\\
&+(1-\theta)(1-r_2)\int_0^\infty \bar{P}^{(2)}(x,z,s)\mu_2(x)dx\\
&+\int_0^\infty \bar{V}(x,z,s)\gamma(x)dx
\end{aligned} \tag{3.42}$$

Performing similar operation on equations (3.33) to (3.36), we get

$$\begin{aligned}
\bar{P}^{(2)}(0,z,s)=&(1-r_1)p\int_0^\infty \bar{P}^{(1)}(x,z,s)\mu_1(x)dx\\
&+p\int_0^\infty \bar{R}^{(1)}(x,z,s)\mu_1(x)dx
\end{aligned} \tag{3.43}$$

$$\bar{R}^{(1)}(0,z,s)=r_1\int_0^\infty \bar{P}^{(1)}(x,z,s)\mu_1(x)dx \tag{3.44}$$

$$\bar{R}^{(2)}(0,z,s)=r_2\int_0^\infty \bar{P}^{(2)}(x,z,s)\mu_2(x)dx \tag{3.45}$$

$$\begin{aligned}
\bar{V}(0,z,s)=&\,\theta(1-r_1)(1-p)\int_0^\infty \bar{P}^{(1)}(x,z,s)\mu_1(x)dx\\
&+\theta(1-r_2)\int_0^\infty \bar{P}^{(2)}(x,z,s)\mu_2(x)dx\\
&+\theta(1-p)\int_0^\infty \bar{R}^{(1)}(x,z,s)\mu_1(x)dx\\
&+\theta\int_0^\infty \bar{R}^{(2)}(x,z,s)\mu_2(x)dx, \quad n\geq 0
\end{aligned} \tag{3.46}$$

Integrating equation (3.37) between 0 and x, we get

$$\bar{P}^{(1)}(x,z,s) = \bar{P}^{(1)}(0,z,s)e^{-[s+\lambda-\lambda C(z)]x-\int\limits_0^x \mu_1(t)dt} \tag{3.47}$$

where $P^{(1)}(0,z,s)$ is given by equation (3.42).

Again integrating equation (3.47) by parts with respect to x, yields

$$\bar{P}^{(1)}(z,s) = \bar{P}^{(1)}(0,z,s)\left[\frac{1-\bar{B}_1(s+\lambda-\lambda C(z))}{s+\lambda-\lambda C(z)}\right] \tag{3.48}$$

where

$$\bar{B}_1(s+\lambda-\lambda C(z)) = \int\limits_0^\infty e^{-[s+\lambda-\lambda C(z)]x}dB_1(x)$$

is the Laplace-Stieltjes transform of the first phase of service time $B_1(x)$. Now multiplying both sides of equation (3.47) by $\mu_1(x)$ and integrating over x, we obtain

$$\int\limits_0^\infty \bar{P}^{(1)}(x,z,s)\mu_1(x)dx = \bar{P}^{(1)}(0,z,s)\bar{B}_1[s+\lambda(1-C(z))] \tag{3.49}$$

Similarly, on integrating equations (3.38) to (3.41) from 0 to x, we get

$$\bar{P}^{(2)}(x,z,s) = \bar{P}^{(2)}(0,z,s)e^{-[s+\lambda-\lambda C(z)]x-\int\limits_0^x \mu_2(t)dt} \tag{3.50}$$

$$\bar{R}^{(1)}(x,z,s) = \bar{R}^{(1)}(0,z,s)e^{-[s+\lambda-\lambda C(z)]x-\int\limits_0^x \mu_1(t)dt} \tag{3.51}$$

$$\bar{R}^{(2)}(x,z,s) = \bar{R}^{(2)}(0,z,s)e^{-[s+\lambda-\lambda C(z)]x-\int\limits_0^x \mu_2(t)dt} \tag{3.52}$$

$$\bar{V}(x,z,s) = \bar{V}(0,z,s)e^{-[s+\lambda-\lambda C(z)]x-\int\limits_0^x \gamma(t)dt} \tag{3.53}$$

where $\bar{P}^{(2)}(0,z,s)$, $\bar{R}^{(1)}(0,z,s)$, $\bar{R}^{(2)}(0,z,s)$ and $\bar{V}(0,z,s)$ are given by equations (3.43) to (3.46). Again integrating equations (3.50) to (3.53) by parts with respect to x, yields

$$\bar{P}^{(2)}(z,s) = \bar{P}^{(2)}(0,z,s)\left[\frac{1-\bar{B}_2(s+\lambda-\lambda C(z))}{s+\lambda-\lambda C(z)}\right] \tag{3.54}$$

$$\bar{R}^{(1)}(z,s) = \bar{R}^{(1)}(0,z,s)\left[\frac{1-\bar{B}_1(s+\lambda-\lambda C(z))}{s+\lambda-\lambda C(z)}\right] \tag{3.55}$$

$$\bar{R}^{(2)}(z,s) = \bar{R}^{(2)}(0,z,s)\left[\frac{1-\bar{B}_2(s+\lambda(1-C(z)))}{s+\lambda-\lambda C(z)}\right] \tag{3.56}$$

$$\bar{V}(z,s) = \bar{V}(0,z,s)\left[\frac{1-\bar{V}(s+\lambda-\lambda C(z))}{s+\lambda-\lambda C(z)}\right] \tag{3.57}$$

where

$$\bar{B}_2(s+\lambda-\lambda C(z)) = \int_0^\infty e^{-[s+\lambda-\lambda C(z)]x} dB_2(x)$$

$$\bar{V}(s+\lambda-\lambda C(z)) = \int_0^\infty e^{-[s+\lambda-\lambda C(z)]x} dV(x)$$

are the Laplace-Stieltjes transform of the second phase of service time $B_2(x)$ and vacation time $V(x)$. Now multiplying both sides of equations (3.50) to (3.53) by $\mu_1(x)$, $\mu_2(x)$ and $\gamma(x)$ integrating over x, we obtain

$$\int_0^\infty \bar{P}^{(2)}(x,z,s)\mu_2(x)dx = \bar{P}^{(2)}(0,z,s)\bar{B}_2[s+\lambda-\lambda C(z)] \tag{3.58}$$

$$\int_0^\infty \bar{R}^{(1)}(x,z,s)\mu_1(x)dx = \bar{R}^{(1)}(0,z,s)\bar{B}_1[s+\lambda-\lambda C(z)] \tag{3.59}$$

$$\int_0^\infty \bar{R}^{(2)}(x,z,s)\mu_2(x)dx = \bar{R}^{(2)}(0,z,s)\bar{B}_2[s+\lambda-\lambda C(z)] \tag{3.60}$$

$$\int_0^\infty \bar{V}(x,z,s)\gamma(x)dx = \bar{V}(0,z,s)\bar{V}[s+\lambda-\lambda C(z)] \tag{3.61}$$

Using equation (3.58) in (3.45), we get

$$\bar{R}^{(2)}(0,z,s) = r_2\bar{B}_2(a)\bar{P}^{(2)}(0,z,s) \tag{3.62}$$

where $a = s + \lambda - \lambda C(z)$.

By using equation (3.49) in (3.44), we get

$$\bar{R}^{(1)}(0,z,s) = r_1\bar{B}_1(a)\bar{P}^{(1)}(0,z,s) \tag{3.63}$$

Using equations (3.49), (3.59) and (3.63) in (3.43), we get

$$\bar{P}^{(2)}(0,z,s) = p\bar{B}_1(a)[1 - r_1 + r_1\bar{B}_1(a)]\bar{P}^{(1)}(0,z,s) \tag{3.64}$$

Using equations (3.49), (3.58) to (3.60), (3.62) to (3.64) in (3.46), we get

$$\begin{aligned}\bar{V}(0,z,s) = \theta\bar{B}_1(a)(1 - r_1 + r_1\bar{B}_1(a))\\ \times [1 - p + p\bar{B}_2(a)(1 - r_2 + r_2\bar{B}_2(a))]\bar{P}^{(1)}(0,z,s)\end{aligned} \tag{3.65}$$

Using equations (3.49), (3.58) to (3.61) in (3.42), we get

$$\begin{aligned}z\bar{P}^{(1)}(0,z,s) =& [1 - s\bar{Q}(s)] + \lambda[C(z) - 1]\bar{Q}(s)\\ &+ (1-\theta)(1-r_1)(1-p)\bar{B}_1(a)\bar{P}^{(1)}(0,z,s)\\ &+ (1-\theta)(1-p)\bar{B}_1(a)\bar{R}^{(1)}(0,z,s)\\ &+ (1-\theta)\bar{B}_2(a)\bar{R}^{(2)}(0,z,s)\\ &+ (1-\theta)(1-r_2)\bar{B}_2(a)\bar{P}^{(2)}(0,z,s)\\ &+ \bar{V}(a)\bar{V}(0,z,s)\end{aligned} \tag{3.66}$$

Similarly using equations (3.62) to (3.65), in (3.66), we get

$$\bar{P}^{(1)}(0,z,s) = \frac{\lambda(C(z) - 1)\bar{Q}(s) + (1 - s\bar{Q}(s))}{Dr} \tag{3.67}$$

where

$$Dr = z - \bar{B}_1(a)(1-r_1+r_1\bar{B}_1(a))(1-\theta+\theta\bar{V}(a))[1-p+p\bar{B}_2(a)(1-r_2+r_2\bar{B}_2(a))],$$

Substituting (3.67) into equations (3.62) to (3.65), we get

$$\bar{P}^{(2)}(0,z,s) = p\bar{B}_1(a)(1-r_1+r_1\bar{B}_1(a))\frac{[(1-s\bar{Q}(s))+\lambda(C(z)-1)\bar{Q}(s)]}{Dr} \tag{3.68}$$

$$\bar{R}^{(1)}(0,z,s) = r_1\bar{B}_1(a)\frac{[(1-s\bar{Q}(s))+\lambda(C(z)-1)\bar{Q}(s)]}{Dr} \tag{3.69}$$

$$\begin{aligned}\bar{R}^{(2)}(0,z,s) &= r_2p\bar{B}_1(a)\bar{B}_2(a)(1-r+r\bar{B}_1(a)) \\ &\quad\times \frac{[(1-s\bar{Q}(s))+\lambda(C(z)-1)\bar{Q}(s)]}{Dr}\end{aligned} \tag{3.70}$$

$$\begin{aligned}\bar{V}(0,z,s) =& \frac{\theta}{Dr}\bar{B}_1(a)(1-r_1+r_1\bar{B}_1(a))(1-p+p\bar{B}_2(a) \\ &\times(1-r_2+r_2\bar{B}_2(a)))[\lambda(C(z)-1)\bar{Q}(s)+(1-s\bar{Q}(s))]\end{aligned} \tag{3.71}$$

Using equations (3.67) to (3.71) in (3.48), (3.54) to (3.57), we get

$$\bar{P}^{(1)}(z,s) =\frac{[(1-s\bar{Q}(s))+\lambda(C(z)-1)\bar{Q}(s)]}{Dr}\frac{[1-\bar{B}_1(a)]}{a} \tag{3.72}$$

$$\begin{aligned}\bar{P}^{(2)}(z,s) =& \frac{p\bar{B}_1(a)[(1-s\bar{Q}(s))+\lambda(C(z)-1)\bar{Q}(s)]}{Dr} \\ &\times(1-r+r\bar{B}_1(a))\frac{[1-\bar{B}_2(a)]}{a}\end{aligned} \tag{3.73}$$

$$\begin{aligned}\bar{R}^{(1)}(z,s) =& \frac{r\bar{B}_1(a)[(1-s\bar{Q}(s))+\lambda(C(z)-1)\bar{Q}(s)]}{Dr} \\ &\times\frac{[1-\bar{B}_1(a)]}{a}\end{aligned} \tag{3.74}$$

$$\begin{aligned}\bar{R}^{(2)}(z,s) &= r_2p\bar{B}_1(a)\bar{B}_2(a)\frac{[(1-s\bar{Q}(s))+\lambda(C(z)-1)\bar{Q}(s)]}{Dr} \\ &\quad\times(1-r+r\bar{B}_1(a))\frac{[1-\bar{B}_2(a)]}{a}\end{aligned} \tag{3.75}$$

$$\begin{aligned}\bar{V}(z,s) =& \frac{\theta}{Dr}\bar{B}_1(a)(1-r_1+r_1\bar{B}_1(a))(1-p+p\bar{B}_2(a)(1-r_2+r_2\bar{B}_2(a))) \\ &\times[\lambda(C(z)-1)\bar{Q}(s)+(1-s\bar{Q}(s))]\left[\frac{1-\bar{V}(a)}{a}\right]\end{aligned} \tag{3.76}$$

Thus $\bar{P}^{(1)}(z,s)$, $\bar{P}^{(2)}(z,s)$, $\bar{R}^{(1)}(z,s)$, $\bar{R}^{(2)}(z,s)$ and $\bar{V}(z,s)$ are completely determined from equations (3.72) to (3.76) which completes the proof of the theorem.

3.5 The steady state results

In this section, we shall derive the steady state probability distribution for our queueing model. To define the steady state probabilities, we suppress the argument t wherever it appears in the time-dependent analysis. This can be obtained by applying the well-known Tauberian property

$$\lim_{s\to 0} s\bar{f}(s) = \lim_{t\to\infty} f(t) \tag{3.77}$$

In order to determine $\bar{P}^{(1)}(z,s)$, $\bar{P}^{(2)}(z,s)$, $\bar{R}^{(1)}(z,s)$, $\bar{R}^{(2)}(z,s)$ and $\bar{V}(z,s)$ completely, we have yet to determine the unknown Q which appears in the numerators of the right hand sides of equations (3.72) to (3.76). For that purpose, we shall use the normalizing condition

$$P^{(1)}(1) + P^{(2)}(1) + R^{(1)}(1) + R^{(2)}(1) + V(1) + Q = 1 \tag{3.78}$$

The steady state probabilities for $M^{[X]}/G/1$ queue with second phase of service, optional re-services and Bernoulli vacation are given by

$$P^{(1)}(1) = \frac{\lambda E(I)E(B_1)Q}{dr}$$

$$P^{(2)}(1) = \frac{p\lambda E(I)E(B_2)Q}{dr}$$

$$R^{(1)}(1) = \frac{r_1\lambda E(I)E(B_1)Q}{dr}$$

$$R^{(2)}(1) = \frac{r_2 p\lambda E(I)E(B_2)Q}{dr}$$

$$V(1) = \frac{\lambda\theta E(I)E(V)Q}{dr}$$

where
$$dr = 1 - \lambda E(I)[(1+r_1)E(B_1) + p(1+r_2)E(B_2) + \theta E(V)], \tag{3.79}$$

$P^{(1)}(1),\ P^{(2)}(1),\ R^{(1)}(1),\ R^{(2)}(1), V(1)$ and Q are the steady state probabilities that the server is providing first phase of service, second phase of optional service, first phase of re-optional service, second phase of re-optional service, server under vacation and idle respectively without regard to the number of customers in the queue.

Thus multiplying both sides of equations (3.72) to (3.76) by s, taking limit as $s \to 0$, applying property (3.77) and simplifying, we obtain

$$P^{(1)}(z) =\frac{[\bar{B}_1(b) - 1]}{D(z)}Q \tag{3.80}$$

$$P^{(2)}(z) =\frac{p\bar{B}_1(b)[1 - r_1 + r_1\bar{B}_1(b)][\bar{B}_2(b) - 1]}{D(z)}Q \tag{3.81}$$

$$R^{(1)}(z) =\frac{r_1\bar{B}_1(b)[\bar{B}_1(b) - 1]}{D(z)}Q \tag{3.82}$$

$$R^{(2)}(z) =\frac{pr_2\bar{B}_1(b)\bar{B}_2(b)[1 - r_1 + r_1\bar{B}_1(b)][\bar{B}_2(b) - 1]}{D(z)}Q \tag{3.83}$$

$$\begin{aligned} V(z) =&\frac{1}{D(z)}[\theta\bar{B}_1(b)(1 - r_1 + r_1\bar{B}_1(b))(1 - p + p\bar{B}_2(b) \\ &\times (1 - r_2 + r_2\bar{B}_2(b)))(\bar{V}(b) - 1)]Q \end{aligned} \tag{3.84}$$

where

$$\begin{aligned} D(z) = z - \bar{B}_1(b)[1 - p + p\bar{B}_2(b)(1 - r_2 + r_2\bar{B}_2(b)) \\ \times (1 - \theta + \theta\bar{V}(b))][1 - r_1 + r_1\bar{B}_1(b)], \end{aligned} \tag{3.85}$$

and $b = \lambda - \lambda C(z)$.

Let $W_q(z)$ denote the probability generating function of the queue size irrespective of the state of the system. Then adding equations (3.80) to (3.84), we obtain

$$W_q(z) = P^{(1)}(z) + P^{(2)}(z) + R^{(1)}(z) + R^{(2)}(z) + V(z)$$

$$
\begin{aligned}
W_q(z) =& \frac{[\bar{B}_1(b)-1]Q}{D(z)} \\
&+ \frac{p\bar{B}_1(b)[1-r_1+r_1\bar{B}_1(b)][\bar{B}_2(b)-1]Q}{D(z)} \\
&+ \frac{r_1\bar{B}_1(b)[\bar{B}_1(b)-1]Q}{D(z)} \\
&+ \frac{pr_2\bar{B}_1(b)\bar{B}_2(b)[1-r_1+r_1\bar{B}_1(b)][\bar{B}_2(b)-1]Q}{D(z)} \\
&+ \frac{\theta\bar{B}_1(b)(1-r_1+r_1\bar{B}_1(b))[1-p+p\bar{B}_2(b)}{D(z)} \\
&\times (1-r_2+r_2\bar{B}_2(b))][\bar{V}(b)-1]Q
\end{aligned} \tag{3.86}
$$

In order to find Q, we use the normalization condition $W_q(1) + Q = 1$. We see that for z=1, $W_q(1)$ is indeterminate of the form 0/0. Therefore, we apply L'Hopital's rule and on simplifying, we get

$$
W_q(1) = \frac{\lambda E(I)[(1+r_1)E(B_1)+p(1+r_2)E(B_2)+\theta E(V)]}{1-\lambda E(I)[(1+r_1)E(B_1)+p(1+r_2)E(B_2)+\theta E(V)]}Q \tag{3.87}
$$

where C(1)= 1, $C'(1) = E(I)$ is mean batch size of the arriving customers, $E(V) = -\bar{V}'(0)$, $E(B_i) = -\bar{B}_i'(0)$ for $i = 1, 2$.

Therefore adding Q to equation (3.87), equating to 1 and simplifying, we get

$$
Q = 1 - \rho \tag{3.88}
$$

and hence the utilization factor ρ of the system is given by

$$
\rho = \lambda E(I)[(1+r_1)E(B_1)+p(1+r_2)E(B_2)+\theta E(V)] \tag{3.89}
$$

where $\rho < 1$ is the stability condition under which the steady state exists. Equation (3.88) gives the probability that the server is idle. By knowing Q from (3.88), we have completely and explicitly determined $W_q(z)$, the probability generating function of the queue size.

3.6 The average queue size and average waiting time

Let L_q denote the average number of customers in the queue. Then

$$L_q = \frac{d}{dz}W_q(z) \ \text{ at } z = 1 \tag{3.90}$$

since this formula gives 0/0 form, then we write $W_q(z)$ given in (3.86) as $W_q(z) = \frac{N(z)}{D(z)}Q$ where

$$\begin{aligned}N(z) =&(\bar{B}_1(b) - 1)(1 + r_1\bar{B}_1(b))\\&+ p\bar{B}_1(b)(1 - r_1 + r_1\bar{B}_1(b))(\bar{B}_2(b) - 1)(1 + r_2\bar{B}_2(b))\\&+ \theta\bar{B}_1(b)(1 - r_1 + r_1\bar{B}_1(b))(\bar{V}(b) - 1)\\&\times (1 - p + p\bar{B}_2(b)(1 - r_2 + r_2\bar{B}_2(b)))\end{aligned}$$

and $D(z)$ is given by equation (3.85)

$$\begin{aligned}N'(z) =&\bar{B}_1'(b)b'(1 + r_1\bar{B}_1(b)) + (\bar{B}_1(b) - 1)(r_1\bar{B}_1'(b)b')\\&+ p\bar{B}_1'(b)b'(\bar{B}_2(b) - 1)(1 + r_2\bar{B}_2(b))(1 - r_1 + r_1\bar{B}_1(b))\\&+ p\bar{B}_1(a)[(r_1\bar{B}_1'(a)b')(\bar{B}_2(b) - 1)(1 + r_2\bar{B}_2(b))\\&+ (1 - r_1 + r_1\bar{B}_1(b))b'\bar{B}_2'(b))(1 + r_2\bar{B}_2(b))\\&+ (1 - r_1 + r_1\bar{B}_1(b))(\bar{B}_2(b) - 1)b'\bar{B}_2'(b))]\\&+ \theta\bar{B}_1'(b)b'(1 - r_1 + r_1\bar{B}_1(b))(\bar{V}(b) - 1)\\&\times (1 - p + p\bar{B}_2(b)(1 - r_2 + r_2\bar{B}_2(b)))\\&+ \theta\bar{B}_1(b)[(r_1\bar{B}_1'(b)b')(\bar{V}(a) - 1)(1 - p + p\bar{B}_2(a)(1 - r_2 + r_2\bar{B}_2(a)))\\&+ (1 - r_1 + r_1\bar{B}_1(b))\bar{V}'(b)b'(1 - p + p\bar{B}_2(b)(1 - r_2 + r_2\bar{B}_2(b)))\\&+ (1 - r_1 + r_1\bar{B}_1(b))(\bar{V}(b) - 1)(p\bar{B}_2'(b)b'(1 - r_2 + r_2\bar{B}_2(b))\\&+ p\bar{B}_2(b)r_2\bar{B}_2'(b)b')]\end{aligned} \tag{3.91}$$

$$
\begin{aligned}
D'(z) =& 1 - \bar{B}_1'(b)b'(1 - r_1 + r_1\bar{B}_1(b))(1 - \theta + \theta\bar{V}(b)) \\
& \times (1 - p + p\bar{B}_2(b)(1 - r_2 + r_2\bar{B}_2(b))) - \bar{B}_1(b)[(r_1\bar{B}_1'(b)b' \\
& \times (1 - \theta + \theta\bar{V}(b))(1 - p + p\bar{B}_2(b)(1 - r_2 + r_2\bar{B}_2(b))) \\
& + (1 - r_1 + r_1\bar{B}_1(b))[\theta\bar{V}'(b)b'(1 - p + p\bar{B}_2(b) \\
& \times (1 - r_2 + r_2\bar{B}_2(b))) + (1 - r_1 + r_1\bar{B}_1(b)) \\
& \times (1 - \theta + \theta\bar{V}(b))(p\bar{B}_2'(b)b'(1 - r_2 + r_2\bar{B}_2(b)) \\
& + pr_2\bar{B}_2(b)\bar{B}_2'(b)b')]
\end{aligned} \tag{3.92}
$$

$$
L_q = \frac{d}{dz}W_q(z) = \left[\frac{D'(1)N''(1) - N'(1)D''(1)}{2(D'(1))^2}\right]Q \tag{3.93}
$$

where primes and double primes in equation (3.93) denote first and second derivative at $z = 1$ respectively. Carrying out the derivative at $z = 1$, we have

$$
N'(1) = \lambda E(I)[(1 + r_1)E(B_1) + p(1 + r_2)E(B_2) + \theta E(V)] \tag{3.94}
$$

$$
\begin{aligned}
N''(1) =& \lambda^2(E(I))^2[(1 + r_1)E(B_1^2) + p(1 + r_2)E(B_2^2) + \theta E(V^2)] \\
& + \lambda E(I(I - 1))[E(B_1)(1 + r_1) + p(1 + r_2)E(B_2) + \theta E(V)] \\
& + 2\lambda^2(E(I))^2[r_1(E(B_1))^2 + pr_2(E(B_2))^2] \\
& + 2\lambda^2(E(I))^2\theta E(V)[(1 + r_1)E(B_1) + p(1 + r_2)E(B_2)] \\
& + 2p\lambda^2(E(I))^2(1 + r_1)(1 + r_2)E(B_1)E(B_2)
\end{aligned} \tag{3.95}
$$

$$
D'(1) = 1 - \lambda E(I)[(1 + r_1)E(B_1) + p(1 + r_2)E(B_2) + \theta E(V)] \tag{3.96}
$$

$$
\begin{aligned}
D''(1) =& -[\lambda^2(E(I))^2[(1 + r_1)E(B_1^2) + \theta E(V^2) + p(1 + r_2)E(B_2^2)] \\
& + \lambda E(I(I - 1))[E(B_1)(1 + r_1) + p(1 + r_2)E(B_2) + \theta E(V)] \\
& + 2\theta\lambda^2(E(I))^2E(V)[(1 + r_1)E(B_1) + p(1 + r_2)E(B_2)] \\
& + 2\lambda^2(E(I))^2[r_1(E(B_1))^2 + pr_2(E(B_2))^2] \\
& + 2p\lambda^2(1 + r_1)(1 + r_2)E(B_1)E(B_2)]
\end{aligned} \tag{3.97}
$$

where $E(B_1^2)$, $E(B_2^2)$ and $E(V^2)$ are the second moment of the service time of phase one, phase two and vacation times respectively. $E(I(I-1))$ is the second factorial moment of the batch size of arriving customers. Then if we substitute the values $N'(1), N''(1), D'(1), D''(1)$ into equation (3.93), we obtain L_q in the closed form.

Further, we find the average system size L using Little's formula. Thus we have

$$L = L_q + \rho \tag{3.98}$$

L_q has been found by equation (3.93) and ρ is obtained from equation (3.89).

Let W_q and W denote the average waiting time in the queue and in the system respectively. Then by using Little's formula, we obtain

$$W_q = \frac{L_q}{\lambda} \tag{3.99}$$

$$W = \frac{L}{\lambda} \tag{3.100}$$

where L_q and L have been found in equations (3.93) and (3.98).

3.7 Particular cases

Case 1: When the server has no option to take vacation and $C(z) = z$, i.e, θ=0, $E(I)$=1 and $E(I(I-1))$=0 then our model reduces to the $M/G/1$ queue with second optional service and optional re-services.

Using this in the main result of (3.88), (3.89) and (3.93), we can find the idle probability Q, utilization factor ρ, and the mean queue size L_q can be simplified to the following expressions.

$$Q = 1 - \lambda[(1 + r_1)E(B_1) + p(1 + r_2)E(B_2)]$$
$$\rho = \lambda[1 + r_1)E(B_1) + p(1 + r_2)E(B_2)]$$

$$L_q = \left[\frac{D'(1)N''(1) - N'(1)D''(1)}{2(D'(1))^2}\right] Q$$

where

$$\begin{aligned}
N'(1) =& \lambda[(1+r_1)E(B_1) + p(1+r_2)E(B_2)] \\
N''(1) =& \lambda^2[(1+r_1)E(B_1^2) + p(1+r_2)E(B_2^2)] \\
& + 2\lambda^2[r_1(E(B_1))^2 + pr_2(E(B_2))^2] \\
& + 2p\lambda^2(1+r_1)(1+r_2)E(B_1)E(B_2)] \\
D'(1) =& 1 - \lambda[(1+r_1)E(B_1) + p(1+r_2)E(B_2)] \\
D''(1) =& - [\lambda^2[(1+r_1)E(B_1^2) + p(1+r_2)E(B_2^2)] \\
& + 2\lambda^2[r_1(E(B_1))^2 + pr_2(E(B_2))^2] \\
& + 2p\lambda^2(1+r_1)(1+r_2)E(B_1)E(B_2)]
\end{aligned}$$

The above result coincide with Madan and Ayman Baklizi (2002).

Case 2: If there is no optional re-service. i.e, $r_1 = r_2 = 0$. Then our model reduces to $M^{[X]}/G/1$ queue with second optional service and Bernoulli vacation.

Using this in the main result of (3.88), (3.89) and (3.93), we can find the idle probability Q, utilization factor ρ, and the mean queue size L_q can be simplified to the following expressions.

$$\begin{aligned}
Q =& 1 - \lambda E(I)[E(B_1) + pE(B_2) + \theta E(V)] \\
\rho =& \lambda E(I)[E(B_1) + pE(B_2) + \theta E(V)]
\end{aligned}$$

$$L_q = \left[\frac{D'(1)N''(1) - N'(1)D''(1)}{2(D'(1))^2}\right] Q$$

where

$$N'(1) = \lambda E(I)[E(B_1) + pE(B_2) + \theta E(V)]$$

$$
\begin{aligned}
N''(1) =& \lambda^2(E(I))^2[E(B_1^2) + pE(B_2^2) + \theta E(V^2)] \\
& + \lambda E(I(I-1))[E(B_1) + pE(B_2) + \theta E(V)] \\
& + 2\lambda^2(E(I))^2\theta E(V)[E(B_1) + pE(B_2)] \\
& + 2p\lambda^2(E(I))^2 E(B_1)E(B_2)] \\
D'(1) =& 1 - \lambda E(I)[E(B_1) + pE(B_2) + \theta E(V)] \\
D''(1) =& -[\lambda^2(E(I))^2[E(B_1^2) + \theta E(V^2) + pE(B_2^2)] \\
& + \lambda E(I(I-1))[E(B_1) + pE(B_2) + \theta E(V)] \\
& + 2\theta\lambda^2(E(I))^2 E(V)[E(B_1) + pE(B_2)] \\
& + 2p\lambda^2 E(B_1)E(B_2)]
\end{aligned}
$$

Case 3: If there is no second optional service, re-service, no first type re-service, no vacation and $C(z) = z$. i.e, $p = 0$, $r_1 = 0$ and $\theta = 0$, $E(I) = 1$ and $E(I(I-1))$=0. Then our model reduces $M/G/1$ queueing system.

Using this in the main result of (3.188), (3.189) and (3.190) we can find the idle probability Q, utilization factor ρ and the mean queue size L_q can be simplified to the following expressions.

$$
\begin{aligned}
Q =& 1 - \lambda E(B_1) \\
\rho =& \lambda E(B_1) \\
L_q =& \left[\frac{D'(1)N''(1) - N'(1)D''(1)}{2(D'(1))^2}\right] Q
\end{aligned}
$$

where

$$
\begin{aligned}
N'(1) =& \lambda E(B_1) \\
N''(1) =& \lambda^2 E(B_1^2) \\
D'(1) =& 1 - \lambda E(B_1) \\
D''(1) =& -\lambda^2 E(B_1)^2
\end{aligned}
$$

The above results coincide with Kashyap and Chaudhry (1988).

3.8 Numerical results

To numerically illustrate the results obtained in this work, we consider that the service times and vacation times are exponentially distributed with rates μ_1, μ_2 and γ.

In order to see the effect of various parameters on server's idle time Q, utilization factor ρ and various other queue characteristics such as L_q, L, W_q, W.

We base our numerical example on the result found in case 1. For this purpose in Table 3.1, we choose the following arbitrary values: r_1= 0.3, r_2= 0.5, μ_1 =4, μ_2 =3, p=0.6 while λ varies from 0.1 to 1.0 such that the stability condition is satisfied.

It clearly shows as long as increasing the arrival rate, the server's idle time decreases while the utilization factor, the average queue size, system size and average waiting time in the queue and system of our queueing model are all increases.

Table 3.1: Computed values of various queue characteristics

λ	Q	ρ	L_q	L	W_q	W
0.1	0.93750	0.06250	0.00353	0.06603	0.03529	0.66029
0.2	0.87500	0.12500	0.01513	0.14012	0.07562	0.70062
0.3	0.81250	0.18750	0.03665	0.22415	0.12215	0.74715
0.4	0.75000	0.25000	0.07058	0.32058	0.17644	0.80144
0.5	0.68750	0.31250	0.12030	0.43280	0.24061	0.86561
0.6	0.62500	0.37500	0.19056	0.56556	0.31760	0.94260
0.7	0.56250	0.43750	0.28819	0.72569	0.41170	1.03670
0.8	0.50000	0.50000	0.42347	0.92347	0.52933	1.15433
0.9	0.43750	0.56250	0.61251	1.17501	0.68057	1.30557
1.0	0.37500	0.62500	0.88222	1.50722	0.88222	1.50722

In Table 3.2, we base our numerical example on result found in case 2. For this purpose we choose the following arbitrary values: $E(I)$=0.3, $E(I(I-1))$= 0.04, μ_1 =5, μ_2 =3, θ= 0.6, $\lambda = 3$, $p = 0.5$ while γ varies from 1 to 10 such that the stability condition is satisfied.

Table 3.2: Computed values of various queue characteristics

γ	Q	ρ	L_q	L	W_q	W
1	0.13000	0.87000	7.73931	8.60931	2.57970	2.86977
2	0.40000	0.60000	1.19700	1.79700	0.39900	0.59900
3	0.49000	0.51000	0.72863	1.23863	0.24288	0.41288
4	0.53500	0.46500	0.57247	1.03747	0.19082	0.34582
5	0.56200	0.43800	0.49654	0.93454	0.16551	0.31151
6	0.58000	0.42000	0.45217	0.87217	0.15072	0.29072
7	0.59286	0.40714	0.42325	0.83039	0.14108	0.27679
8	0.60250	0.39750	0.40297	0.80046	0.13432	0.26682
9	0.61000	0.39000	0.38799	0.77798	0.12933	0.25933
10	0.61600	0.38400	0.37649	0.76048	0.12550	0.25349

It clearly shows as long as increasing the vacation rate, the server's idle time increases while the utilization factor, the average queue size, the average queue size, system size and average waiting time in the queue and system of our queueing model are all decreases.

Model II: $M^{[X]}/G/1$ queue with two types of service, optional re-service and Bernoulli vacation

Model II differ from Model I in such a way that the customer has the option of choosing either type 1 service with probability p_1 or type 2 service with probability p_2. After completion of either type 1 or type 2 service, a customer has the option to repeat or not to repeat the same type of service. As soon as the service of a customer is completed, the server will take a vacation with probability θ or may continue to stay in the system with probability $1-\theta$.

Model II is described as follows. The model description is given in section 3.9. Definitions and equations governing the system are given in section 3.10. The time dependent solution have been obtained in section 3.11 and corresponding steady state results have been derived explicitly in section 3.12. Average queue size, system size and average waiting time in the queue and the system are computed in section 3.13. Some particular cases and numerical results are discussed in section 3.14 and 3.15 respectively.

3.9 Model description

We assume the following to describe the queueing model of our study.

a) Customers arrive at the system in batches of variable size in a compound Poisson process and they are provided one by one service on a first come - first served basis. Let $\lambda c_i dt$ $(i \geq 1)$ be the first order probability that a batch of i customers arrives at the system during a short interval of time $(t, t + dt]$, where $0 \leq c_i \leq 1$, $\sum\limits_{i=1}^{\infty} c_i = 1$ and $\lambda > 0$ is the arrival rate of batches.

b) There is a single server who provides either type 1 service with probability p_1 or type 2 service with probability p_2 for all customers, as soon as the service of a customer is completed, he may opt to repeat the type 1 service with probability r_1 or may not repeat with probability $(1 - r_1)$. Similarly after taking the type 2 service he may opt to repeat the type 2 with probability r_2 or may not repeat with probability $(1 - r_2)$. Further, we assume that this option of repeating the type 1 or the type 2 service can be availed once.

c) The service time follows a general (arbitrary) distribution with distribution function $B_i(s)$ and density function $b_i(s)$. Let $\mu_i(x)dx$ be the conditional probability density of service completion during the interval $(x, x + dx]$, given that the elapsed service time is x, so that

$$\mu_i(x) = \frac{b_i(x)}{1 - B_i(x)}, \quad i = 1, 2,$$

and therefore,

$$b_i(s) = \mu_i(s) e^{-\int_0^s \mu_i(x)dx}, \quad i = 1, 2.$$

d) As soon as each service is over, the server may take a vacation with probability θ or may continue to stay in the system with probability $1 - \theta$.

e) The server's vacation time follows a general (arbitrary) distribution with distribution function $V(t)$ and density function $v(t)$. Let $\gamma(x)dx$ be the conditional probability density of vacation completion during the interval $(x, x + dx]$, given that the elapsed vacation time is x, so that

$$\gamma(x) = \frac{v(x)}{1 - V(x)},$$

and therefore,

$$v(t) = \gamma(t) e^{-\int_0^t \gamma(x)dx}$$

f) Various stochastic processes involved in the system are assumed to be independent of each other.

3.10 Definitions and equations governing the system

We define

$P_n^{(1)}(x,t)$ = Probability that at time t, the server is active providing type 1 service and there are n $(n \geq 0)$ customers in the queue excluding the one customer in the service being served and the elapsed service time is x. $P_n^{(1)}(t) = \int_0^\infty P_n^{(1)}(x,t)dx$ denotes the probability that at time t there are n customers in the queue excluding the one customer in the type 1 service irrespective of the value of x.

$P_n^{(2)}(x,t)$ = Probability that at time t, the server is active providing type 2 service and there are n $(n \geq 0)$ customers in the queue excluding the one customer in the service being served and the elapsed service time is x. $P_n^{(2)}(t) = \int_0^\infty P_n^{(2)}(x,t)dx$ denotes the probability that at time t there are n customers in the queue excluding the one customer in the type 2 service irrespective of the value of x.

$R_n^{(1)}(x,t)$ = Probability that at time t, the server is active providing type 1 re-service and there are n $(n \geq 0)$ customers in the queue excluding the one customer who is repeating type 1 service and the elapsed re-service time is x. Consequently $R_n^{(1)}(t) = \int_0^\infty R_n^{(1)}(x,t)dx$ denotes the probability that at time t there are n customers in the queue excluding the one customer who is repeating type 1 service irrespective of the value of x.

$R_n^{(2)}(x,t)$ = Probability that at time t, the server is active providing type 2 re-service and there are n $(n \geq 0)$ customers in the queue excluding the one customer who is repeating type 2 service and the elapsed re-service time is x. Consequently $R_n^{(2)}(t) = \int_0^\infty R_n^{(2)}(x,t)dx$ denotes the probability that at time t there are n customers in the queue excluding the one customer who is repeating type 2 service irrespective of the value of x.

$V_n(x,t)$ = Probability that at time t, the server is under vacation with elapsed vacation time is x and there are n $(n \geq 0)$ customers in the queue. Accordingly $V_n(t)=\int_0^\infty V_n(x,t)dx$ denotes the probability that at time t there are n customers in the queue and the server is under vacation irrespective of the value of x.

$Q(t)$ = Probability that at time t, there are no customers in the system and the server is idle but available in the system.

The model is then, governed by the following set of differential- difference equations:

$$\frac{\partial}{\partial x}P_0^{(1)}(x,t) + \frac{\partial}{\partial t}P_0^{(1)}(x,t) + (\lambda + \mu_1(x))P_0^{(1)}(x,t) = 0 \tag{3.101}$$

$$\frac{\partial}{\partial x}P_n^{(1)}(x,t) + \frac{\partial}{\partial t}P_n^{(1)}(x,t) + (\lambda + \mu_1(x))P_n^{(1)}(x,t) = \lambda\sum_{k=1}^{n} c_k P_{n-k}^{(1)}(x,t), \quad n \geq 1 \tag{3.102}$$

$$\frac{\partial}{\partial x}P_0^{(2)}(x,t) + \frac{\partial}{\partial t}P_0^{(2)}(x,t) + [\lambda + \mu_2(x)]P_0^{(2)}(x,t) = 0 \tag{3.103}$$

$$\frac{\partial}{\partial x}P_n^{(2)}(x,t) + \frac{\partial}{\partial t}P_n^{(2)}(x,t) + [\lambda + \mu_2(x)]P_n^{(2)}(x,t) = \lambda\sum_{k=1}^{n} c_k P_{n-k}^{(2)}(x,t), \quad n \geq 1 \tag{3.104}$$

$$\frac{\partial}{\partial x}R_0^{(1)}(x,t) + \frac{\partial}{\partial t}R_0^{(1)}(x,t) + [\lambda + \mu_1(x)]R_0^{(1)}(x,t) = 0 \tag{3.105}$$

$$\frac{\partial}{\partial x}R_n^{(1)}(x,t) + \frac{\partial}{\partial t}R_n^{(1)}(x,t) + [\lambda + \mu_1(x)]R_n^{(1)}(x,t) = \lambda\sum_{k=1}^{n} c_k R_{n-k}^{(1)}(x,t), \quad n \geq 1 \tag{3.106}$$

$$\frac{\partial}{\partial x}R_0^{(2)}(x,t) + \frac{\partial}{\partial t}R_0^{(2)}(x,t) + [\lambda + \mu_2(x)]R_0^{(2)}(x,t) = 0 \tag{3.107}$$

$$\frac{\partial}{\partial x}R_n^{(2)}(x,t) + \frac{\partial}{\partial t}R_n^{(2)}(x,t) + [\lambda + \mu_2(x)]R_n^{(2)}(x,t) = \lambda\sum_{k=1}^{n} c_k R_{n-k}^{(2)}(x,t), \quad n \geq 1 \tag{3.108}$$

$$\frac{\partial}{\partial x}V_0(x,t) + \frac{\partial}{\partial t}V_0(x,t) + [\lambda + \gamma(x)]V_0(x,t) = 0 \tag{3.109}$$

$$\frac{\partial}{\partial x}V_n(x,t) + \frac{\partial}{\partial t}V_n(x,t) + [\lambda + \gamma(x)]V_n(x,t) = \lambda\sum_{k=1}^{n} c_k V_{n-k}(x,t), \quad n \geq 1 \tag{3.110}$$

$$
\begin{aligned}
\frac{d}{dt}Q(t) = \lambda Q(t) &+ (1-r_1)(1-\theta)\int_0^\infty P_0^{(1)}(x,t)\mu_1(x)dx \\
&+ (1-r_2)(1-\theta)\int_0^\infty P_0^{(2)}(x,t)\mu_2(x)dx \\
&+ (1-\theta)\int_0^\infty R_0^{(1)}(x,t)\mu_1(x)dx \\
&+ (1-\theta)\int_0^\infty R_0^{(2)}(x,t)\mu_2(x)dx + \int_0^\infty V_0(x,t)\gamma(x)dx \qquad (3.111)
\end{aligned}
$$

The above equations are to be solved subject to the following boundary conditions

$$
\begin{aligned}
P_n^{(1)}(0,t) = p_1\lambda c_{n+1}Q(t) &+ p_1(1-r_1)(1-\theta)\int_0^\infty P_{n+1}^{(1)}(x,t)\mu_1(x)dx \\
&+ p_1(1-r_2)(1-\theta)\int_0^\infty P_{n+1}^{(2)}(x,t)\mu_2(x)dx \\
&+ p_1(1-\theta)\int_0^\infty R_{n+1}^{(1)}(x,t)\mu_1(x)dx \\
&+ p_1(1-\theta)\int_0^\infty R_{n+1}^{(2)}(x,t)\mu_2(x)dx \\
&+ p_1\int_0^\infty V_{n+1}(x,t)\gamma(x)dx, \quad n \geq 0 \qquad (3.112)
\end{aligned}
$$

$$
\begin{aligned}
P_n^{(2)}(0,t) = p_2\lambda c_{n+1}Q(t) &+ p_2(1-r_1)(1-\theta)\int_0^\infty P_{n+1}^{(1)}(x,t)\mu_1(x)dx \\
&+ p_2(1-r_2)(1-\theta)\int_0^\infty P_{n+1}^{(2)}(x,t)\mu_2(x)dx \\
&+ p_2(1-\theta)\int_0^\infty R_{n+1}^{(1)}(x,t)\mu_1(x)dx \\
&+ p_2(1-\theta)\int_0^\infty R_{n+1}^{(2)}(x,t)\mu_2(x)dx \\
&+ p_2\int_0^\infty V_{n+1}(x,t)\gamma(x)dx, \quad n \geq 0 \qquad (3.113)
\end{aligned}
$$

$$
R_n^{(1)}(0,t) = r_1\int_0^\infty P_n^{(1)}(x,t)\mu_1(x)dx, \quad n \geq 0 \qquad (3.114)
$$

$$
R_n^{(2)}(0,t) = r_2\int_0^\infty P_n^{(2)}(x,t)\mu_2(x)dx, \quad n \geq 0 \qquad (3.115)
$$

$$
V_n(0,t) = \theta(1-r_2)\int_0^\infty P_n^{(2)}(x,t)\mu_2(x)dx
$$

$$+ \theta(1-r_1)\int_0^\infty P_n^{(1)}(x,t)\mu_1(x)dx$$
$$+ \theta\int_0^\infty R_n^{(1)}(x,t)\mu_1(x)dx$$
$$+ \theta\int_0^\infty R_n^{(2)}(x,t)\mu_2(x)dx, \quad n \geq 0 \tag{3.116}$$

We assume that initially there are no customers in the system and the server is idle. So the initial conditions are

$$P_n^{(i)}(0) = R_n^{(i)}(0) = V_n(0) = 0 \ \text{ for } \ i = 1,2, \quad n \geq 0 \text{ and } Q(0) = 1. \tag{3.117}$$

3.11 Probability generating functions of queue length: The time-dependent solution

In this section, we obtain the transient solution for the above set of differential - difference equations.

Theorem: *The system of differential difference equations to describe an $M^{[X]}/G/1$ queue with two types of service, optional re-service and Bernoulli vacation are given by equations (3.101) to (3.116) with initial conditions (3.117) and the generating functions of transient solution are given by equations (3.173) to (3.177).*

Proof : We define the probability generating functions, for i =1, 2.

$$P^{(i)}(x,z,t) = \sum_{n=0}^{\infty} z^n P_n^{(i)}(x,t);\ P^{(i)}(z,t) = \sum_{n=0}^{\infty} z^n P_n^{(i)}(t);\ C(z) = \sum_{n=1}^{\infty} c_n z^n;$$

$$R^{(i)}(x,z,t) = \sum_{n=0}^{\infty} z^n R_n^{(i)}(x,t); \quad R^{(i)}(z,t) = \sum_{n=0}^{\infty} z^n R_n^{(i)}(t);$$

$$V(x,z,t) = \sum_{n=0}^{\infty} z^n V_n(x,t); \quad V(z,t) = \sum_{n=0}^{\infty} z^n V_n(t) \tag{3.118}$$

which are convergent inside the circle given by $\mid z \mid \leq 1$ and define the Laplace transform of a function $f(t)$ as

$$\bar{f}(s) = \int_0^{\infty} e^{-st} f(t)dt, \quad \Re(s) > 0 \tag{3.119}$$

We take the Laplace transform of equations (3.101) to (3.116) and using (3.117), we get

$$\frac{\partial}{\partial x}\bar{P}_0^{(1)}(x,s) + [s+\lambda+\mu_1(x)]\bar{P}_0^{(1)}(x,s) = 0 \tag{3.120}$$

$$\frac{\partial}{\partial x}\bar{P}_n^{(1)}(x,s) + [s+\lambda+\mu_1(x)]\bar{P}_n^{(1)}(x,s) = \lambda\sum_{k=1}^{n} c_k \bar{P}_{n-k}^{(1)}(x,s), n \geq 1 \tag{3.121}$$

$$\frac{\partial}{\partial x}\bar{P}_0^{(2)}(x,t) + [s+\lambda+\mu_2(x)]\bar{P}_0^{(2)}(x,s) = 0 \tag{3.122}$$

$$\frac{\partial}{\partial x}\bar{P}_n^{(2)}(x,s) + [s+\lambda+\mu_2(x)]\bar{P}_n^{(2)}(x,s) = \lambda\sum_{k=1}^{n} c_k \bar{P}_{n-k}^{(2)}(x,s), n \geq 1 \tag{3.123}$$

$$\frac{\partial}{\partial x}\bar{R}_0^{(1)}(x,s) + [s+\lambda+\mu_1(x)]\bar{R}_0^{(1)}(x,s) = 0 \tag{3.124}$$

$$\frac{\partial}{\partial x}\bar{R}_n^{(1)}(x,s) + [s+\lambda+\mu_1(x)]\bar{R}_n^{(1)}(x,s) = \lambda\sum_{k=1}^{n} c_k \bar{R}_{n-k}^{(1)}(x,s), \quad n \geq 1 \tag{3.125}$$

$$\frac{\partial}{\partial x}\bar{R}_0^{(2)}(x,s) + [s+\lambda+\mu_2(x)]\bar{R}_0^{(2)}(x,s) = 0 \tag{3.126}$$

$$\frac{\partial}{\partial x}\bar{R}_n^{(2)}(x,s) + [s+\lambda+\mu_2(x)]\bar{R}_n^{(2)}(x,s) = \lambda\sum_{k=1}^{n} c_k \bar{R}_{n-k}^{(2)}(x,s), \quad n \geq 1 \tag{3.127}$$

$$\frac{\partial}{\partial x}\bar{V}_0(x,s) + [s+\lambda+\gamma(x)]\bar{V}_0(x,s) = 0 \tag{3.128}$$

$$\frac{\partial}{\partial x}\bar{V}_n(x,s) + [s+\lambda+\gamma(x)]\bar{V}_n(x,s) = \lambda\sum_{k=1}^{n} c_k \bar{V}_{n-k}(x,s), \ n \geq 1 \tag{3.129}$$

$$\begin{aligned}(s+\lambda)\bar{Q}(s) = 1 &+ (1-r_1)(1-\theta)\int_0^{\infty} \bar{P}_0^{(1)}(x,s)\mu_1(x)dx \\ &+ (1-r_2)(1-\theta)\int_0^{\infty} \bar{P}_0^{2}(x,s)\mu_2(x)dx \\ &+ (1-\theta)\int_0^{\infty} \bar{R}_0^{(1)}(x,s)\mu_1(x)dx \\ &+ (1-\theta)\int_0^{\infty} \bar{R}_0^{(2)}(x,s)\mu_2(x)dx + \int_0^{\infty} V_0(x,s)\gamma(x)dx\end{aligned} \tag{3.130}$$

$$
\begin{aligned}
\bar{P}_n^{(1)}(0,s) = {} & p_1\lambda c_{n+1}\bar{Q}(s) + p_1(1-r_1)(1-\theta)\int_0^\infty \bar{P}_{n+1}^{(1)}(x,s)\mu_1(x)dx \\
& + p_1(1-r_2)(1-\theta)\int_0^\infty \bar{P}_{n+1}^{(2)}(x,s)\mu_2(x)dx \\
& + p_1(1-\theta)\int_0^\infty \bar{R}_{n+1}^{(1)}(x,s)\mu_1(x)dx \\
& + p_1(1-\theta)\int_0^\infty \bar{R}_{n+1}^{(2)}(x,s)\mu_2(x)dx \\
& + p_1\int_0^\infty V_{n+1}(x,s)\gamma(x)dx, \quad n \geq 0 \qquad (3.131)
\end{aligned}
$$

$$
\begin{aligned}
\bar{P}_n^{(2)}(0,s) = {} & p_2\lambda c_{n+1}\bar{Q}(s) + p_2(1-r_1)(1-\theta)\int_0^\infty \bar{P}_{n+1}^{(1)}(x,s)\mu_1(x)dx \\
& + p_2(1-r_2)(1-\theta)\int_0^\infty \bar{P}_{n+1}^{(2)}(x,s)\mu_2(x)dx \\
& + p_2(1-\theta)\int_0^\infty \bar{R}_{n+1}^{(1)}(x,s)\mu_1(x)dx \\
& + p_2(1-\theta)\int_0^\infty \bar{R}_{n+1}^{(2)}(x,s)\mu_2(x)dx \\
& + p_2\int_0^\infty V_{n+1}(x,s)\gamma(x)dx, \quad n \geq 0 \qquad (3.132)
\end{aligned}
$$

$$
\bar{R}_n^{(1)}(0,s) = r_1\int_0^\infty \bar{P}_n^{(1)}(x,s)\mu_1(x)dx \qquad (3.133)
$$

$$
\bar{R}_n^{(2)}(0,s) = r_2\int_0^\infty \bar{P}_n^{(2)}(x,s)\mu_2(x)dx \qquad (3.134)
$$

$$
\begin{aligned}
\bar{V}_n(0,s) = {} & \theta(1-r_2)\int_0^\infty \bar{P}_n^{(2)}(x,s)\mu_2(x)dx \\
& + \theta(1-r_1)\int_0^\infty \bar{P}_n^{(1)}(x,s)\mu_1(x)dx + \theta\int_0^\infty \bar{R}_n^{(1)}(x,s)\mu_1(x)dx \\
& + \theta\int_0^\infty \bar{R}_n^{(2)}(x,s)\mu_2(x)dx, \quad n \geq 0 \qquad (3.135)
\end{aligned}
$$

Now multiplying equations (3.121), (3.123), (3.125), (3.127) and (3.129) by suitable powers of z, adding to equations (3.120), (3.122), (3.124), (3.126) and (3.128) summing over n from 0 to ∞ and using the generating functions defined in (3.118), we get

$$\frac{\partial}{\partial x}\bar{P}^{(1)}(x,z,s)+[s+\lambda-\lambda C(z)+\mu_1(x)]\bar{P}^{(1)}(x,z,s)=0 \tag{3.136}$$

$$\frac{\partial}{\partial x}\bar{P}^{(2)}(x,z,s)+[s+\lambda-\lambda C(z)+\mu_2(x)]\bar{P}^{(2)}(x,z,s)=0 \tag{3.137}$$

$$\frac{\partial}{\partial x}\bar{R}^{(1)}(x,z,s)+[s+\lambda-\lambda C(z)+\mu_1(x)]\bar{R}^{(1)}(x,z,s)=0 \tag{3.138}$$

$$\frac{\partial}{\partial x}\bar{R}^{(2)}(x,z,s)+[s+\lambda-\lambda C(z)+\mu_2(x)]\bar{R}^{(2)}(x,z,s)=0 \tag{3.139}$$

$$\frac{\partial}{\partial x}\bar{V}(x,z,s)+[s+\lambda-\lambda C(z)+\gamma(x)]\bar{V}(x,z,s)=0 \tag{3.140}$$

For the boundary conditions, we multiply both sides of equation (3.131) by z^n summing over n from 0 to ∞, and use the equation (3.130), we get

$$\begin{aligned} z\bar{P}^{(1)}(0,z,s) &= p_1\lambda C(z)\bar{Q}(s)+p_1[1-(s+\lambda)\bar{Q}(s)] \\ &+p_1(1-r_1)(1-\theta)\int_0^\infty \bar{P}^{(1)}(x,z,s)\mu_1(x)dx \\ &+p_1(1-r_2)(1-\theta)\int_0^\infty \bar{P}^{(2)}(x,z,s)\mu_2(x)dx \\ &+p_1(1-\theta)\int_0^\infty \bar{R}^{(1)}(x,z,s)\mu_1(x)dx \\ &+p_1(1-\theta)\int_0^\infty \bar{R}^{(2)}(x,z,s)\mu_2(x)dx \\ &+p_1\int_0^\infty V(x,z,s)\gamma(x)dx, \quad n\geq 0 \end{aligned} \tag{3.141}$$

Performing similar operation on equations (3.132) to (3.135), we get

$$\begin{aligned} z\bar{P}^{(2)}(0,z,s) &= p_2\lambda C(z)\bar{Q}(s)+p_2[1-(s+\lambda)\bar{Q}(s)] \\ &+p_2(1-r_1)(1-\theta)\int_0^\infty \bar{P}^{(1)}(x,z,s)\mu_1(x)dx \\ &+p_2(1-r_2)(1-\theta)\int_0^\infty \bar{P}^{(2)}(x,z,s)\mu_2(x)dx \\ &+p_2(1-\theta)\int_0^\infty \bar{R}^{(1)}(x,z,s)\mu_1(x)dx \end{aligned}$$

$$+ p_2(1-\theta)\int_0^\infty \bar{R}^{(2)}(x,z,s)\mu_2(x)dx$$

$$+ p_2\int_0^\infty \bar{V}(x,z,s)\gamma(x)dx, \quad n \geq 0 \tag{3.142}$$

$$\bar{R}^{(1)}(0,z,s) = r_1\int_0^\infty \bar{P}^{(1)}(x,z,s)\mu_1(x)dx \tag{3.143}$$

$$\bar{R}^{(2)}(0,z,s) = r_2\int_0^\infty \bar{P}^{(2)}(x,z,s)\mu_2(x)dx \tag{3.144}$$

$$\bar{V}_n(0,z,s) = \theta(1-r_1)\int_0^\infty \bar{P}_n^{(1)}(x,z,s)\mu_1(x)dx$$

$$+ \theta\int_0^\infty \bar{R}_n^{(1)}(x,z,s)\mu_1(x)dx$$

$$+ \theta(1-r_2)\int_0^\infty \bar{P}_n^{(2)}(x,z,s)\mu_2(x)dx$$

$$+ \theta\int_0^\infty \bar{R}_n^{(2)}(x,z,s)\mu_2(x)dx, \quad n \geq 0 \tag{3.145}$$

Integrating equations (3.136) to (3.140) between 0 and x, we obtain

$$\bar{P}^{(1)}(x,z,s) = \bar{P}^{(1)}(0,z,s)e^{-[s+\lambda-\lambda C(z)]x - \int_0^x \mu_1(t)dt} \tag{3.146}$$

$$\bar{P}^{(2)}(x,z,s) = \bar{P}^{(2)}(0,z,s)e^{-[s+\lambda-\lambda C(z)]x - \int_0^x \mu_2(t)dt} \tag{3.147}$$

$$\bar{R}^{(1)}(x,z,s) = \bar{R}^{(1)}(0,z,s)e^{-[s+\lambda-\lambda C(z)]x - \int_0^x \mu_1(t)dt} \tag{3.148}$$

$$\bar{R}^{(2)}(x,z,s) = \bar{R}^{(2)}(0,z,s)e^{-[s+\lambda-\lambda C(z)]x - \int_0^x \mu_2(t)dt} \tag{3.149}$$

$$\bar{V}(x,z,s) = \bar{V}(0,z,s)e^{-[s+\lambda-\lambda C(z)]x - \int_0^x \gamma(t)dt} \tag{3.150}$$

Again integrating equations (3.146) to (3.150) with respect to x, we have

$$\bar{P}^{(1)}(z,s) = \bar{P}^{(1)}(0,z,s)\left[\frac{1-\bar{B}_1(s+\lambda-\lambda C(z))}{s+\lambda-\lambda C(z)}\right] \tag{3.151}$$

$$\bar{P}^{(2)}(z,s) = \bar{P}^{(2)}(0,z,s)\left[\frac{1-\bar{B}_2(s+\lambda-\lambda C(z))}{s+\lambda-\lambda C(z)}\right] \tag{3.152}$$

$$\bar{R}^{(1)}(z,s) = \bar{R}^{(1)}(0,z,s)\left[\frac{1-\bar{B}_1(s+\lambda-\lambda C(z))}{s+\lambda-\lambda C(z)}\right] \tag{3.153}$$

$$\bar{R}^{(2)}(z,s) = \bar{R}^{(2)}(0,z,s)\left[\frac{1-\bar{B}_2(s+\lambda-\lambda C(z))}{s+\lambda-\lambda C(z)}\right] \tag{3.154}$$

$$\bar{V}(z,s) = \bar{V}(0,z,s)\left[\frac{1-\bar{V}(s+\lambda-\lambda C(z))}{s+\lambda-\lambda C(z)}\right] \tag{3.155}$$

where

$$\bar{B}_1(s+\lambda-\lambda C(z)) = \int_0^\infty e^{-[s+\lambda-\lambda C(z)]x} dB_1(x)$$

$$\bar{B}_2(s+\lambda-\lambda C(z)) = \int_0^\infty e^{-[s+\lambda-\lambda C(z)]x} dB_2(x)$$

$$\bar{V}(s+\lambda-\lambda C(z)) = \int_0^\infty e^{-[s+\lambda-\lambda C(z)]x} dV(x)$$

are the Laplace-Stieltjes transform of the type 1 service time $B_1(x)$, type 2 service time $B_2(x)$ and vacation time $V(x)$. Now multiplying both sides of equations (3.146) to (3.150) by $\mu_1(x)$, $\mu_2(x)$, $\mu_1(x)$, $\mu_2(x)$ and $\gamma(x)$ respectively and integrating over x, we obtain

$$\int_0^\infty \bar{P}^{(1)}(x,z,s)\mu_1(x)dx = \bar{P}^{(1)}(0,z,s)\bar{B}_1[s+\lambda-\lambda C(z)] \tag{3.156}$$

$$\int_0^\infty \bar{P}^{(2)}(x,z,s)\mu_2(x)dx = \bar{P}^{(2)}(0,z,s)\bar{B}_2[s+\lambda-\lambda C(z)] \tag{3.157}$$

$$\int_0^\infty \bar{R}^{(1)}(x,z,s)\mu_1(x)dx = \bar{R}^{(1)}(0,z,s)\bar{B}_1[s+\lambda-\lambda C(z)] \tag{3.158}$$

$$\int_0^\infty \bar{R}^{(2)}(x,z,s)\mu_2(x)dx = \bar{R}^{(2)}(0,z,s)\bar{B}_2[s+\lambda-\lambda C(z)] \tag{3.159}$$

$$\int_0^\infty \bar{V}(x,z,s)\gamma(x)dx = \bar{V}(0,z,s)\bar{V}[s+\lambda-\lambda C(z)] \tag{3.160}$$

Using equations (3.156) and (3.157) in (3.143) and (3.144), we get

$$\bar{R}^{(1)}(0,z,s) = r_1\bar{B}_1(a)\bar{P}^{(1)}(0,z,s) \tag{3.161}$$

$$\bar{R}^{(2)}(0,z,s) = r_2\bar{B}_2(a)\bar{P}^{(2)}(0,z,s) \tag{3.162}$$

where $a = s + \lambda - \lambda C(z)$.

Using equations (3.156) to (3.159) in (3.145), we get

$$\begin{aligned}\bar{V}(0,z,s) = &\ \theta(1-r_1)\bar{B}_1(a)\bar{P}^{(1)}(0,z,s) \\ &+ \theta(1-r_2)\bar{B}_2(a)\bar{P}^{(2)}(0,z,s) \\ &+ \theta\bar{B}_1(a)\bar{R}^{(1)}(0,z,s) + \theta\bar{B}_2(a)\bar{R}^{(2)}(0,z,s)\end{aligned} \tag{3.163}$$

Using equations (3.161) and (3.162) in the above equation, we have

$$\begin{aligned}\bar{V}(0,z,s) = &\ \theta\bar{B}_1(a)(1-r_1+r_1\bar{B}_1(a))\bar{P}^{(1)}(0,z,s) \\ &+ \theta\bar{B}_2(a)(1-r_2+r_2\bar{B}_2(a))\bar{P}^{(2)}(0,z,s)\end{aligned} \tag{3.164}$$

Using equations (3.156) to (3.164) in (3.141) and (3.142), we get

$$\begin{aligned}[z - p_1\bar{B}_1(a)A]\bar{P}^{(1)}(0,z,s) = &\ p_1[1-(s+\lambda)\bar{Q}(s)] + \lambda p_1 C(z)\bar{Q}(s) \\ &+ p_1\bar{B}_2(a)B\bar{P}^{(2)}(0,z,s)\end{aligned} \tag{3.165}$$

$$\begin{aligned}[z - p_2\bar{B}_2(a)B]\bar{P}^{(2)}(0,z,s) = &\ p_2[1-(s+\lambda)\bar{Q}(s)] + \lambda p_2 C(z)\bar{Q}(s) \\ &+ p_2\bar{B}_1(a)A\bar{P}^{(1)}(0,z,s)\end{aligned} \tag{3.166}$$

where

$$A = (1-r_1+r_1\bar{B}_1(a))(1-\theta+\theta\bar{V}(a))$$

and

$$B = (1-r_2+r_2\bar{B}_2(a))(1-\theta+\theta\bar{V}(a))$$

From equations (3.165) and (3.166), we get

$$\bar{P}^{(1)}(0,z,s) = \frac{p_1[1-s\bar{Q}(s)] + p_1\lambda(C(z)-1)\bar{Q}(s)}{Dr} \tag{3.167}$$

where

$$Dr = z - (1 - \theta + \theta\bar{V}(a))[p_1\bar{B}_1(a)(1 - r_1 + r_1\bar{B}_1(a)) \\ + p_2\bar{B}_2(a)(1 - r_2 + r_2\bar{B}_2(a))] \tag{3.168}$$

$$\bar{P}^{(2)}(0, z, s) = \frac{p_2[1 - s\bar{Q}(s)] + \lambda p_2(C(z) - 1)\bar{Q}(s)}{Dr} \tag{3.169}$$

By substituting equations (3.167) and (3.169) in (3.161), (3.162) and (3.164), we get

$$\bar{R}^{(1)}(0, z, s) = \frac{r_1\bar{B}_1(a)[p_1(1 - s\bar{Q}(s)) + p_1\lambda(C(z) - 1)\bar{Q}(s)]}{Dr} \tag{3.170}$$

$$\bar{R}^{(2)}(0, z, s) = \frac{r_2\bar{B}_2(a)[p_2(1 - s\bar{Q}(s)) + p_2\lambda(C(z) - 1)\bar{Q}(s)]}{Dr} \tag{3.171}$$

$$\bar{V}(0, z, s) = \frac{\theta}{Dr}[p_1\bar{B}_1(a)(1 - r_1 + r_1\bar{B}_1(a)) + p_2\bar{B}_2(a) \\ \times (1 - r_2 + r_2\bar{B}_2(a))][\lambda(C(z) - 1)\bar{Q}(s) + (1 - s\bar{Q}(s))] \tag{3.172}$$

By sustituting equations (3.167), (3.169) to (3.172) in (3.151) to (3.155), we have

$$\bar{P}^{(1)}(z, s) = \frac{[\lambda p_1(C(z) - 1)\bar{Q}(s) + p_1(1 - s\bar{Q}(s))]}{Dr}\left[\frac{1 - \bar{B}_1(a)}{a}\right] \tag{3.173}$$

$$\bar{P}^{(2)}(z, s) = \frac{[\lambda p_2(C(z) - 1)\bar{Q}(s) + p_2(1 - s\bar{Q}(s))]}{Dr}\left[\frac{1 - \bar{B}_2(a)}{a}\right] \tag{3.174}$$

$$\bar{R}^{(1)}(z, s) = \frac{r_1\bar{B}_1(a)[\lambda p_1(C(z) - 1)\bar{Q}(s) + p_1(1 - s\bar{Q}(s))]}{Dr}\left[\frac{1 - \bar{B}_1(a)}{a}\right] \tag{3.175}$$

$$\bar{R}^{(2)}(z, s) = \frac{r_2\bar{B}_2(a)[\lambda p_2(C(z) - 1)\bar{Q}(s) + p_2(1 - s\bar{Q}(s))]}{Dr}\left[\frac{1 - \bar{B}_2(a)}{a}\right] \tag{3.176}$$

$$\bar{V}(z, s) = \frac{\theta}{Dr}[p_1\bar{B}_1(a)(1 - r_1 + r_1\bar{B}_1(a)) + p_2\bar{B}_2(a)(1 - r_2 + r_2\bar{B}_2(a))] \\ \times [\lambda(C(z) - 1)\bar{Q}(s) + (1 - s\bar{Q}(s))]\left[\frac{1 - \bar{V}(a)}{a}\right] \tag{3.177}$$

Thus $\bar{P}^{(1)}(z,s)$, $\bar{P}^{(2)}(z,s)$, $\bar{R}^{(1)}(z,s)$, $\bar{R}^{(2)}(z,s)$ and $\bar{V}(z,s)$ are completely determined from equations (3.173) to (3.177).

3.12 The steady state results

In this section, we shall derive the steady state probability distribution for our queueing model. To define the steady state probabilities, we suppress the argument t wherever it appears in the time-dependent analysis. This can be obtained by applying the Tauberian property,

$$\lim_{s\to 0} s\bar{f}(s) = \lim_{t\to\infty} f(t) \tag{3.178}$$

In order to determine $\bar{P}^{(1)}(z,s)$, $\bar{P}^{(2)}(z,s)$, $\bar{R}^{(1)}(z,s)$, $\bar{R}^{(2)}(z,s)$ and $\bar{V}(z,s)$ completely, we have yet to determine the unknown Q which appears in the numerators of the right hand sides of equations (3.173) to (3.177). For that purpose, we shall use the normalizing condition

$$P^{(1)}(1) + P^{(2)}(1) + R^{(1)}(1) + R^{(2)}(1) + V(1) + Q = 1$$

The steady state probabilities for $M^{[X]}/G/1$ queue with two types of service, optional re-service and Bernoulli vacation are given by

$$\begin{aligned}
P^{(1)}(1) =& \frac{\lambda p_1 E(I)E(B_1)Q}{dr}\\
P^{(2)}(1) =& \frac{\lambda p_2 E(I)E(B_2)Q}{dr}\\
R^{(1)}(1) =& \frac{\lambda r_1 p_1 E(I)E(B_1)Q}{dr}\\
R^{(2)}(1) =& \frac{\lambda r_2 p_2 E(I)E(B_2)Q}{dr}\\
V(1) =& \frac{\lambda \theta E(I)E(V)Q}{dr}
\end{aligned}$$

where

$$dr = 1 - \lambda E(I)[p_1(1+r_1)E(B_1) + p_2(1+r_2)E(B_2) + \theta E(V)], \qquad (3.179)$$

$P^{(1)}(1),\ P^{(2)}(1),\ R^{(1)}(1),\ R^{(2)}(1),$ V(1) and Q are the steady state probabilities that the server is providing type 1 service, type 2 service, type 1 re-optional service, type 2 re-optional service, server under vacation and idle respectively without regard to the number of customers in the queue.

Thus multiplying both sides of equations (3.173) to (3.177) by s, taking limit as $s \to 0$, applying property (3.178) and simplifying, we obtain

$$P^{(1)}(z) = \frac{p_1[\bar{B}_1(b) - 1]Q}{D(z)} \qquad (3.180)$$

$$P^{(2)}(z) = \frac{p_2[\bar{B}_2(b) - 1]Q}{D(z)} \qquad (3.181)$$

$$R^{(1)}(z) = \frac{p_1 r_1 \bar{B}_1(b)[\bar{B}_1(b) - 1]Q}{D(z)} \qquad (3.182)$$

$$R^{(2)}(z) = \frac{p_2 r_2 \bar{B}_2(b)[\bar{B}_2(b) - 1]Q}{D(z)} \qquad (3.183)$$

$$V(z) = \frac{\theta[p_1\bar{B}_1(b)(1 - r_1 + r_1\bar{B}_1(b)) + p_2\bar{B}_2(b)(1 - r_2 + r_2\bar{B}_2(b))][\bar{V}(b) - 1]}{D(z)} \qquad (3.184)$$

where

$$D(z) = z - (1 - \theta + \theta\bar{V}(b))[p_1\bar{B}_1(b)(1 - r_1 + r_1\bar{B}_1(b)) + p_2\bar{B}_2(b)(1 - r_2 + r_2\bar{B}_2(b))] \qquad (3.185)$$

and $b = \lambda - \lambda C(z)$.

Let $W_q(z)$ denote the probability generating function of the queue size irrespective of the state of the system. Then adding equations (3.180) to (3.184), we obtain

$$W_q(z) = P^{(1)}(z) + P^{(2)}(z) + R^{(1)}(z) + R^{(2)}(z) + V(z)$$

$$W_q(z) = \frac{p_1[\bar{B}_1(b)-1]Q}{D(z)} + \frac{p_2[\bar{B}_2(b)-1]Q}{D(z)} + \frac{p_1 r_1 \bar{B}_1(b)[\bar{B}_1(b)-1]Q}{D(z)}$$
$$+\frac{p_2 r_2 \bar{B}_2(b)[\bar{B}_2(b)-1]Q}{D(z)}$$
$$+\frac{\theta[p_1\bar{B}_1(b)(1-r_1+r_1\bar{B}_1(b))+p_2\bar{B}_2(b)(1-r_2+r_2\bar{B}_2(b))][\bar{V}(b)-1]}{D(z)} \tag{3.186}$$

In order to find Q, we use the normalization condition $W_q(1) + Q = 1$. We see that for z=1, $W_q(1)$ is indeterminate of the form 0/0. Therefore, we apply L'Hopital's rule and on simplifying, we get

$$W_q(1) = \frac{\lambda E(I)[p_1(1+r_1)E(B_1) + p_2(1+r_2)E(B_2) + \theta E(V)]Q}{dr} \tag{3.187}$$

where C(1)= 1, $C'(1) = E(I)$ is mean batch size of the arriving customers, $E(V) = -\bar{V}'(0)$, $E(B_i) = -\bar{B}_i'(0)$ for i = 1, 2.

Therefore adding Q to equation (3.187), equating to 1 and simplifying, we get

$$Q = 1 - \rho \tag{3.188}$$

and hence the utilization factor ρ of the system is given by

$$\rho = \lambda E(I)[p_1(1+r_1)E(B_1) + p_2(1+r_2)E(B_2) + \theta E(V)] \tag{3.189}$$

where $\rho < 1$ is the stability condition under which the steady state exists. Equation (3.188) gives the probability that the server is idle. By knowing Q from (3.188), we have completely and explicitly determined $W_q(z)$, the probability generating function of the queue size.

3.13 The average queue size and average waiting time

Let L_q denote the mean number of customers in the queue. Then

$$L_q = \frac{d}{dz} W_q(z) \ \text{ at } z = 1$$

since this formula gives 0/0 form, then we write $W_q(z)$ given in (3.186) as $W_q(z) = \frac{N(z)}{D(z)} Q$ where

$$\begin{aligned} N(z) &= p_1(\bar{B}_1(b) - 1)(1 + r_1\bar{B}_1(b)) + p_2(\bar{B}_2(b) - 1)(1 + r_2\bar{B}_2(b)) \\ &\quad + \theta[p_1\bar{B}_1(b)(1 - r_1 + r_1\bar{B}_1(b)) + p_2\bar{B}_2(b)(1 - r_2 + r_2\bar{B}_2(b))](\bar{V}(b) - 1) \end{aligned}$$

and $D(z)$ given in equation (3.185).

$$\begin{aligned} N'(z) =& p_1\bar{B}_1'(b)b'(1 + r_1\bar{B}_1(b)) + p_1(\bar{B}_1(b) - 1)r_1\bar{B}_1'(b)b' \\ & + p_2\bar{B}_2'(b)b'(1 + r_2\bar{B}_2(b)) + p_2(\bar{B}_2(b) - 1)r_2\bar{B}_2'(b)b' \\ & + \theta\bar{V}'(b)b'[p_1\bar{B}_1(b)(1 - r_1 + r_1\bar{B}_1(b)) \\ & + p_2\bar{B}_2(b)(1 - r_2 + r_2\bar{B}_2(b))] + \theta(\bar{V}(b) - 1) \\ & \times [p_1\bar{B}_1'(b)b'(1 - r_1 + r_1\bar{B}_1(b)) + p_1\bar{B}_1(b)r_1\bar{B}_1'(b)b' \\ & + p_2\bar{B}_2'(b)b'(1 - r_2 + r_2\bar{B}_2(b)) + p_2\bar{B}_2(b)r_2\bar{B}_1'(b)b'] \\ D'(z) =& 1 - \theta\bar{V}'(b)(b')[p_1\bar{B}_1(b)(1 - r_1 + r_1\bar{B}_1(b)) + p_2\bar{B}_2(b)(1 - r_2 + r_2\bar{B}_2(b))] \\ & - (1 - \theta + \theta\bar{V}(b))[p_1\bar{B}_1'(b)(b')(1 - r_1 + r_1\bar{B}_1(b)) + p_1\bar{B}_1(b)r_1\bar{B}_1'(b)b' \\ & + p_2\bar{B}_2'(b)b'(1 - r_2 + r_2\bar{B}_2(b)) + p_2\bar{B}_2(b)r_2\bar{B}_1'(b)b'] \\ N''(z) =& [p_1(\bar{B}_1''(b)(b')^2 + b''\bar{B}_1'(b))(1 + r_1\bar{B}_1(b)) \\ & + p_1r_1(\bar{B}_1(b) - 1)(\bar{B}_1''(b)(b')^2 + b''\bar{B}_1'(b)) \\ & + 2p_1\bar{B}_1'(b)b'r_1\bar{B}_1'(b)b' \\ & + p_2(\bar{B}_2''(b)(b')^2 + b''\bar{B}_2'(b))(1 + r_2\bar{B}_2(b)) \\ & + p_2(\bar{B}_2(b) - 1)r_2(\bar{B}_2''(b)(b')^2 + b''\bar{B}_2'(b)) \\ & + 2p_2\bar{B}_2'(b)b'r_2\bar{B}_2'(b)b' \\ & + \theta(\bar{V}''(b)(b')^2 + b''\bar{V}'(b))[p_1\bar{B}_1(b)(1 - r_1 + r_1\bar{B}_1(b)) \end{aligned}$$

$$
\begin{aligned}
&+ p_2\bar{B}_2(b)(1-r_1+r_1\bar{B}_2(b))] \\
&+ 2\theta\bar{V}'(b)b'[p_1\bar{B}_1'(b)b'(1-r_1+r_1\bar{B}_1(b)) + p_1\bar{B}_1(b)r_1\bar{B}_1'(b)b' \\
&+ p_2\bar{B}_2'(b)b'(1-r_2+r_2\bar{B}_2(b)) + p_1\bar{B}_2(b)r_2\bar{B}_2'(b)b'] \\
&+ \theta(\bar{V}(b)-1)p_1[(1-r_1+r_1\bar{B}_1(b))(\bar{B}_1''(b)(b')^2+\bar{B}_1'(b)b'') \\
&+ 2p_1r_1\bar{B}_1'(b)(b')^2 + p_1r_1\bar{B}_1(b)(\bar{B}_1''(b)(b')^2+\bar{B}_1'(b)b'') \\
&+ p_2(1-r_2+r_2\bar{B}_2(b))(\bar{B}_2''(b)(b')^2+\bar{B}_2'(b)b'') \\
&+ 2p_2r_2\bar{B}_2'(b)(b')^2 + p_2r_2\bar{B}_2(b)(\bar{B}_2''(b)(b')^2+\bar{B}_2'(b)b'')] \\
D''(z) =& -2\theta\bar{V}'(b)b'(p_1\bar{B}_1'(b)b'(1-r_1+r_1\bar{B}_1(b)) + p_1\bar{B}_1(b)r_1\bar{B}_1'(b)b' \\
&+ p_2\bar{B}_2'(b)b'(1-r_2+r_2\bar{B}_2(b)) + p_2\bar{B}_2(b)r_2\bar{B}_2'(b)b') - \theta[(b')^2\bar{V}''(b)+b''\bar{V}'(b)] \\
&\times (p_1\bar{B}_1(b)(1-r_1+r_1\bar{B}_1(b)) + p_2\bar{B}_2(b)(1-r_2+r_2\bar{B}_2(b))) \\
&- (1-\theta+\theta\bar{V}(b)))[p_1(1-r_1+r_1\bar{B}_1(b))(\bar{B}_1''(b)(b')^2+\bar{B}_1'(b)b'') \\
&+ 2p_1r_1\bar{B}_1(b)(b')^2 + p_1r_1\bar{B}_1(b)(\bar{B}_1''(b)(b')^2+\bar{B}_1'(b)b'') \\
&+ p_2(1-r_2+r_2\bar{B}_2(b))(\bar{B}_2''(b)(b')^2+\bar{B}_2'(b)b'') \\
&+ 2p_2r_2\bar{B}_2'(b)(b')^2 + p_2r_2\bar{B}_2(b)(\bar{B}_2''(b)(b')^2+\bar{B}_2'(b)b'')]
\end{aligned}
$$

Then, we use

$$
\begin{aligned}
L_q &= \lim_{z\to 1}\frac{d}{dz}W_q(z) \\
&= \lim_{z\to 1}\left[\frac{D'(z)N''(z)-N'(z)D''(z)}{2(D'(z))^2}\right]Q \\
L_q &= \left[\frac{D'(1)N''(1)-N'(1)D''(1)}{2(D'(1))^2}\right]Q \qquad (3.190)
\end{aligned}
$$

where primes and double primes in equation (3.190) denote first and second derivative at $z=1$ respectively. Carrying out the derivative at $z=1$, we have

$$
\begin{aligned}
N'(1) =& \lambda E(I)[p_1(1+r_1)E(B_1) + p_2(1+r_2)E(B_2) + \theta E(V)] \\
N''(1) =& \lambda^2(E(I))^2[p_1(1+r_1)E(B_1^2) + p_2(1+r_2)E(B_2^2) + \theta E(V^2)] \\
&+ \lambda E(I(I-1))[p_1(1+r_1)E(B_1) + p_2(1+r_2)E(B_2) + \theta E(V)]
\end{aligned}
$$

$$
\begin{aligned}
&+ 2\lambda^2(E(I))^2[p_1r_1(E(B_1))^2 + p_2r_2(E(B_2))^2] \\
&+ 2\theta\lambda^2(E(I))^2E(V)[p_1(1+r_1)E(B_1) + p_2(1+r_2)E(B_2)])
\end{aligned}
$$

$$
\begin{aligned}
D'(1) =& 1 - \lambda E(I)[p_1(1+r_1)E(B_1) + p_2(1+r_2)E(B_2) + \theta E(V)] \\
D''(1) =& - 2\lambda^2(E(I))^2\theta E(V)[p_1(1+r_1)E(B_1) + p_2(1+r_2)E(B_2)] \\
&- \lambda^2(E(I))^2[\theta E(V^2) + p_1(1+r_1)E(B_1^2) + p_2(1+r_2)E(B_2^2)] \\
&- \lambda E(I(I-1))[\theta E(V) + p_1(1+r_1)E(B_1) + p_2(1+r_2)E(B_2)] \\
&- 2\lambda^2(E(I))^2[p_1r_1(E(B_1))^2 + p_2r_2(E(B_2))^2]
\end{aligned}
$$

where $E(B_1^2)$, $E(B_2^2)$ and $E(V^2)$ are the second moment of type 1 service, type 2 service and vacation time respectively. $E(I(I-1))$ is the second factorial moment of the batch size of arriving customers. Further, we find the average system size L by using Little's formula. Thus we have

$$
L = L_q + \rho \tag{3.191}
$$

where L_q has been found by equation (3.190) and ρ is obtained from equation (3.189).

Let W_q and W denote the average waiting time in the queue and in the system respectively. Then by using Little's formula, we obtain,

$$
W_q = \frac{L_q}{\lambda}
$$

$$
W = \frac{L}{\lambda}
$$

where L_q and L have been found in equations (3.190) and (3.191).

3.14 Particular cases

Case 1: If server has no vacation i.e, θ=0. Then our model reduces to the $M^{[X]}/G/1$ queue with two types of service and optional re-service.

Using this in the main result of (3.188), (3.189) and (3.190), we can find the the idle probability Q, utilization factor ρ and the mean queue size L_q can be simplified to the following expressions.

$$\begin{aligned} Q =& 1 - \lambda E(I)[p_1(1+r_1)E(B_1) + p_2(1+r_2)E(B_2)] \\ \rho =& \lambda E(I)[p_1(1+r_1)E(B_1) + p_2(1+r_2)E(B_2)] \end{aligned}$$

$$L_q = \left[\frac{D'(1)N''(1) - N'(1)D''(1)}{2(D'(1))^2}\right] Q$$

where

$$\begin{aligned} N'(1) =& \lambda E(I)[p_1(1+r_1)E(B_1) + p_2(1+r_2)E(B_2)] \\ N''(1) =& \lambda^2(E(I))^2[p_1(1+r_1)E(B_1^2) + p_2(1+r_2)E(B_2^2)] \\ & + \lambda E(I(I-1))[p_1(1+r_1)E(B_1) + p_2(1+r_2)E(B_2)] \\ & + 2\lambda^2(E(I))^2[p_1r_1(E(B_1))^2 + p_2r_2(E(B_2))^2] \\ D'(1) =& 1 - \lambda E(I)[p_1(1+r_1)E(B_1) + p_2(1+r_2)E(B_2)] \\ D''(1) =& -\lambda^2(E(I))^2[p_1(1+r_1)E(B_1^2) + p_2(1+r_2)E(B_2^2)] \\ & - \lambda E(I(I-1))[p_1(1+r_1)E(B_1) + p_2(1+r_2)E(B_2)] \\ & - 2\lambda^2(E(I))^2[p_1r_1(E(B_1))^2 + p_2r_2(E(B_2))^2] \end{aligned}$$

The above result coincides with results given by Madan et al. (2004).

Case 2: If there is no second type of service i.e, $p_2 = 0$. Then our model reduces to $M^{[X]}/G/1$ queue with re-service and Bernoulli vacation.

Using this in the main result of (3.188), (3.189) and (3.190) we can find the idle probability Q, utilization factor ρ and the mean queue size L_q can be simplified to the following expressions.

$$Q = 1 - \lambda E(I)[(1+r_1)E(B_1) + \theta E(V)]$$

$$\rho = \lambda E(I)[(1+r_1)E(B_1)+\theta E(V)]$$

$$L_q = \left[\frac{D'(1)N''(1)-N'(1)D''(1)}{2(D'(1))^2}\right]Q$$

where

$$\begin{aligned}
N'(1) =& \lambda E(I)[(1+r_1)E(B_1)+\theta E(V)] \\
N''(1) =& \lambda^2(E(I))^2[(1+r_1)E(B_1^2)+\theta E(V^2)] \\
&+ \lambda E(I(I-1))[(1+r_1)E(B_1)+\theta E(V)] \\
&+ 2\lambda^2(E(I))^2 r_1(E(B_1))^2 \\
&+ 2\theta\lambda^2(E(I))^2E(V)(1+r_1)E(B_1) \\
D'(1) =& 1-\lambda E(I)[(1+r_1)E(B_1)+\theta E(V)] \\
D''(1) =& -2\lambda^2(E(I))^2\theta E(V)[(1+r_1)E(B_1)] \\
&- \lambda^2(E(I))^2[\theta E(V^2)+(1+r_1)E(B_1^2)] \\
&- \lambda E(I(I-1))[\theta E(V)+(1+r_1)E(B_1)] \\
&- 2\lambda^2(E(I))^2 r_1(E(B_1))^2
\end{aligned}$$

Case 3: If there is no second type of service, re-service, no first type re-service, no vacation and $C(z) = z$. i.e, $p_2 = 0$, r_1= 0 and $\theta = 0$, $E(I) = 1$ and $E(I(I-1))$=0. Then our model reduces $M/G/1$ queueing system.

Using this in the main result of (3.188), (3.189) and (3.190), we can find the idle probability Q, utilization factor ρ and the mean queue size L_q can be simplified to the following expressions.

$$Q = 1-\lambda E(B_1)$$

$$\rho = \lambda E(B_1)$$

$$L_q = \left[\frac{D'(1)N''(1)-N'(1)D''(1)}{2(D'(1))^2}\right]Q$$

where

$$N'(1) = \lambda E(B_1)$$
$$N''(1) = \lambda^2 E(B_1^2)$$
$$D'(1) = 1 - \lambda E(B_1)$$
$$D''(1) = - \lambda^2 E(B_1)^2)$$

we note that the above results coincide with the results given by Kashyap and Chaudhry (1988).

3.15 Numerical results

To numerically illustrate the results obtained in this work, we consider that the service times and vacation times are exponentially distributed with rates μ_1, μ_2 and γ.

In order to see the effect of various parameters on server's idle time Q, utilization factor ρ and various other queue characteristics such as L_q, L, W_q, W.

We base our numerical example on the result found in case 1. For this purpose in Table 3.3, we choose the following arbitrary values: $E(I)$= 0.3, $E(I(I-1))$= 0.04 , r_1= 0.4, r_2= 0.5, μ_1 =6, μ_2 =4 and p_1= 0.4, p_2= 0.6 while λ varies from 0.1 to 1.0 such that the stability condition is satisfied.

It clearly shows as long as increasing the arrival rate, the server's idle time decreases while the utilization factor, the average queue size, system size and average waiting time in the queue and system of our queueing model are all increases.

We base our numerical example on the result found in case 2. For this purpose in Table 3.4, we choose the following arbitrary values: r_1= 0.3, $E(I)$=0.3, $E(I(I-1))$= 0.04, μ_1 =4, θ= 0.6, $\lambda = 2$ while γ varies from 1 to 10 such that the stability condition is satisfied.

Table 3.3: Computed values of various queue characteristics

λ	Q	ρ	L_q	L	W_q	W
0.1	0.990450	0.009550	0.000729	0.010279	0.007291	0.102791
0.2	0.980900	0.019100	0.001647	0.020747	0.008234	0.103734
0.3	0.971350	0.028650	0.002759	0.031409	0.009195	0.104695
0.4	0.961800	0.038200	0.004070	0.042270	0.010175	0.105675
0.5	0.952250	0.047750	0.005588	0.053338	0.011175	0.106675
0.6	0.942700	0.057300	0.007317	0.064617	0.012195	0.107695
0.7	0.933150	0.066850	0.009266	0.076116	0.013237	0.108737
0.8	0.923600	0.076400	0.011439	0.087839	0.014299	0.109799
0.9	0.914050	0.085950	0.013846	0.099796	0.015384	0.110884
1.0	0.904500	0.095500	0.016492	0.111992	0.016492	0.111992

Table 3.4: Computed values of various queue characteristics

γ	Q	ρ	L_q	L	W_q	W
1	0.445000	0.555000	0.537798	1.092798	0.268899	0.546399
2	0.625000	0.375000	0.207760	0.582760	0.103880	0.291380
3	0.685000	0.315000	0.141372	0.450372	0.070686	0.228186
4	0.715000	0.285000	0.114969	0.399969	0.057484	0.199984
5	0.733000	0.267000	0.101191	0.368191	0.050596	0.184096
6	0.745000	0.255000	0.092846	0.347846	0.046423	0.173923
7	0.753570	0.246430	0.087290	0.333718	0.043645	0.166859
8	0.760000	0.240000	0.083342	0.323342	0.041671	0.161671
9	0.765000	0.235000	0.080401	0.315401	0.040200	0.157700
10	0.769000	0.231000	0.078129	0.309129	0.039064	0.154564

It clearly shows as long as increasing the vacation rate, the server's idle time increases while the utilization factor, average queue size, system size, average waiting time in the queue and system of our queueing model are all decreases.

CHAPTER FOUR

$M^{[X]}/G/1$ Feedback Queue with Server Vacation and Balking

CHAPTER

FOUR

$M^{[X]}/G/1$ FEEDBACK QUEUE WITH SERVER VACATION AND BALKING

4.1 Introduction

The $M^{[X]}/G/1$ queue has been studied by numerous authors including Gross and Harris (1985), Baba (1987), Madan and Al-Rawwash (2005), Badamchi Zadeh and Shankar (2008) and Deepak Gupta et al. (2011). Queueing systems with servers vacations have been studied extensively. A comprehensive review of vacation models, methods, results, examples and applications can be found in the survey of Doshi (1986).

In real life, many queueing situations arise in which there may be tendency of customers to be discouraged by a long queue. As a result, the customers either decide not join the queue (i.e. balk) or depart after joining the queue without getting served due to impatience (i.e. renege). The importance of this system appears in many real life problems such as the situations involving impatient telephone switchboard customers and the inventory systems that store perishable goods.

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Haight (1957) first presented the M/M/1 queue with balking. Madan (2012) analyzed the steady state batch arrival queueing system with balking and re-service in a vacation queue, having two types of heterogeneous services. Kumar and Sharma (2012) have studied Markovian queueing model with balking and reneging.

In this chapter, we consider $M^{[X]}/G/1$ feedback queue with server vacation and balking. Customers arrive to the service station in batches of variable size, but are served one by one where the arrival follows Poisson. An arriving batch may join with probability b or balks (refuses to join) the system with probability $(1-b)$ during the period of servers busy or vacation times. As soon as the completion of service, if the customer is dissatisfied with his service, he can immediately join the tail of the original queue with probability r or he leaves the system with probability $(1 - r)$ without re-joining the system. At each service completion epoch, the server may opt to take vacation with probability p or else with probability $(1 - p)$ stay in the system for the next service. The service and vacation periods are assumed to be general (arbitrary) distribution.

Here we derive time dependent probability generating functions in terms of Laplace transforms. We also derive the average system size and average waiting time. Some particular cases and numerical results are also discussed.

The rest of the chapter is organized as follows. The model description is given in section 4.2. Definitions and equations governing the model are given in section 4.3. The time dependent solution have been obtained in section 4.4 and corresponding steady state results have been derived explicitly in section 4.5. Average system size and average waiting time are computed in section 4.6. Some particular cases and numerical results are discussed in section 4.7 and 4.8 respectively.

4.2 Model description

We assume the following to describe the queueing model of our study.

a) Customers arrive at the system in batches of variable size in a compound Poisson process and they are provided one by one service on a 'first come - first served basis'. Let $\lambda c_i dt$ $(i = 1, 2, \ldots)$ be the first order probability that a batch of i customers arrives at the system during a short interval of time $(t, t + dt]$, where $0 \leq c_i \leq 1$, $\sum\limits_{i=1}^{\infty} c_i = 1$ and $\lambda > 0$ is the arrival rate of batches.

b) In real life, many queueing situations arise in which there may be tendency of customers to be discouraged by a long queue. As a result, the customers either decide not to join the queue (balks i.e. refuses to join) with probability $(1 - b)$ during the period of server's busy/vacation or may join the system with probability b.

c) As soon as the completion of service, if the customer is dissatisfied with his service, he can immediately join the tail of the original queue as a feedback customer for receiving the same service with probability r or he leaves the system with probability $(1 - r)$ without re-joining the system.

d) The service time follows a general (arbitrary) distribution with distribution function $B(s)$ and density function $b(s)$. Let $\mu(x)dx$ be the conditional probability density of service completion during the interval $(x, x + dx]$, given that the elapsed service time is x, so that

$$\mu(x) = \frac{b(x)}{1 - B(x)},$$

and therefore,

$$b(s) = \mu(s) e^{-\int\limits_0^s \mu(x)dx}.$$

e) As soon as each service of a customer is completed, then with probability p, the server may decide to take a vacation or with probability $(1-p)$ he may decide to continue to be available for the next service.

f) The server vacation time follows a general (arbitrary) distribution with distribution function $V(t)$ and density function $v(t)$. Let $\beta(x)dx$ be the conditional probability density of vacation completion during the interval $(x, x+dx]$, given that the elapsed vacation time is x, so that

$$\beta(x) = \frac{v(x)}{1-V(x)},$$

and therefore,

$$v(t) = \beta(t)e^{-\int\limits_0^t \beta(x)dx}.$$

g) Various stochastic processes involved in the system are assumed to be independent of each other.

4.3 Definitions and equations governing the system

We define

$P_n(x,t)$ = Probability that at time t, the server is active providing service and there are n $(n \geq 0)$ customers in the system and the elapsed service time is x. Consequently $P_n(t) = \int\limits_0^\infty P_n(x,t)dx$ denotes the probability that at time t there are n customers in the system irrespective of the value of x.

$V_n(x,t)$ = Probability that at time t, the server is under vacation with elapsed vacation time is x and there are n $(n \geq 0)$ customers in the system. Consequently $V_n(t)$= $\int\limits_0^\infty V_n(x,t)dx$ denotes the probability that at time t there are n customers in the queue and the server is under vacation irrespective of the value of x.

$Q(t)$ = Probability that at time t, there are no customers in the system and the server is idle but available in the system.

The model is then, governed by the following set of differential-difference equations:

$$\frac{\partial}{\partial x}P_n(x,t) + \frac{\partial}{\partial t}P_n(x,t) + [\lambda + \mu(x)]P_n(x,t) = \lambda(1-b)P_n(x,t) + \lambda b\sum_{k=1}^{n} c_k P_{n-k}(x,t), \quad n \geq 1 \tag{4.1}$$

$$\frac{\partial}{\partial x}V_0(x,t) + \frac{\partial}{\partial t}V_0(x,t) + [\lambda + \beta(x)]V_0(x,t) = \lambda(1-b)V_0(x,t) \tag{4.2}$$

$$\frac{\partial}{\partial x}V_n(x,t) + \frac{\partial}{\partial t}V_n(x,t) + [\lambda + \beta(x)]V_n(x,t) = \lambda(1-b)V_n(x,t), + \lambda b\sum_{k=1}^{n} c_k V_{n-k}(x,t), \quad n \geq 1 \tag{4.3}$$

$$\frac{d}{dt}Q(t) + \lambda Q(t) = \lambda(1-b)Q(t) + (1-r)(1-p)\int_0^\infty P_1(x,t)\mu(x)dx + \int_0^\infty V_0(x,t)\beta(x)dx \tag{4.4}$$

The above set of equations are to be solved subject to the following boundary conditions:

$$P_n(0,t) = \lambda b c_n Q(t) + (1-r)(1-p)\int_0^\infty P_{n+1}(x,t)\mu(x)dx + r(1-p)\int_0^\infty P_n(x,t)\mu(x)dx + \int_0^\infty V_n(x,t)\beta(x)dx, \quad n \geq 1 \tag{4.5}$$

$$V_0(0,t) = (1-r)p\int_0^\infty P_1(x,t)\mu(x)dx \tag{4.6}$$

$$V_n(0,t) = (1-r)p\int_0^\infty P_{n+1}(x,t)\mu(x)dx + rp\int_0^\infty P_n(x,t)\mu(x)dx, \quad n \geq 1 \tag{4.7}$$

We assume that initially there are no customers in the system and the server is idle. So the initial conditions are

$$P_n(0) = V_n(0) = 0, \quad n \geq 1 \text{ and } Q(0) = 1. \tag{4.8}$$

4.4 Generating functions of the system length: The time-dependent solution

In this section, we obtain the transient solution for the above set of differential-difference equations.

Theorem: *The system of differential-difference equations to describe an* $M^{[X]}/G/1$ *feedback queue with server vacation and balking are given by equations (4.1) to (4.7) with initial conditions (4.8) and the generating functions of transient solution are given by equation (4.33) and (4.34).*

Proof: We define the probability generating functions,

$$P(x,z,t) = \sum_{n=1}^{\infty} z^n P_n(x,t); \quad P(z,t) = \sum_{n=1}^{\infty} z^n P_n(t); \tag{4.9}$$

$$V(x,z,t) = \sum_{n=0}^{\infty} z^n V_n(x,t); \quad V(z,t) = \sum_{n=0}^{\infty} z^n V_n(t); \quad C(z) = \sum_{n=1}^{\infty} c_n z^n; \tag{4.10}$$

which are convergent inside the circle given by $|z| \leq 1$ and define the Laplace transform of a function $f(t)$ as

$$\bar{f}(s) = \int_0^{\infty} e^{-st} f(t)dt, \quad \Re(s) > 0$$

Taking the Laplace transform of equations (4.1) to (4.7) and using (4.8), we obtain

$$\frac{\partial}{\partial x}\bar{P}_n(x,s) + [s + \lambda + \mu(x)]\bar{P}_n(x,s) = \lambda(1-b)\bar{P}_n(x,s) + \lambda b \sum_{k=1}^{n} c_k \bar{P}_{n-k}(x,s), \quad n \geq 1 \tag{4.11}$$

$$\frac{\partial}{\partial x}\bar{V}_0(x,s) + [s+\lambda+\beta(x)]\bar{V}_0(x,s) = \lambda(1-b)\bar{V}_0(x,s) \tag{4.12}$$

$$\frac{\partial}{\partial x}\bar{V}_n(x,s) + [s+\lambda+\beta(x)]\bar{V}_n(x,s) = \lambda(1-b)\bar{V}_n(x,s), + \lambda b\sum_{k=1}^{n} c_k\bar{V}_{n-k}(x,s), \quad n\geq 1 \tag{4.13}$$

$$(s+\lambda b)\bar{Q}(s) = 1 + (1-r)(1-p)\int_0^\infty \bar{P}_1(x,s)\mu(x)dx + \int_0^\infty \bar{V}_0(x,s)\beta(x)dx \tag{4.14}$$

$$\bar{P}_n(0,s) = \lambda bc_n\bar{Q}(s) + (1-r)(1-p)\int_0^\infty \bar{P}_{n+1}(x,s)\mu(x)dx + r(1-p)\int_0^\infty \bar{P}_n(x,s)\mu(x)dx + \int_0^\infty \bar{V}_n(x,s)\beta(x)dx, \quad n\geq 1 \tag{4.15}$$

$$\bar{V}_0(0,s) = (1-r)p\int_0^\infty \bar{P}_1(x,s)\mu(x)dx \tag{4.16}$$

$$\bar{V}_n(0,s) = (1-r)p\int_0^\infty \bar{P}_{n+1}(x,s)\mu(x)dx + rp\int_0^\infty \bar{P}_n(x,s)\mu(x)dx, \quad n\geq 1 \tag{4.17}$$

Now multiplying equation (4.11) by z^n and summing over n from 1 to ∞, and using equation (4.9) and (4.10), we get

$$\frac{\partial}{\partial x}\bar{P}(x,z,s) + [s+\lambda b-\lambda bC(z)+\mu(x)]\bar{P}(x,z,s) = 0 \tag{4.18}$$

Now multiplying equation (4.13) by z^n and summing over n from 1 to ∞, adding to equation (4.12) and using equation (4.10), we get

$$\frac{\partial}{\partial x}\bar{V}(x,z,s) + [s+\lambda b-\lambda bC(z)+\beta(x)]\bar{V}(x,z,s) = 0 \tag{4.19}$$

For the boundary condition, we multiply both sides of equation (4.15) by z^n

summing over n from 1 to ∞ and use the equation (4.14), we get

$$\begin{aligned} z\bar{P}(0,z,s) =& z(1-s\bar{Q}(s)) + \lambda bz(C(z)-1)\bar{Q}(s) \\ &+ (1-r+rz)(1-p)\int_0^\infty \bar{P}(x,z,s)\mu(x)dx \\ &+ z\int_0^\infty \bar{V}(x,z,s)\beta(x)dx \end{aligned} \tag{4.20}$$

Performing similar operation on equations (4.16) and (4.17), we get

$$z\bar{V}(0,z,s) = (1-r+rz)p\int_0^\infty \bar{P}(x,z,s)\mu(x)dx \tag{4.21}$$

Integrating equations (4.18) and (4.19) between 0 and x, we get

$$\bar{P}(x,z,s) = \bar{P}(0,z,s)e^{-[s+\lambda b-\lambda bC(z)]x-\int_0^x \mu(t)dt} \tag{4.22}$$

$$\bar{V}(x,z,s) = \bar{V}(0,z,s)e^{-[s+\lambda b-\lambda bC(z)]x-\int_0^x \beta(t)dt} \tag{4.23}$$

Again integrating equations (4.22) and (4.23) by parts with respect to x, yields

$$\bar{P}(z,s) = \bar{P}(0,z,s)\left[\frac{1-\bar{B}(s+\lambda b-\lambda bC(z))}{s+\lambda b-\lambda bC(z)}\right] \tag{4.24}$$

$$\bar{V}(z,s) = \bar{V}(0,z,s)\left[\frac{1-\bar{V}(s+\lambda b-\lambda bC(z))}{s+\lambda b-\lambda bC(z)}\right] \tag{4.25}$$

where

$$\bar{B}(s+\lambda b-\lambda bC(z)) = \int_0^\infty e^{-[s+\lambda b-\lambda bC(z)]x}dB(x) \tag{4.26}$$

$$\bar{V}(s+\lambda b-\lambda bC(z)) = \int_0^\infty e^{-[s+\lambda b-\lambda bC(z)]x}dV(x) \tag{4.27}$$

are the Laplace-Stieltjes transform of the service time $B(x)$ and vacation time $V(x)$.

Now multiplying both sides of equations (4.22) and (4.23) by $\mu(x)$ and

$\beta(x)$ respectively and integrating over x, we obtain

$$\int_0^\infty \bar{P}(x,z,s)\mu(x)dx = \bar{P}(0,z,s)\bar{B}[s+\lambda b-\lambda bC(z)] \tag{4.28}$$

$$\int_0^\infty \bar{V}(x,z,s)\beta(x)dx = \bar{V}(0,z,s)\bar{V}[s+\lambda b-\lambda bC(z)] \tag{4.29}$$

Using equation (4.28) in (4.21), we get

$$z\bar{V}(0,z,s) = (1-r+rz)p\bar{B}(a)\bar{P}(0,z,s) \tag{4.30}$$

where $a = s+\lambda b-\lambda b(z)$.

Similarly using equations (4.28), (4.29) and (4.30) in (4.20), we get

$$\begin{aligned} z\bar{P}(0,z,s) =& z[1-s\bar{Q}(s)] + \lambda bz[C(z)-1]\bar{Q}(s) \\ &+ (1-r+rz)(1-p)\bar{B}(a)\bar{P}(0,z,s) \\ &+ (1-p)(1-r+rz)\bar{V}(a)\bar{B}(a)\bar{P}(0,z,s) \end{aligned}$$

$$\bar{P}(0,z,s) = \frac{z[1-s\bar{Q}(s)] + \lambda bz[C(z)-1]\bar{Q}(s)}{z-(1-p+p\bar{V}(a))(1-r+rz)\bar{B}(a)} \tag{4.31}$$

Using equation (4.31) in (4.30), we get

$$\bar{V}(0,z,s) = \frac{(1-r+rz)p\bar{B}(a)[(1-s\bar{Q}(s)) + \lambda b(C(z)-1)\bar{Q}(s)]}{z-(1-p+p\bar{V}(a))(1-r+rz)\bar{B}(a)} \tag{4.32}$$

Using equations (4.31) and (4.32) in (4.24) and (4.25), we get

$$\bar{P}(z,s) = \frac{[z(1-s\bar{Q}(s)) + \lambda bz(C(z)-1)\bar{Q}(s)]}{z-(1-p+p\bar{V}(a))(1-r+rz)\bar{B}(a)}\left[\frac{1-\bar{B}(a)}{a}\right] \tag{4.33}$$

$$\begin{aligned} \bar{V}(z,s) =& \frac{p(1-r+rz)\bar{B}(a)[(1-s\bar{Q}(s)) + \lambda b(C(z)-1)\bar{Q}(s)]}{z-(1-p+p\bar{V}(a))(1-r+rz)\bar{B}(a)} \\ &\times \left[\frac{1-\bar{V}(a)}{a}\right] \end{aligned} \tag{4.34}$$

Thus $\bar{P}(z,s)$ and $\bar{V}(z,s)$ are completely determined from equations (4.33) and (4.34) which completes the proof of the theorem.

4.5 The steady state results

In this section, we shall derive the steady state probability distribution for our queueing model. To define the steady state probabilities, we suppress the argument t wherever it appears in the time-dependent analysis. This can be obtained by applying the well-known Tauberian property

$$\lim_{s\to 0} s\bar{f}(s) = \lim_{t\to\infty} f(t) \tag{4.35}$$

In order to determine $\bar{P}(z,s)$ and $\bar{V}(z,s)$ completely, we have yet to determine the unknown Q which appears in the numerators of the right hand sides of equations (4.33) and (4.34). For that purpose, we shall use the normalizing condition

$$P(1) + V(1) + Q = 1 \tag{4.36}$$

The steady state probabilities for an $M^{[X]}/G/1$ feedback queue with server vacation and balking are given by

$$P(1) = \frac{\lambda b E(I) E(B) Q}{1 - r - \lambda b E(I) E(B) - \lambda b p E(I) E(V)} \tag{4.37}$$

$$V(1) = \frac{\lambda b p E(I) E(V) Q}{1 - r - \lambda b E(I) E(B) - \lambda b p E(I) E(V)} \tag{4.38}$$

$P(1)$, $V(1)$ and Q are the steady state probabilities that the server is providing service, server under vacation and server under idle respectively without regard to the number of customers in the system.

Multiplying both sides of equations (4.33) and (4.34) by s, taking limit as $s \to 0$, applying property (4.35) and simplifying, we obtain

$$P(z) = \frac{z[\bar{B}(f(z)) - 1]Q}{D(z)} \tag{4.39}$$

$$V(z) = \frac{p(1 - r + rz)\bar{B}(f(z))[\bar{V}(f(z)) - 1]Q}{D(z)} \tag{4.40}$$

where

$$D(z) = z - \bar{B}(f(z))(1 - r + rz)[1 - p + p\bar{V}(f(z))], \tag{4.41}$$

and $f(z) = \lambda b - \lambda b C(z)$.

Let $W(z)$ denote the probability generating function of the system size irrespective of the state of the system. Then adding equations (4.39) and (4.40), we obtain

$$W(z) = P(z) + V(z)$$

$$W(z) = \frac{[z(\bar{B}(f(z)) - 1) + p(1 - r + rz)\bar{B}(f(z))(\bar{V}(f(z)) - 1)]Q}{D(z)} \tag{4.42}$$

In order to find Q, we use the normalization condition $W(1) + Q = 1$. We see that for z=1, $W(1)$ is indeterminate of the form 0/0. Therefore, we apply L'Hopital's rule and on simplifying, we get

$$W(1) = \frac{\lambda b E(I)(E(B) + pE(V))Q}{1 - r - \lambda b E(I)(E(B) + pE(V))} \tag{4.43}$$

where C(1)= 1, $C'(1) = E(I)$ is mean batch size of the arriving customers, $E(B) = -\bar{B}'(0)$ and $E(V) = -\bar{V}'(0)$.

Therefore adding Q to equation (4.43), equating to 1 and simplifying, we get

$$Q = 1 - \rho \tag{4.44}$$

and hence the utilization factor ρ of the system is given by

$$\rho = \frac{\lambda b E(I)(E(B) + pE(V))}{1 - r}, \quad r \neq 1 \tag{4.45}$$

where $\rho < 1$ is the stability condition under which the steady state exists. Equation (4.44) gives the probability that the server is idle. By knowing Q from (4.44), we have completely and explicitly determined $W(z)$, the probability generating function of the system size.

4.6 Average system size and average waiting time

Let L denote the mean number of customers in the system. Then

$$L = \frac{d}{dz}W(z) \text{ at } z = 1$$

since this formula gives 0/0 form, then we write $W(z)$ given in (4.42) as $W(z) = \frac{N(z)}{D(z)}\mathrm{Q}$ where

$$N(z) = z(\bar{B}(f(z)) - 1) + p(1 - r + rz)\bar{B}(f(z))(\bar{V}(f(z)) - 1)$$

and $D(z)$ is given in equation (4.41).

$$\begin{aligned}
N'(z) =& (\bar{B}(f(z)) - 1) + z\bar{B}'(f(z))(f'(z)) + pr\bar{B}(f(z))(\bar{V}(f(z)) - 1) \\
& + p(1 - r + rz)\bar{B}'(f(z))(f'(z))(\bar{V}(f(z)) - 1) \\
& + p(1 - r + rz)\bar{B}(f(z))\bar{V}'(f(z))f'(z) \\
N''(z) =& 2\bar{B}'(f(z))f'(z) + z(\bar{B}''(f(z))(f'(z))^2 + (\bar{B}'(f(z))f''(z)) \\
& + 2pr\bar{B}'(f(z))f'(z)(\bar{V}(f(z)) - 1) \\
& + 2pr\bar{B}(f(z))(\bar{V}'(f(z))f'(z) \\
& + p(1 - r + rz)[\bar{B}''(f(z))(f'(z))^2 \\
& + \bar{B}'(f(z))f''(z)](\bar{V}(f(z)) - 1) \\
& + 2p(1 - r + rz)\bar{B}'(f(z))f'(z)\bar{V}'(f(z))f'(z) \\
& + p(1 - r + rz)\bar{B}(f(z))[\bar{V}''(f(z))(f'(z))^2 + \bar{V}'(f(z))(f''(z)] \\
D'(z) =& 1 - \bar{B}'(f(z)(f'(z))(1 - r + rz)(1 - p + p\bar{V}(f(z)) \\
& - \bar{B}(f(z))r(1 - p + p\bar{V}(f(z)) \\
& - \bar{B}(f(z))(1 - r + rz)p(\bar{V}'(f(z))(f'(z)) \\
D''(z) =& -[\bar{B}''(f(z))f'(z)^2 + \bar{B}'(f(z))f''(z)](1 - r + rz) \\
& \times (1 - p + p\bar{V}(f(z))) - 2\bar{B}'(f(z)f'(z)r(1 - p + p\bar{V}(f(z)))
\end{aligned}$$

$$
\begin{aligned}
&- 2\bar{B}'(f(z))f'(z)(1-r+rz)p\bar{V}'(f(z))f'(z) \\
&- 2\bar{B}(f(z))rp\bar{V}'(f(z))f'(z) \\
&- \bar{B}(f(z))(1-r+rz)p[\bar{V}''(f(z))(f'(z))^2 + \bar{V}'(f(z))f''(z)]
\end{aligned}
$$

Then, we use

$$L = \lim_{z\to 1} \frac{d}{dz}W(z) = \left[\frac{D'(1)N''(1) - N'(1)D''(1)}{2(D'(1))^2}\right] Q \tag{4.46}$$

where primes and double primes in equation(4.46) denote first and second derivative at $z = 1$ respectively. Carrying out the derivative at $z = 1$, we have

$$N'(1) = \lambda b E(I)[E(B) + pE(V)] \tag{4.47}$$

$$
\begin{aligned}
N''(1) =& \lambda^2 b^2 (E(I))^2 [E(B^2) + pE(V^2)] \\
&+ \lambda b E(I(I-1))[E(B) + pE(V)] \\
&+ 2\lambda b p E(I)E(V)(r + \lambda b E(I)E(B)) + 2\lambda b E(I)E(B)
\end{aligned} \tag{4.48}
$$

$$D'(1) = 1 - r - \lambda b E(I)E(B) - \lambda b p E(I)E(V) \tag{4.49}$$

$$
\begin{aligned}
D''(1) =& -[\lambda^2 b^2 (E(I))^2 [E(B^2) + pE(V^2)] \\
&+ \lambda b E(I(I-1))[E(B) + pE(V)] \\
&+ 2\lambda b r(E(I))[E(B) + pE(V)] \\
&+ 2\lambda^2 b^2 p(E(I))^2 E(B)E(V)]
\end{aligned} \tag{4.50}
$$

where $E(B^2)$, $E(V^2)$ are the second moment of the service time and vacation time. $E(I(I-1))$ is the second factorial moment of the batch size of arriving customers. Then if we substitute the values $N'(1), N''(1), D'(1), D''(1)$ from equations (4.47) to (4.50) into equation (4.46), we obtain L in the closed form.

Let W denote the average waiting time in the system. Then by using Little's formula, we obtain,

$$W = \frac{L}{\lambda} \tag{4.51}$$

where L have been found in equation (4.46).

4.7 Particular cases

Case 1: If there is no balking in the queueing system. i.e, b=1.

Then our model reduces to the $M^{[X]}/G/1$ queue with feedback and vacation. Using this in the main result of (4.44), (4.45) and (4.46), we can find the idle probability Q, utilization factor ρ and the average system size L can be simplified to the following expressions.

$$\begin{aligned} Q =& 1-\frac{\lambda E(I)(E(B)+pE(V))}{1-r} \\ \rho =& \frac{\lambda E(I)(E(B)+pE(V))}{1-r} \\ L =& \left[\frac{D'(1)N''(1)-N'(1)D''(1)}{2(D'(1))^2}\right]Q \end{aligned}$$

where

$$\begin{aligned} N'(1) =& \lambda E(I)[E(B)+pE(V)] \\ N''(1) =& \lambda^2(E(I))^2[E(B^2)+pE(V^2)] \\ &+\lambda E(I(I-1))[E(B)+pE(V)] \\ &+2\lambda pE(I)E(V)(r+\lambda E(I)E(B))+2\lambda E(I)E(B) \\ D'(1) =& 1-r-\lambda E(I)E(B)-\lambda pE(I)E(V) \\ D''(1) =& -[\lambda^2(E(I))^2[E(B^2)+pE(V^2)] \\ &+\lambda E(I(I-1))[E(B)+pE(V)] \\ &+2\lambda r(E(I))[E(B)+pE(V)] \\ &+2\lambda^2 p(E(I))^2E(B)E(V)] \end{aligned}$$

The above equations coincide with result given by Madan and Al-Rawwash (2005).

Case 2: If there is no feedback and no balking in the queueing system. i.e, r=0 and b=1.

Then our model reduces to the $M^{[X]}/G/1$ queue with vacation. Using this in the main result of (4.44), (4.45) and (4.46) we can find the idle probability Q, utilization factor ρ and the average system size L can be simplified to the following expressions.

$$Q = 1 - \lambda E(I)(E(B) + pE(V)) \tag{4.52}$$

$$\rho = \lambda E(I)(E(B) + pE(V)) \tag{4.53}$$

$$L = \left[\frac{D'(1)N''(1) - N'(1)D''(1)}{2(D'(1))^2}\right] Q \tag{4.54}$$

where

$$\begin{aligned}
N'(1) =& \lambda E(I)[E(B) + pE(V)] \\
N''(1) =& \lambda^2(E(I))^2[E(B^2) + pE(V^2)] \\
&+ \lambda E(I(I-1))[E(B) + pE(V)] \\
&+ 2p\lambda^2(E(I))^2E(V)E(B) + 2\lambda E(I)E(B) \\
D'(1) =& 1 - \lambda E(I)E(B) - \lambda pE(I)E(V) \\
D''(1) =& -[\lambda^2(E(I))^2[E(B^2) + pE(V^2)] \\
&+ \lambda E(I(I-1))[E(B) + pE(V)] \\
&+ 2\lambda^2 p(E(I))^2E(B)E(V)]
\end{aligned}$$

Case 3: If there is no feedback and no balking and no vacation. In this case, we put r=0, b=1, p=0.

Then our model reduces to the $M^{[X]}/G/1$ queue with no feedback and no balking and no vacation. Using this in the main result of (4.44), (4.45) and (4.46) we can find the idle probability Q, utilization factor ρ and the mean system size can be simplified to the following expressions.

$$Q = 1 - \lambda E(I)E(B) \tag{4.55}$$

$$\rho = \lambda E(I)E(B) \tag{4.56}$$

$$L = \left[\frac{D'(1)N''(1) - N'(1)D''(1)}{2(D'(1))^2} \right] Q \tag{4.57}$$

where

$$\begin{aligned}
N'(1) =& \lambda E(I)E(B) \\
N''(1) =& \lambda^2 (E(I))^2 E(B^2) + \lambda E(I(I-1))E(B) + 2\lambda E(I)E(B) \\
D'(1) =& 1 - \lambda E(I)E(B) \\
D''(1) =& -\lambda^2 (E(I))^2 E(B^2) + \lambda E(I(I-1))E(B)
\end{aligned}$$

We note that the above results agree with known resuts of the $M^{[X]}/G/1$ queue with no vacation, no feedback and no balking.

4.8 Numerical results

To numerically illustrate the results obtained in this work, we consider that the service time and vacation time are exponentially distributed with rates μ and γ.

We base our numerical example on the result found in case 1. For this purpose in Table 4.1, we choose the following arbitrary values: r=0.4, p=0.5, μ=6, $\gamma = 4$, $E(I)$=0.3, $E(I(I-1))$= 0.04, while λ varies from 0.1 to 1.0 such that the stability condition is satisfied.

It clearly shows as long as increasing the arrival rate, the server's idle time decreases while the utilization factor, the average system size and average waiting time of our queueing model are all increases.

In Table 4.2 we choose $E(I)$=0.2, $E(I(I-1))$= 0.03, $\lambda = 2$, r=0.3, p=0.6, μ=2, while γ varies from 1 to 10 such that the stability condition is satisfied.

It clearly shows as long as increasing the vacation rate, the server's idle time increases while the utilization factor, the average system size and average waiting time of our queueing model are all decreases.

Table 4.1: Computed values of various queue characteristics

λ	Q	ρ	L	W
0.1	0.985417	0.014583	0.012028	0.120278
0.2	0.970833	0.029167	0.024514	0.122568
0.3	0.956250	0.043750	0.037478	0.124927
0.4	0.941667	0.058333	0.050944	0.127360
0.5	0.927083	0.072917	0.064934	0.129869
0.6	0.912500	0.087500	0.079475	0.132458
0.7	0.897917	0.102083	0.094592	0.135131
0.8	0.883333	0.116667	0.110314	0.137893
0.9	0.868750	0.131250	0.126673	0.140747
1.0	0.854167	0.145833	0.143699	0.143699

Table 4.2: Computed values of various queue characteristics

γ	Q	ρ	L	W
1	0.371429	0.628571	1.542308	0.771154
2	0.542857	0.457143	0.747368	0.373684
3	0.600000	0.400000	0.608730	0.304365
4	0.628571	0.371429	0.553409	0.276705
5	0.645714	0.354286	0.523982	0.261991
6	0.657143	0.342857	0.505797	0.252899
7	0.665306	0.334694	0.493471	0.246735
8	0.671429	0.328571	0.484574	0.242287
9	0.676190	0.323810	0.477856	0.238928
10	0.680000	0.320000	0.472605	0.236303

CHAPTER FIVE

$M^{[X]}/G/1$ Queue with Service Interruption and Extended Server Vacation

CHAPTER

FIVE

$M^{[X]}/G/1$ QUEUE WITH SERVICE INTERRUPTION AND EXTENDED SERVER VACATION

5.1 Introduction

During the last three or four decades, queueing models with vacations had been the subject of interest to queueing theorists of deep study because of their applicability and theoretical structures in real life situations such as manufacturing and production systems, computer and communication systems, service and distribution systems, etc.

Queuing systems with server vacations have been studied by numerous researchers including Baba (1986), Shanthikumar (1988), Lee (1989), Choi and Park (1990), Madan (1991), Borthakur and Chaudhury (1997), Chaudhury (2000) and Choudhury and Tadj (2011).

In queueing theory, periods of temporary service unavailability are referred to as server vacations, server interruptions or server breakdowns. Queueing models with service interruptions have proved to be a useful abstraction in situations where a service facility is shared by multiple queues or where the facility is subject to failure. Queuing systems with service interruptions are considered by Avi-Itzhak and Naor (1963), Thiruvengadam (1963), Baskar et al. (2011), Balamani (2012) and Maragatha Sundari and Srinivasan (2012b).

In this chapter, we consider $M^{[X]}/G/1$ queueing system with service interruption and extended server vacation. We assume that the customers arrive to the service station in batches of variable size, but are served one by one. While serving the customer, we assume interruptions arrive at random and assumed to occur according to a Poisson process with mean rate α. Let β be the server rate of attending interruption are exponentially distributed. Also we assume, the customer whose service is interrupted goes back to the head of the queue where the arrivals are Poisson. The vacation period has three heterogeneous phases. After every service completion the server takes phase one vacation of random length with probability p or to continue to stay in the system with probability $1-p$. As soon as the completion of phase one vacation, the server may take phase two vacation with probability q or to join in the system with probability $1-q$, after phase two vacation again the server has the option to take phase three vacation with probability r or to join in the system with probability $1-r$. We assume that the service times and vacation times have a general (arbitrary) distribution.

Here we derive time dependent probability generating functions in terms of Laplace transforms. We also derive the average queue size, system size and average waiting time in the queue, the system. Some particular cases and numerical results are also discussed.

The rest of the chapter is organized as follows. Analysis of the model is given in section 5.2. Definitions and equations governing the system are given in section 5.3. The time dependent solution have been obtained in section 5.4 and corresponding steady state results have been derived explicitly in section 5.5. Average queue size, system size and average waiting time in the queue, the system are computed in section 5.6 and 5.7 respectively. Particular cases and numerical results are discussed in section 5.8 and 5.9 respectively.

5.2 Analysis of the model

We assume the following to describe the queueing model of our study.

a) Customers arrive at the system in batches of variable size in a compound Poisson process and they are provided one by one service on a first come - first served basis. Let $\lambda c_i dt$ $(i = 1, 2, \ldots)$ be the first order probability that a batch of i customers arrives at the system during a short interval of time $(t, t + dt]$, where $0 \leq c_i \leq 1$, $\sum_{i=1}^{\infty} c_i = 1$ and $\lambda > 0$ is the arrival rate of batches.

b) A single server provides service to all arriving customer, with the service time having general (arbitrary) distribution. Let $B(v)$ and $b(v)$ be the distribution and the density function of the service time respectively.

c) Let $\mu(x)dx$ be the conditional probability density of service completion during the interval $(x, x + dx]$, given that the elapsed service time is x, so that

$$\mu(x) = \frac{b(x)}{1 - B(x)},$$

and therefore,

$$b(s) = \mu(s)e^{-\int_0^s \mu(x)dx}$$

d) We assume interruptions arrive at random while serving the customers and assumed to occur according to a Poisson process with mean rate $\alpha > 0$. Let β be the server rate of attending interruption are exponentially distributed. Further we assume that once the interruption arrives, the customer whose service is interrupted comes back to the head of the queue.

e) After service completion, the server may take phase one vacation with probability p or continue to stay in the system with probability $1-p$. As soon as the completion of phase one vacation, the server may take phase two vacation with probability q or continue to join in the system with probability $1-q$, after phase two vacation, again the server has the option to take phase three vacation with probability r or to join in the system with probability $1-r$.

f) The server's vacation time follows a general (arbitrary) distribution with distribution function $V_i(t)$ and density function $v_i(t)$. Let $\gamma_i(x)dx$ be the conditional probability density of vacation completion during the interval $(x, x+dx]$, given that the elapsed vacation time is x, so that

$$\gamma_i(x) = \frac{v_i(x)}{1 - V_i(x)}, \quad i = 1, 2, 3$$

and therefore,

$$v_i(t) = \gamma_i(t) e^{-\int_0^t \gamma_i(x)dx}, \; i = 1, 2, 3.$$

g) Various stochastic processes involved in the system are assumed to be independent of each other.

5.3 Definitions and equations governing the system

We define

$P_n(x,t)$ = Probability that at time t, the server is active providing service and there are n ($n \geq 0$) customers in the queue excluding the one being served and the elapsed service time is x. Consequently $P_n(t) = \int_0^\infty P_n(x,t)dx$ denotes the probability that at time t there are n customers in the queue excluding one customer in the service irrespective of the value of x.

$V_n^{(1)}(x,t)$ = Probability that at time t, the server is under phase one vacation with elapsed vacation time is x and there are n $(n \geq 0)$ customers in the queue. Consequently $V_n^{(1)}(t)=\int\limits_0^\infty V_n^{(1)}(x,t)dx$ denotes the probability that at time t there are n customers in the queue and the server is under phase one vacation irrespective of the value of x.

$V_n^{(2)}(x,t)$ = Probability that at time t, the server is under phase two vacation with elapsed vacation time is x and there are n $(n \geq 0)$ customers in the queue. Consequently $V_n^{(2)}(t)=\int\limits_0^\infty V_n^{(2)}(x,t)dx$ denotes the probability that at time t there are n customers in the queue and the server is under phase two vacation irrespective of the value of x.

$V_n^{(3)}(x,t)$ = Probability that at time t, the server is under phase three vacation with elapsed vacation time is x and there are n $(n \geq 0)$ customers in the queue. Consequently $V_n^{(3)}(t)=\int\limits_0^\infty V_n^{(3)}(x,t)dx$ denotes the probability that at time t there are n customers in the queue and the server is under phase three vacation irrespective of the value of x.

$R_n(t)$= Probability that at time t, the server is inactive due to the arrival of interruption while there are n $(n \geq 0)$ customers in the queue.

$Q(t)$ = Probability that at time t, there are no customers in the system and the server is idle but available in the system.

According to the mathematical model mentioned above, the system has the following set of differential-difference equations:

$$\frac{\partial}{\partial x}P_0(x,t)+\frac{\partial}{\partial t}P_0(x,t)+[\lambda+\alpha+\mu(x)]P_0(x,t)=0 \tag{5.1}$$

$$\frac{\partial}{\partial x}P_n(x,t)+\frac{\partial}{\partial t}P_n(x,t)+[\lambda+\alpha+\mu(x)]P_n(x,t)=\lambda\sum_{k=1}^{n}c_kP_{n-k}(x,t),\quad n\geq 1 \tag{5.2}$$

$$\frac{\partial}{\partial x}V_0^{(1)}(x,t)+\frac{\partial}{\partial t}V_0^{(1)}(x,t)+[\lambda+\gamma_1(x)]V_0^{(1)}(x,t)=0 \tag{5.3}$$

$$\frac{\partial}{\partial x}V_n^{(1)}(x,t)+\frac{\partial}{\partial t}V_n^{(1)}(x,t)+[\lambda+\gamma_1(x)]V_n^{(1)}(x,t)=\lambda\sum_{k=1}^{n}c_kV_{n-k}^{(1)}(x,t),\quad n\geq 1 \tag{5.4}$$

$$\frac{\partial}{\partial x}V_0^{(2)}(x,t)+\frac{\partial}{\partial t}V_0^{(2)}(x,t)+[\lambda+\gamma_2(x)]V_0^{(2)}(x,t)=0 \tag{5.5}$$

$$\frac{\partial}{\partial x}V_n^{(2)}(x,t)+\frac{\partial}{\partial t}V_n^{(2)}(x,t)+[\lambda+\gamma_2(x)]V_n^{(2)}(x,t)=\lambda\sum_{k=1}^{n}c_kV_{n-k}^{(2)}(x,t),\quad n\geq 1 \tag{5.6}$$

$$\frac{\partial}{\partial x}V_0^{(3)}(x,t)+\frac{\partial}{\partial t}V_0^{(3)}(x,t)+[\lambda+\gamma_3(x)]V_0^{(3)}(x,t)=0 \tag{5.7}$$

$$\frac{\partial}{\partial x}V_n^{(3)}(x,t)+\frac{\partial}{\partial t}V_n^{(3)}(x,t)+[\lambda+\gamma_3(x)]V_n^{(3)}(x,t)=\lambda\sum_{k=1}^{n}c_kV_{n-k}^{(3)}(x,t),\quad n\geq 1 \tag{5.8}$$

$$\frac{d}{dt}R_0(t)=-(\lambda+\beta)R_0(t) \tag{5.9}$$

$$\frac{d}{dt}R_n(t)=-(\lambda+\beta)R_n(t)+\lambda\sum_{k=1}^{n}c_kR_{n-k}(t)+\alpha\int_0^\infty P_{n-1}(x,t)dx \tag{5.10}$$

$$\begin{aligned}\frac{d}{dt}Q(t)=&-\lambda Q(t)+\beta R_0(t)+(1-p)\int_0^\infty \mu(x)P_0(x,t)dx\\&+(1-q)\int_0^\infty \gamma_1(x)V_0^{(1)}(x,t)dx\\&+(1-r)\int_0^\infty \gamma_2(x)V_0^{(2)}(x,t)dx+\int_0^\infty \gamma_3(x)V_0^{(3)}(x,t)dx\end{aligned} \tag{5.11}$$

The above equations are to be solved subject to the following boundary conditions:

$$
\begin{aligned}
P_n(0,t) =&\lambda c_{n+1}Q(t) + (1-p)\int_0^\infty \mu(x)P_{n+1}(x,t)dx \\
&+ \beta R_{n+1}(t) + (1-q)\int_0^\infty \gamma_1(x)V_{n+1}^{(1)}(x,t)dx \\
&+ (1-r)\int_0^\infty \gamma_2(x)V_{n+1}^{(2)}(x,t)dx \\
&+ \int_0^\infty \gamma_3(x)V_{n+1}^{(3)}(x,t)dx \qquad (5.12)\\
V_n^{(1)}(0,t) =&p\int_0^\infty \mu(x)P_n(x,t)dx, \quad n \geq 0 \qquad (5.13)\\
V_n^{(2)}(0,t) =&q\int_0^\infty \gamma_1(x)V_n^{(1)}(x,t)dx, \quad n \geq 0 \qquad (5.14)\\
V_n^{(3)}(0,t) =&r\int_0^\infty \gamma_2(x)V_n^{(2)}(x,t)dx, \quad n \geq 0 \qquad (5.15)
\end{aligned}
$$

We assume that initially there are no customers in the system and the server is idle. So the initial conditions are

$$
\begin{aligned}
Q(0) = 1,\ V_n^{(i)}(0) =&0,\ R_n(0) = 0 \\
P_n(0) =&0 \ \text{ for } n \geq 0 \text{ and } \quad i = 1, 2, 3. \qquad (5.16)
\end{aligned}
$$

5.4 Generating functions of the queue length: The time-dependent solution

In this section, we obtain the transient solution for the above set of differential-difference equations.

Theorem: *The system of differential difference equations to describe an $M^{[X]}/G/1$ queue with service interruption and extended vacation are given by equations (5.1) to (5.15) with initial conditions (5.16) and the generating* *functions of transient solution are given by equation (5.63) to (5.67).*

Proof: We define the probability generating functions for i=1, 2, 3.

$$\begin{aligned} P(x,z,t) &= \sum_{n=0}^{\infty} z^n P_n(x,t); P(z,t) = \sum_{n=0}^{\infty} z^n P_n(t); \\ R(z,t) &= \sum_{n=0}^{\infty} z^n R_n(t); C(z) = \sum_{n=1}^{\infty} c_n z^n; \\ V^{(i)}(x,z,t) &= \sum_{n=0}^{\infty} z^n V_n^{(i)}(x,t); V^{(i)}(z,t) = \sum_{n=0}^{\infty} z^n V_n^{(i)}(t); \end{aligned} \tag{5.17}$$

which are convergent inside the circle given by $\mid z \mid \leq 1$ and define the Laplace transform of a function $f(t)$ as

$$\bar{f}(s) = \int_0^{\infty} e^{-st} f(t)dt, \quad \Re(s) > 0.$$

We take the Laplace transform of equations (5.1) to (5.15) and using (5.16), we obtain

$$\frac{\partial}{\partial x}\bar{P}_0(x,s) + (s+\lambda+\alpha+\mu(x))\bar{P}_0(x,s) = 0 \tag{5.18}$$

$$\frac{\partial}{\partial x}\bar{P}_n(x,s) + (s+\lambda+\alpha+\mu(x))\bar{P}_n(x,s) = \lambda\sum_{k=1}^{n} c_k \bar{P}_{n-k}(x,s), \; n \geq 1 \tag{5.19}$$

$$\frac{\partial}{\partial x}\bar{V}_0^{(1)}(x,s) + (s+\lambda+\gamma_1(x))\bar{V}_0^{(1)}(x,s) = 0 \tag{5.20}$$

$$\frac{\partial}{\partial x}\bar{V}_n^{(1)}(x,s) + (s+\lambda+\gamma_1(x))\bar{V}_n^{(1)}(x,s) = \lambda\sum_{k=1}^{n} c_k \bar{V}_{n-k}^{(1)}(x,s), \; n \geq 1 \tag{5.21}$$

$$\frac{\partial}{\partial x}\bar{V}_0^{(2)}(x,s) + (s+\lambda+\gamma_2(x))\bar{V}_0^{(2)}(x,s) = 0 \tag{5.22}$$

$$\frac{\partial}{\partial x}\bar{V}_n^{(2)}(x,s) + (s+\lambda+\gamma_2(x))\bar{V}_n^{(2)}(x,s) = \lambda\sum_{k=1}^{n} c_k \bar{V}_{n-k}^{(2)}(x,s), \; n \geq 1 \tag{5.23}$$

$$\frac{\partial}{\partial x}\bar{V}_0^{(3)}(x,s) + (s+\lambda+\gamma_3(x))\bar{V}_0^{(3)}(x,s) = 0 \tag{5.24}$$

$$\frac{\partial}{\partial x}\bar{V}_n^{(3)}(x,s) + (s+\lambda+\gamma_3(x))\bar{V}_n^{(3)}(x,s) = \lambda\sum_{k=1}^{n} c_k \bar{V}_{n-k}^{(3)}(x,s), \; n \geq 1 \tag{5.25}$$

$$(s+\lambda+\beta)\bar{R}_0(s)=0 \tag{5.26}$$

$$(s+\lambda+\beta)\bar{R}_n(s)=\lambda\sum_{k=1}^{n}c_k\bar{R}_{n-k}(s)+\alpha\int_0^\infty \bar{P}_{n-1}(x,s)dx,\ n\geq 1 \tag{5.27}$$

$$\begin{aligned}(s+\lambda)\bar{Q}(s)=1+\beta\bar{R}_0(s)+(1-p)\int_0^\infty \mu(x)\bar{P}_0(x,s)dx\\ +(1-q)\int_0^\infty \gamma_1(x)\bar{V}_0^{(1)}(x,s)dx\\ +(1-r)\int_0^\infty \gamma_2(x)\bar{V}_0^{(2)}(x,s)dx\\ +\int_0^\infty \gamma_3(x)\bar{V}_0^{(3)}(x,s)dx\end{aligned} \tag{5.28}$$

$$\begin{aligned}\bar{P}_n(0,s)=\lambda c_{n+1}\bar{Q}(s)+(1-p)\int_0^\infty \mu(x)\bar{P}_{n+1}(x,s)dx\\ +\beta\bar{R}_{n+1}(s)+(1-q)\int_0^\infty \gamma_1(x)\bar{V}_{n+1}^{(1)}(x,s)dx\\ +(1-r)\int_0^\infty \gamma_2(x)\bar{V}_{n+1}^{(2)}(x,s)dx\\ +\int_0^\infty \gamma_3(x)\bar{V}_{n+1}^{(3)}(x,s)dx\end{aligned} \tag{5.29}$$

$$\bar{V}_n^{(1)}(0,s)=p\int_0^\infty \bar{P}_n(x,s)\mu(x)dx,\quad n\geq 0 \tag{5.30}$$

$$\bar{V}_n^{(2)}(0,s)=q\int_0^\infty \bar{V}_n^{(1)}(x,s)\gamma_1(x)dx,\quad n\geq 0 \tag{5.31}$$

$$\bar{V}_n^{(3)}(0,s)=r\int_0^\infty \bar{V}_n^{(2)}(x,s)\gamma_2(x)dx,\quad n\geq 0. \tag{5.32}$$

Now multiplying equations (5.19), (5.21), (5.23), (5.25), (5.27) by z^n and summing over n from 1 to ∞, adding to equations (5.18), (5.20), (5.22), (5.24), (5.26) and using the generating functions defined in equations (5.17), we get

$$\frac{\partial}{\partial x}\bar{P}(x,z,s)+[s+\lambda-\lambda C(z)+\alpha+\mu(x)]\bar{P}(x,z,s)=0 \tag{5.33}$$

$$\frac{\partial}{\partial x}\bar{V}^{(1)}(x,z,s)+[s+\lambda-\lambda C(z)+\gamma_1(x)]\bar{V}^{(1)}(x,z,s)=0 \tag{5.34}$$

$$\frac{\partial}{\partial x}\bar{V}^{(2)}(x,z,s)+[s+\lambda-\lambda C(z)+\gamma_2(x)]\bar{V}^{(2)}(x,z,s)=0 \tag{5.35}$$

$$\frac{\partial}{\partial x}\bar{V}^{(3)}(x,z,s)+[s+\lambda-\lambda C(z)+\gamma_3(x)]\bar{V}^{(3)}(x,z,s)=0 \tag{5.36}$$

$$(s+\lambda-\lambda C(z)+\beta)\bar{R}(z,s)=\alpha z\int_0^\infty \bar{P}(x,z,s)dx \tag{5.37}$$

For the boundary conditions, we multiply both sides of equation (5.29) by z^n summing over n from 0 to ∞, and use the equation (5.17), we get

$$\begin{aligned}
z\bar{P}(0,z,s) = {} & \lambda C(z)\bar{Q}(s)+\beta\bar{R}(z,s)-\beta\bar{R}_0(s)\\
& +(1-p)\int_0^\infty \mu(x)\bar{P}(x,z,s)dx\\
& -(1-p)\int_0^\infty \mu(x)\bar{P}_0(x,s)dx\\
& +(1-q)\int_0^\infty \gamma_1(x)\bar{V}^{(1)}(x,z,s)dx\\
& -(1-q)\int_0^\infty \gamma_1(x)\bar{V}_0^{(1)}(x,s)dx\\
& +(1-r)\int_0^\infty \gamma_2(x)\bar{V}^{(2)}(x,z,s)dx\\
& -(1-r)\int_0^\infty \gamma_2(x)\bar{V}_0^{(2)}(x,s)dx\\
& +\int_0^\infty \gamma_3(x)\bar{V}_0^{(3)}(x,z,s)dx-\int_0^\infty \gamma_3(x)\bar{V}_0(x,s)dx
\end{aligned}$$

Using equation (5.28), the above equation becomes

$$\begin{aligned}
z\bar{P}(0,z,s) = {} & [1-s\bar{Q}(s)]+\lambda(C(z)-1)\bar{Q}(s)+\beta\bar{R}(z,s)\\
& +(1-p)\int_0^\infty \mu(x)\bar{P}(x,z,s)dx\\
& +(1-q)\int_0^\infty \gamma_1(x)\bar{V}^{(1)}(x,z,s)dx\\
& +(1-r)\int_0^\infty \gamma_2(x)\bar{V}^{(2)}(x,z,s)dx\\
& +\int_0^\infty \gamma_3(x)\bar{V}^{(3)}(x,z,s)dx
\end{aligned} \tag{5.38}$$

Performing similar operation on equations (5.30) to (5.32), we get

$$\bar{V}^{(1)}(0,z,s) = p\int_0^\infty \mu(x)\bar{P}(x,z,s)dx \tag{5.39}$$

$$\bar{V}^{(2)}(0,z,s) = q\int_0^\infty \gamma_1(x)\bar{V}^{(1)}(x,z,s)dx \tag{5.40}$$

$$\bar{V}^{(3)}(0,z,s) = r\int_0^\infty \gamma_2(x)\bar{V}^{(2)}(x,z,s)dx \tag{5.41}$$

Integrating equation (5.33) between 0 and x, we get

$$\bar{P}(x,z,s) = \bar{P}(0,z,s)e^{-[s+\lambda-\lambda C(z)+\alpha]x-\int\limits_0^x \mu(t)dt} \tag{5.42}$$

where $\bar{P}(0,z,s)$ is given by equation (5.38).

Again integrating equation (5.42) by parts with respect to x, yields

$$\bar{P}(z,s) = \bar{P}(0,z,s)\left[\frac{1-\bar{B}(s+\lambda-\lambda C(z)+\alpha)}{s+\lambda-\lambda C(z)+\alpha}\right] \tag{5.43}$$

where

$$\bar{B}(s+\lambda-\lambda C(z)+\alpha) = \int_0^\infty e^{-[s+\lambda-\lambda C(z)+\alpha]x}dB(x)$$

is the Laplace-Stieltjes transform of the service time $B(x)$.

Now multiplying both sides of equation (5.42) by $\mu(x)$ and integrating over x, we obtain

$$\int_0^\infty \bar{P}(x,z,s)\mu(x)dx = \bar{P}(0,z,s)\bar{B}[s+\lambda-\lambda C(z)+\alpha] \tag{5.44}$$

Similarly, on integrating equations (5.34) to (5.36) from 0 to x, we get

$$\bar{V}^{(1)}(x,z,s) = \bar{V}^{(1)}(0,z,s)e^{-[s+\lambda-\lambda C(z)]x-\int\limits_0^x \gamma_1(t)dt} \tag{5.45}$$

$$\bar{V}^{(2)}(x,z,s) = \bar{V}^{(2)}(0,z,s)e^{-[s+\lambda-\lambda C(z)]x-\int\limits_0^x \gamma_2(t)dt} \tag{5.46}$$

$$\bar{V}^{(3)}(x,z,s) = \bar{V}^{(3)}(0,z,s)e^{-[s+\lambda-\lambda C(z)]x-\int\limits_0^x \gamma_3(t)dt} \tag{5.47}$$

where $\bar{V}^{(1)}(0,z,s)$, $\bar{V}^{(2)}(0,z,s)$, and $\bar{V}^{(3)}(0,z,s)$ are given by equations (5.39) to (5.41).

Again integrating equations (5.45) to (5.47) by parts with respect to x, yields

$$\bar{V}^{(1)}(z,s) = \bar{V}^{(1)}(0,z,s)\left[\frac{1-\bar{V}_1(s+\lambda-\lambda C(z))}{s+\lambda-\lambda C(z)}\right] \tag{5.48}$$

$$\bar{V}^{(2)}(z,s) = \bar{V}^{(2)}(0,z,s)\left[\frac{1-\bar{V}_2(s+\lambda-\lambda C(z))}{s+\lambda-\lambda C(z)}\right] \tag{5.49}$$

$$\bar{V}^{(3)}(z,s) = \bar{V}^{(3)}(0,z,s)\left[\frac{1-\bar{V}_3(s+\lambda-\lambda C(z))}{s+\lambda-\lambda C(z)}\right] \tag{5.50}$$

where

$$\bar{V}_1(s+\lambda-\lambda C(z)) = \int_0^\infty e^{-[s+\lambda-\lambda C(z)]x} dV_1(x)$$

$$\bar{V}_2(s+\lambda-\lambda C(z)) = \int_0^\infty e^{-[s+\lambda-\lambda C(z)]x} dV_2(x)$$

$$\bar{V}_3(s+\lambda-\lambda C(z)) = \int_0^\infty e^{-[s+\lambda-\lambda C(z)]x} dV_3(x)$$

are the Laplace-Stieltjes transform of phase one, phase two and phase three vacation times $V_1(x)$, $V_2(x)$ and $V_3(x)$ respectively.

Now multiplying both sides of equations (5.45), (5.46), (5.47) by $\gamma_1(x)$, $\gamma_2(x)$ and $\gamma_3(x)$ and integrating over x, we obtain

$$\int_0^\infty \bar{V}^{(1)}(x,z,s)\gamma_1(x)dx = \bar{V}^{(1)}(0,z,s)\bar{V}_1[s+\lambda-\lambda C(z)] \tag{5.51}$$

$$\int_0^\infty \bar{V}^{(2)}(x,z,s)\gamma_2(x)dx = \bar{V}^{(2)}(0,z,s)\bar{V}_2[s+\lambda-\lambda C(z)] \tag{5.52}$$

$$\int_0^\infty \bar{V}^{(3)}(x,z,s)\gamma_3(x)dx = \bar{V}^{(3)}(0,z,s)\bar{V}_3[s+\lambda-\lambda C(z)] \tag{5.53}$$

Using equation (5.44) in equation (5.39), we get

$$\bar{V}^{(1)}(0,z,s) = p\bar{B}(a)\bar{P}(0,z,s) \tag{5.54}$$

Now using equations (5.51) and (5.54) in (5.40), we get

$$\bar{V}^{(2)}(0,z,s) = \ pq\bar{V}_1(a_1)\bar{B}(a)\bar{P}(0,z,s) \tag{5.55}$$

where $a = s+\lambda-\lambda C(z)+\alpha$ and $a_1 = s+\lambda-\lambda C(z)$.

By using equations (5.52) and (5.55) in (5.41), we get

$$\bar{V}^{(3)}(0,z,s) = \ pqr\bar{V}_1(a_1)\bar{V}_2(a_1)\bar{B}(a)\bar{P}(0,z,s) \tag{5.56}$$

Using equations (5.44), (5.51) to (5.56) in (5.38), we get

$$[z-\bar{B}(a)(1-p+p\bar{V}_1(a_1)(1-q+q\bar{V}_2(a_1)(1-r+r\bar{V}_3(a_1))))]\bar{P}(0,z,s)$$

$$= [1-s\bar{Q}(s)] + \lambda(C(z)-1)\bar{Q}(s) + \beta\bar{R}(z,s) \tag{5.57}$$

From (5.37), we get

$$\bar{R}(z,s) = \frac{\alpha z}{a_2}\bar{P}(0,z,s)\left[\frac{1-\bar{B}(a)}{a}\right] \tag{5.58}$$

Now using equation (5.58) in (5.57), we have

$$\bar{P}(0,z,s) = \frac{a_2a[(1-s\bar{Q}(s)) + \lambda(C(z)-1)\bar{Q}(s)]}{aa_2[z-\bar{B}(a)(1-p+p\bar{V}_1(a)a_3)] - \alpha z\beta(1-\bar{B}(a))} \tag{5.59}$$

where $a_2 = s+\lambda-\lambda C(z)+\beta$, $a_3 = \ 1-q+q\bar{V}_2(a_1)a_4$, $a_4 = \ 1-r+r\bar{V}_3(a_1)$.

Using equation (5.59), in equations (5.54), (5.55) and (5.56), we get

$$\bar{V}^{(1)}(0,z,s) = \frac{p\bar{B}(a)a_2a[(1-s\bar{Q}(s)) + \lambda(C(z)-1)\bar{Q}(s)]}{aa_2[z-\bar{B}(a)(1-p+p\bar{V}_1(a)a_3)] - \alpha z\beta(1-\bar{B}(a))} \tag{5.60}$$

$$\bar{V}^{(2)}(0,z,s) = \frac{pq\bar{V}_1(a_1)\bar{B}(a)a_2a}{aa_2[z-\bar{B}(a)(1-p+p\bar{V}_1(a)a_3)] - \alpha z\beta(1-\bar{B}(a))} \times [(1-s\bar{Q}(s)) + \lambda(C(z)-1)\bar{Q}(s)] \tag{5.61}$$

$$\bar{V}^{(3)}(0,z,s) = \frac{pqr\bar{V}_1(a_1)\bar{V}_2(a_1)\bar{B}(a)a_2a}{aa_2[z-\bar{B}(a)(1-p+p\bar{V}_1(a)a_3)] - \alpha z\beta(1-\bar{B}(a))} \times [(1-s\bar{Q}(s)) + \lambda(C(z)-1)\bar{Q}(s)] \tag{5.62}$$

Using equations (5.59) to (5.62) in equations (5.43), (5.48), (5.49), (5.50) and (5.58), we get

$$\bar{P}(z,s) = \frac{a_2(1-\bar{B}(a))[(1-s\bar{Q}(s)) + \lambda(C(z)-1)\bar{Q}(s)]}{aa_2[z-\bar{B}(a)(1-p+p\bar{V}_1(a)a_3)] - \alpha z\beta(1-\bar{B}(a))} \tag{5.63}$$

$$\bar{V}^{(1)}(z,s) = \frac{p\bar{B}(a)aa_2[1-s\bar{Q}(s) + \lambda(C(z)-1)\bar{Q}(s)]}{aa_2[z-\bar{B}(a)(1-p+p\bar{V}_1(a)a_3)] - \alpha z\beta(1-\bar{B}(a))} \times \left[\frac{1-\bar{V}_1(a_1)}{a_1}\right] \tag{5.64}$$

$$\bar{V}^{(2)}(z,s) = \frac{pq\bar{B}(a)aa_2\bar{V}_1(a_1)[1-s\bar{Q}(s) + \lambda(C(z)-1)\bar{Q}(s)]}{aa_2[z-\bar{B}(a)(1-p+p\bar{V}_1(a)a_3)] - \alpha z\beta(1-\bar{B}(a))} \times \left[\frac{1-\bar{V}_2(a_1)}{a_1}\right] \tag{5.65}$$

$$\bar{V}^{(3)}(z,s) = \frac{pqr\bar{B}(a)aa_2\bar{V}_1(a_1)\bar{V}_2(a_1)[(1-s\bar{Q}(s)) + \lambda(C(z)-1)\bar{Q}(s)]}{aa_2[z-\bar{B}(a)(1-p+p\bar{V}_1(a)a_3)] - \alpha z\beta(1-\bar{B}(a))} \times \left[\frac{1-\bar{V}_3(a_1)}{a_1}\right] \tag{5.66}$$

$$\bar{R}(z,s) = \frac{\alpha z(1-\bar{B}(a))[1-s\bar{Q}(s) + \lambda(C(z)-1)\bar{Q}(s)]}{aa_2[z-\bar{B}(a)(1-p+p\bar{V}_1(a)a_3)] - \alpha z\beta(1-\bar{B}(a))} \tag{5.67}$$

Thus $\bar{P}(z,s)$, $\bar{V}^{(1)}(z,s)$, $\bar{V}^{(2)}(z,s), \bar{V}^{(3)}(z,s)$ and $\bar{R}(z,s)$ are completely determined from equations (5.63) to (5.67) which completes the proof of the theorem.

5.5 The steady state results

In this section, we shall derive the steady state probability distribution for our queueing model. These probabilities are obtained by suppress the argument t wherever it appears in the time-dependent analysis. This can be obtained by applying the Tauberian property

$$\lim_{s\to 0} s\bar{f}(s) = \lim_{t\to\infty} f(t)$$

In order to determine $\bar{P}(z,s)$, $\bar{V}^{(1)}(z,s)$, $\bar{V}^{(2)}(z,s)$, $\bar{V}^{(3)}(z,s)$ and $\bar{R}(z,s)$ completely, we have yet to determine the unknown Q which appears in the numerators of the right hand sides of equations (5.63) to (5.67). For that purpose, we shall use the normalizing condition

$$P(1) + V^{(1)}(1) + V^{(2)}(1) + V^{(3)}(1) + R(1) + Q = 1$$

The steady state probabilities for an $M^{[X]}/G/1$ queue with service interruption and extended server vacation are given by

$$\begin{aligned}
P(1) =& \frac{\lambda E(I)\beta[1-\bar{B}(\alpha)]Q}{Dr}\\
V^{(1)}(1) =& \frac{\lambda p\alpha\beta E(I)\bar{B}(\alpha)E(V_1)Q}{Dr}\\
V^{(2)}(1) =& \frac{\lambda pq\alpha\beta E(I)\bar{B}(\alpha)E(V_2)Q}{Dr}\\
V^{(3)}(1) =& \frac{\lambda pqr\alpha\beta E(I)\bar{B}(\alpha)E(V_3)Q}{Dr}\\
R(1) =& \frac{\lambda\alpha E(I)[1-\bar{B}(\alpha)]Q}{Dr}
\end{aligned}$$

where

$$\begin{aligned}
Dr =& -\lambda E(I)(\alpha+\beta)[1-\bar{B}(\alpha)] + \alpha\beta\bar{B}(\alpha)\\
& \times [1 - \lambda pE(I)(E(V_1) + q(E(V_2) + rE(V_3)))] \qquad (5.68)
\end{aligned}$$

$P(1),\ V^{(1)}(1),\ V^{(2)}(1), V^{(3)}(1),\ R(1)$ and Q are the steady state probabilities that the server is providing service, server under phase one vacation, phase two vacation, phase three vacation, server under interruption and server under idle respectively without regard to the number of customers in the queue.

Multiplying both sides of equations (5.63) to (5.67) by s, taking limit as $s \to 0$, applying Tauberian property and simplifying, we obtain

$$P(z) = \frac{f_1(z)(1-\bar{B})\lambda(C(z)-1)Q}{D(z)} \tag{5.69}$$

$$V^{(1)}(z) = \frac{pf_1(z)f_2(z)\bar{B}[\bar{V}_1 - 1]Q}{D(z)} \tag{5.70}$$

$$V^{(2)}(z) = \frac{pqf_1(z)f_2(z)\bar{V}_1\bar{B}[\bar{V}_2 - 1]Q}{D(z)} \tag{5.71}$$

$$V^{(3)}(z) = \frac{pqrf_1(z)f_2(z)\bar{V}_1\bar{V}_2\bar{B}[\bar{V}_3 - 1]Q}{D(z)} \tag{5.72}$$

$$R(z) = \frac{\lambda\alpha z(1-\bar{B})(C(z)-1)Q}{D(z)} \tag{5.73}$$

where

$$D(z) = f_1(z)f_2(z)[z - \bar{B}(1 - p + p\bar{V}_1 f_3(z))] - \alpha z\beta(1-\bar{B}), \tag{5.74}$$

$f_1(z) = \lambda - \lambda C(z) + \beta,\ f_2(z) = \lambda - \lambda C(z) + \alpha,\ f_3(z) = 1 - q + q\bar{V}_2 f_4(z)$ $f_4(z) = 1 - r + r\bar{V}_3$, $\bar{B} = \bar{B}(f_2(z))$, $\bar{V}_1 = \bar{V}_1(\lambda - \lambda C(z))$, $\bar{V}_2 = \bar{V}_2(\lambda - \lambda C(z))$ and $\bar{V}_3 = \bar{V}_3(\lambda - \lambda C(z))$.

Let $W_q(z)$ denote the probability generating function of the queue size irrespective of the state of the system. Then adding equations (5.69) to (5.73), we obtain

$$W_q(z) = P(z) + V^{(1)}(z) + V^{(2)}(z) + V^{(3)}(z)$$

$$
\begin{aligned}
W_q(z) =& \frac{f_1(z)(1-\bar{B})\lambda(C(z)-1)Q}{D(z)} \\
&+\frac{pf_1(z)f_2(z)\bar{B}[\bar{V}_1-1]Q}{D(z)} \\
&+\frac{pqf_1(z)f_2(z)\bar{V}_1\bar{B}[\bar{V}_2-1]Q}{D(z)} \\
&+\frac{pqrf_1(z)f_2(z)\bar{V}_1\bar{V}_2\bar{B}[\bar{V}_3-1]Q}{D(z)} \\
&+\frac{\lambda\alpha z(1-\bar{B})(C(z)-1)Q}{D(z)}
\end{aligned} \tag{5.75}
$$

we see that for z=1, $W_q(1)$ is indeterminate of the form 0/0. Therefore, we apply L'Hopital's rule and on simplifying, we obtain

$$
W_q(1) = \frac{\lambda E(I)[(\alpha+\beta)(1-\bar{B}(\alpha)) + p\alpha\beta\bar{B}(\alpha)(E(V_1)+q(E(V_2)+rE(V_3)))]Q}{Dr} \tag{5.76}
$$

where Dr is given by equation (5.68). C(1)= 1, $C'(1) = E(I)$ is mean batch size of the arriving customers, $-\bar{B}'(0) = E(B), -\bar{V}_i'(0) = E(V_i)$, $i = 1, 2, 3$.

Therefore adding Q to above equation and equating to 1 and simplifying, we get

$$
Q = 1-\rho \tag{5.77}
$$

and hence the utilization factor ρ of the system is given by

$$
\rho = \lambda pE(I)[E(V_1)+q(E(V_2)+rE(V_3))] + \frac{\lambda E(I)}{\bar{B}(\alpha)}(\frac{1}{\beta}+\frac{1}{\alpha})[1-\bar{B}(\alpha)] \tag{5.78}
$$

where $\rho < 1$ is the stability condition under which the steady state exists. Equation (5.77) gives the probability that the server is idle. Substituting Q from (5.77) into (5.75), we have completely and explicitly determined $W_q(z)$, the probability generating function of the queue size.

5.6 The average queue size and average system size

Let L_q denote the average number of customers in the queue under the steady state. Then

$$L_q = \frac{d}{dz} W_q(z) \quad at\ z = 1$$

since this formula gives 0/0 form, then we write $W_q(z)$ given in (5.75) as $W_q(z) = \frac{N(z)}{D(z)}$Q where

$$\begin{aligned} N(z) =& \lambda(C(z)-1)(1-\bar{B})(f_1(z)+\alpha z) + p f_1(z) f_2(z)\bar{B} \\ & \times [\bar{V}_1 - 1 + q\bar{V}_1(\bar{V}_2(1-r+r\bar{V}_3)-1)] \end{aligned}$$

$D(z)$ is given by the equation (5.74).

$$\begin{aligned} N'(z) =& (1-\bar{B})(f_1(z)+\alpha z)\lambda C'(z) \\ & + \lambda(C(z)-1)(f_1(z)+\alpha z)\bar{B}'(\alpha)\lambda C'(z) \\ & + \lambda(C(z)-1)(1-\bar{B})(-\lambda C'(z)+\alpha) \\ & + p(f_1'(z)f_2(z)\bar{B} + f_1(z)f_2'(z)\bar{B} + f_1(z)f_2(z)\bar{B}'(-\lambda C'(z))) \\ & \times [\bar{V}_1 - 1 + q\bar{V}_1(\bar{V}_2(1-r+r\bar{V}_3)-1)] \\ & + p f_1(z) f_2(z)\bar{B}[\bar{V}_1'(-\lambda C'(z)) + q\bar{V}_1'(-\lambda C'(z))(\bar{V}_2(1-r+r\bar{V}_3)-1) \\ & + q\bar{V}_1(\bar{V}_2'(-\lambda C'(z))(1-r+r\bar{V}_3) + \bar{V}_2 r\bar{V}_3'(-\lambda C'(z)))] \\ N''(z) =& (1-\bar{B})(f_1(z)+\alpha z)\lambda C''(z) + 2\lambda C'(z)\bar{B}'(\alpha)\lambda C'(z)(f_1(z)+\alpha z) \\ & + 2\lambda C'(z)(1-\bar{B})(-\lambda C'(z)+\alpha) + \lambda(C(z)-1)(f_1'(z)+\alpha z) \\ & \times [\bar{B}''(-\lambda C'(z))^2 + \lambda C''(z)\bar{B}'] + 2\lambda^2(C(z)-1)C'(z)\bar{B}'(f_1'(z)+\alpha) \\ & + \lambda(C(z)-1)(1-\bar{B})f_1''(z) \\ & + 2p(f_1'(z)f_2(z)\bar{B} + f_1(z)f_2'(z)\bar{B} + f_1(z)f_2(z)\bar{B}'(-\lambda C'(z))) \end{aligned}$$

$$
\begin{aligned}
&\times [\bar{V}_1'(-\lambda C'(z)) + q\bar{V}_1'(-\lambda C'(z))(\bar{V}_2(1-r+r\bar{V}_3)-1)\\
&+ q\bar{V}_1(\bar{V}_2'(-\lambda C'(z))(1-r+r\bar{V}_3) + \bar{V}_2 r\bar{V}_3'(-\lambda C'(z))]\\
&+ p(f_1''(z)f_2(z)\bar{B} + 2f_1(z)f_2(z)\bar{B} + 2f_1'(z)f_2(z)\bar{B}(-\lambda C'(z))\\
&+ 2f_1(z)f_2'(z)\bar{B}(-\lambda C'(z)) + f_1'(z)f_2''(z)\bar{B} + f_1(z)f_2(z)\\
&\times (\bar{B}''(-\lambda C'(z))^2 + \bar{B}(-\lambda C''(z))))[\bar{V}_1 - 1 + q\bar{V}_1(\bar{V}_2(1-r+r\bar{V}_3)-1)]\\
&+ pf_1(z)f_2(z)\bar{B}[\bar{V}_1''(-\lambda C'(z))^2 + \bar{V}_1'(-\lambda C''(z))\\
&+ q\bar{V}_1''(-\lambda C'(z))^2(\bar{V}_2(1-r+r\bar{V}_3)-1)\\
&+ q\bar{V}_1'(-\lambda C''(z))(\bar{V}_2(1-r+r\bar{V}_3)-1)\\
&+ 2q\bar{V}_1'(-\lambda C'(z))(\bar{V}_2'(-\lambda C'(z))(1-r+r\bar{V}_3) + \bar{V}_2 r\bar{V}_3'(-\lambda C'(z))\\
&+ q\bar{V}_1(\bar{V}_2''(-\lambda C'(z))^2(1-r+r\bar{V}_3) + \bar{V}_2'(-\lambda C''(z))(1-r+r\bar{V}_3)\\
&+ 2\bar{V}_2' r\bar{V}_3'\lambda^2(C'(z))^2 + \bar{V}_2 r\bar{V}_3''\lambda^2(C'(z))^2 + r\bar{V}_2\bar{V}_3'(-\lambda C''(z))))
\end{aligned}
$$

$$
\begin{aligned}
D'(z) =& [f_1'(z)f_2(z) + f_1(z)f_2'(z)][z - \bar{B}(1-p+p\bar{V}_1 f_3(z))]\\
&+ f_1(z)f_2(z)[1 - \bar{B}'(-\lambda C'(z))(1-p+p\bar{V}_1 f_3(z))\\
&- \bar{B}(p\bar{V}_1'(-\lambda C'(z))f_3(z) + p\bar{V}_1 f_3'(z))]\\
&- \alpha\beta(1-\bar{B}) - \alpha\beta z\lambda\bar{B}'(C'(z))
\end{aligned}
$$

$$
\begin{aligned}
D''(z) =& 2[f_1'(z)f_2(z) + f_1(z)f_2'(z)][1 - \bar{B}'(-\lambda C'(z))(1-p+p\bar{V}_1 f_3(z))\\
&- \bar{B}(p\bar{V}_1'(-\lambda C'(z))f_3(z)) + p\bar{V}_1 f_3'(z))]\\
&+ [f_1''(z)f_2(z) + 2f_1'(z)f_2'(z) + f_1(z)f_2''(z)][z - \bar{B}(1-p+p\bar{V}_1 f_3(z))]\\
&+ f_1(z)f_2(z)[-2\bar{B}'(-\lambda C'(z))(p\bar{V}_1'(-\lambda C'(z))f_3(z) + p\bar{V}_1 f_3'(z))\\
&- (1-p+p\bar{V}_1 f_3(z))(\bar{B}''(-\lambda C'(z))^2 + \bar{B}'(-\lambda C''(z)))\\
&- \bar{B}p(\bar{V}_1''(-\lambda C'(z))^2 f_3(z) + p\bar{V}_1'(-\lambda C''(z))f_3(z)\\
&+ 2p\bar{V}_1'(-\lambda C'(z))f_3'(z) + p\bar{V}_1 f_3''(z))]\\
&- 2\alpha\beta\bar{B}'(\lambda C'(z)) - \alpha\beta z[\bar{B}''(\lambda(C'(z))^2 + \bar{B}'(\alpha)(\lambda C''(z))]
\end{aligned}
$$

$$
L_q = \lim_{z\to 1}\frac{d}{dz}W_q(z) = \left[\frac{D'(1)N''(1) - N'(1)D''(1)}{2(D'(1))^2}\right]Q \qquad (5.79)
$$

where primes and double primes in (5.79) denote first and second derivative at $z=1$ respectively. Carrying out the derivative at $z=1$, we have

$$
\begin{aligned}
N'(1) =& \lambda E(I)(\alpha+\beta)[1-\bar{B}(\alpha)] \\
&+ \lambda p\alpha\beta E(I)\bar{B}(\alpha)[E(V_1)+q(E(V_2)+rE(V_3))]
\end{aligned} \tag{5.80}
$$

$$
\begin{aligned}
N''(1) =& \lambda E(I(I-1))(\alpha+\beta)[1-\bar{B}(\alpha)] \\
&+ 2\lambda^2(E(I))^2(\alpha+\beta)\bar{B}'(\alpha) \\
&+ 2\lambda E(I)(1-\bar{B}(\alpha))[\alpha-\lambda E(I)] \\
&- 2\lambda^2 p(E(I))^2[(\alpha+\beta)\bar{B}(\alpha) \\
&+ \alpha\beta\bar{B}'(\alpha)][E(V_1)+q(E(V_2)+rE(V_3))] \\
&+ \alpha\beta p\bar{B}(\alpha)[\lambda^2(E(I))^2(E(V_1^2)+q(E(V_2^2)+rE(V_3^2))) \\
&+ \lambda E(I(I-1))(E(V_1)+q(E(V_2)+rE(V_3))) \\
&+ 2q\lambda^2(E(I))^2E(V_1)(E(V_2)+rE(V_3)) \\
&+ 2qr\lambda^2(E(I))^2E(V_2)E(V_3)]
\end{aligned} \tag{5.81}
$$

$$
\begin{aligned}
D'(1) =& \alpha\beta\bar{B}(\alpha)[1-\lambda E(I)p(E(V_1)+q(E(V_2)+rE(V_3)))] \\
&- \lambda E(I)(\alpha+\beta)[1-\bar{B}(\alpha)]
\end{aligned} \tag{5.82}
$$

$$
\begin{aligned}
D''(1) =& -2\lambda E(I)(\alpha+\beta)[1+\lambda E(I)\bar{B}'(\alpha)-\bar{B}(\alpha)(\lambda pE(I)E(V_1) \\
&+ \lambda pqE(I)(E(V_2)+rE(V_3)))] + [2\lambda^2(E(I))^2 \\
&- \lambda\alpha E(I(I-1)) - \lambda\beta E(I(I-1))][1-\bar{B}(\alpha)] \\
&+ 2\alpha\beta\lambda E(I)\bar{B}'(\alpha)(\lambda pE(I)E(V_1)+\lambda pqE(I)(E(V_2)+rE(V_3))) \\
&- \alpha\beta[\lambda^2(E(I))^2\bar{B}''(\alpha)-\lambda E(I(I-1))\bar{B}'(\alpha)] \\
&- \alpha\beta\bar{B}(\alpha)[p\lambda^2(E(I))^2(E(V_1^2)+q(E(V_2^2) \\
&+ rE(V_3^2))) + \lambda pE(I(I-1))(E(V_1)+q(E(V_2)+rE(V_3))) \\
&+ 2pq\lambda^2(E(I))^2E(V_1)(E(V_2)+rE(V_3)) \\
&+ 2pqr\lambda^2(E(I))^2E(V_2)E(V_3)] - 2\lambda\alpha\beta E(I)\bar{B}'(\alpha) \\
&- \alpha\beta[\lambda^2(E(I))^2\bar{B}''(\alpha)+\lambda E(I(I-1))\bar{B}'(\alpha)]
\end{aligned} \tag{5.83}
$$

where $E(B^2)$, $E(V_1^2)$, $E(V_2^2)$, $E(V_3^2)$ are the second moment of service time and vacation times respectively. $E(I(I-1))$ is the second factorial moment of the batch size of arriving customers. Then if we substitute the values $N'(1), N''(1), D'(1), D''(1)$ from equations (5.80) to (5.83) into equations (5.79), we obtain L_q in the closed form.

Further, we find the average system size L by using Little's formula. Thus we have

$$L = L_q + \rho \tag{5.84}$$

where L_q has been found by equation (5.79) and ρ is obtained from equation (5.78).

5.7 The average waiting time

Let W_q and W denote the average waiting time in the queue and in the system respectively. Then by using Little's formula, we obtain

$$W_q = \frac{L_q}{\lambda}$$

$$W = \frac{L}{\lambda}$$

where L_q and L have been found in equations (5.78) and (5.84).

5.8 Particular cases

Case 1: If there is no third phase of extended vacation. i.e, r=0.

Then our model reduces to a single server $M^{[X]}/G/1$ queue with service interruption and two phases of server vacation.

In this case, we find the idle probability Q, utilization factor ρ and the average queue size L_q can be simplified to the following expressions.

$$\begin{aligned}
Q =& 1-\lambda pE(I)[E(V_1)+qE(V_2)]-\frac{\lambda E(I)}{\bar{B}(\alpha)}(\frac{1}{\beta}+\frac{1}{\alpha})[1-\bar{B}(\alpha)]\\
\rho =& \lambda pE(I)[E(V_1)+qE(V_2)]+\frac{\lambda E(I)}{\bar{B}(\alpha)}(\frac{1}{\beta}+\frac{1}{\alpha})[1-\bar{B}(\alpha)]\\
L_q =& \left[\frac{D'(1)N''(1)-N'(1)D''(1)}{2(D'(1))^2}\right]Q
\end{aligned}$$

where

$$\begin{aligned}
N'(1) =& \lambda E(I)(\alpha+\beta)[1-\bar{B}(\alpha)]\\
&+\lambda p\alpha\beta E(I)\bar{B}(\alpha)[E(V_1)+qE(V_2)]\\
N''(1) =& \lambda E(I(I-1))(\alpha+\beta)[1-\bar{B}(\alpha)]\\
&+2\lambda^2(E(I))^2(\alpha+\beta)\bar{B}'(\alpha)\\
&+2\lambda E(I)(1-\bar{B}(\alpha))[\alpha-\lambda E(I)]\\
&-2\lambda^2 p(E(I))^2[(\alpha+\beta)\bar{B}(\alpha)\\
&+\alpha\beta\bar{B}'(\alpha)][E(V_1)+qE(V_2)]\\
&+\alpha\beta p\bar{B}(\alpha)[\lambda^2(E(I))^2(E(V_1^2)+qEV_2^2))\\
&+\lambda E(I(I-1))(E(V_1)+qE(V_2))\\
&+2q\lambda^2(E(I))^2E(V_1)E(V_2)\\
D'(1) =& \alpha\beta\bar{B}(\alpha)[1-\lambda E(I)p(E(V_1)+qE(V_2)]\\
&-\lambda E(I)(\alpha+\beta)[1-\bar{B}(\alpha)]\\
D''(1) =& -2\lambda E(I)(\alpha+\beta)[1+\lambda E(I)\bar{B}'(\alpha)-\bar{B}(\alpha)(\lambda pE(I)E(V_1)\\
&+\lambda pqE(I)E(V_2))]+[2\lambda^2(E(I))^2\\
&-\lambda\alpha E(I(I-1))-\lambda\beta E(I(I-1))][1-\bar{B}(\alpha)]\\
&+2\alpha\beta\lambda E(I)\bar{B}'(\alpha)(\lambda pE(I)E(V_1)+\lambda pqE(I)E(V_2))\\
&-\alpha\beta[\lambda^2(E(I))^2\bar{B}''(\alpha)-\lambda E(I(I-1))\bar{B}'(\alpha)]
\end{aligned}$$

$$
\begin{aligned}
&- \alpha\beta\bar{B}(\alpha)[p\lambda^2(E(I))^2(E(V_1^2) + qE(V_2^2)\\
&+ \lambda pE(I(I-1))(E(V_1) + qE(V_2))\\
&+ 2pq\lambda^2(E(I))^2E(V_1)E(V_2)] - 2\lambda\alpha\beta E(I)\bar{B}'(\alpha)\\
&- \alpha\beta[\lambda^2(E(I))^2\bar{B}''(\alpha) + \lambda E(I(I-1))\bar{B}'(\alpha)]
\end{aligned}
$$

Case 2: If there is no second phase and third phase extended vacation and $C(z) = z$ i.e, $q = r = 0$, $E(I) = 1$ and $E(I(I-1)) = 0$.

Then our model reduces to a single server M/G/1 queue with service interruption and Bernoulli schedule server vacation.

In this case we find the idle probability Q, utilization factor ρ and the average queue size L_q can be simplified to the following expressions.

$$
\begin{aligned}
Q =& 1 - \lambda pE(V_1) - \frac{\lambda}{\bar{B}(\alpha)}(\frac{1}{\beta} + \frac{1}{\alpha})[1 - \bar{B}(\alpha)]\\
\rho =& \lambda pE(V_1) + \frac{\lambda}{\bar{B}(\alpha)}(\frac{1}{\beta} + \frac{1}{\alpha})[1 - \bar{B}(\alpha)]\\
L_q =& \left[\frac{D'(1)N''(1) - N'(1)D''(1)}{2(D'(1))^2}\right] Q
\end{aligned}
$$

where

$$
\begin{aligned}
N'(1) =& \lambda(\alpha+\beta)[1 - \bar{B}(\alpha)] + \lambda p\alpha\beta\bar{B}(\alpha)E(V_1)\\
N''(1) =& 2\lambda^2(\alpha+\beta)\bar{B}'(\alpha) + 2\lambda(1 - \bar{B}(\alpha))[\alpha - \lambda]\\
&- 2\lambda^2 p[(\alpha+\beta)\bar{B}(\alpha) + \alpha\beta\bar{B}'(\alpha)]E(V_1)\\
&+ \alpha\beta p\bar{B}(\alpha)\lambda^2E(V_1^2)\\
D'(1) =& \alpha\beta\bar{B}(\alpha)(1 - \lambda pE(V_1)) - \lambda(\alpha+\beta)[1 - \bar{B}(\alpha)]\\
D''(1) =& - 2\lambda(\alpha+\beta)[1 + \lambda\bar{B}'(\alpha) - \bar{B}(\alpha)\lambda pE(V_1)]\\
&+ 2\lambda^2[1 - \bar{B}(\alpha)] + 2p\alpha\beta\lambda^2\bar{B}'(\alpha)E(V_1)\\
&- \alpha\beta\lambda^2\bar{B}''(\alpha) - \alpha\beta\bar{B}(\alpha)p\lambda^2E(V_1^2)\\
&- 2\lambda\alpha\beta\bar{B}'(\alpha) - \alpha\beta\lambda^2\bar{B}''(\alpha)
\end{aligned}
$$

The above equations coincides with Balamani (2012).

Case 3: When the vacation follows exponential distribution for case 2 then the results coincide with Baskar et al. (2011).

5.9 Numerical results

To numerically illustrate the results obtained in this work, we consider that the service time and vacation times are exponentially distributed with rates μ and γ.

In order to see the effect of various parameters on server's idle time Q, utilization factor ρ and various other queue characteristics such as L, W, L_q, W_q. We base our numerical example on the result found in case 2.

For this purpose in Table 5.1, we can choose the following arbitrary values: α= 2, β= 4, μ =8, γ =3, p=0.7 while λ varies from 0.1 to 1.0 such that the stability condition is satisfied.

It clearly shows as long as increasing the arrival rate, the server's idle time decreases while the utilization factor, the average queue size, system size and the average waiting time in the queue and the system of our queueing model are all increases.

In Table 5.2, we choose the following values: α= 6, β= 5, μ =7, λ =0.7, p=0.3 while γ varies from 1 to 10 such that the stability condition is satisfied.

It clearly shows as long as increasing the vacation rate, the server's idle time increases while the utilization factor, average queue size, system size and average waiting time in the queue and system of our queueing model are all decreases.

Table 5.1: Computed values of various queue characteristics

λ	Q	ρ	L_q	L	W_q	W
0.1	0.957917	0.042083	0.008058	0.050141	0.080576	0.501409
0.2	0.915833	0.084167	0.020099	0.104266	0.100497	0.521330
0.3	0.873750	0.126250	0.036759	0.163009	0.122529	0.543362
0.4	0.831667	0.168333	0.058797	0.227131	0.146993	0.567827
0.5	0.789583	0.210417	0.087139	0.297556	0.174279	0.595112
0.6	0.747500	0.252500	0.122917	0.375417	0.204862	0.625695
0.7	0.705417	0.294583	0.167533	0.462116	0.239332	0.660166
0.8	0.663333	0.336667	0.222744	0.559411	0.278430	0.699264
0.9	0.621250	0.378750	0.290787	0.669537	0.323097	0.743930
1.0	0.579167	0.420833	0.374544	0.795378	0.374544	0.795378

Table 5.2: Computed values of various queue characteristics

γ	ρ	Q	L_q	L	W_q	W
1	0.430000	0.570000	0.577708	1.007708	0.825297	1.439583
2	0.325000	0.675000	0.308051	0.633051	0.440073	0.904359
3	0.290000	0.710000	0.258894	0.548894	0.369849	0.784135
4	0.272500	0.727500	0.240299	0.512799	0.343285	0.732570
5	0.262000	0.738000	0.230893	0.492893	0.329848	0.704133
6	0.255000	0.745000	0.225318	0.480318	0.321883	0.686169
7	0.250000	0.750000	0.221666	0.471666	0.316666	0.673809
8	0.246200	0.753750	0.219104	0.465354	0.313006	0.664792
9	0.243300	0.756667	0.217215	0.460548	0.310307	0.657926
10	0.241000	0.759000	0.215767	0.456767	0.308239	0.652525

CHAPTER SIX

$M^{[X]}/G/1$ Queue with Two Types of Service, Multiple Vacation, Random Breakdown and Restricted Admissibility

CHAPTER

SIX

$M^{[X]}/G/1$ QUEUE WITH TWO TYPES OF SERVICE, MULTIPLE VACATION, RANDOM BREAKDOWN AND RESTRICTED ADMISSIBILITY

6.1 Introduction

A queueing system might suddenly break down and hence the server will not be able to continue providing service unless the system is repaired. Tang (1997), Madan et al. (2003), Thangaraj and Vanitha (2010a), Khalaf et al. (2011), Deepak Gupta et al. (2011) and Kalidass and Kasturi (2012) have studied different queueing system subject to random breakdowns.

Vacation queues have been studied by numerous researchers including Doshi (1986), Takagi (1990), Chae et al. (2001). Borthakur and Chaudhury (1997) and Hur and Ahn (2005) have studied vacation queues with batch arrivals. Queue with multiple vacations has been studied by Rosenberg and Yechiali (1993), Tian and Zhang (2002), Jeyakumar and Arumuganathan (2011) and Maragatha Sundari and Srinivasan (2012a). Thangaraj and Vanitha (2010b)

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have studied a single server M/G/1 feedback queue with two types of service having general distribution.

In some queueing systems with batch arrival there is a restriction such that not all batches are allowed to join the system at all time. This policy is named as restricted admissibility. Choudhury and Madan (2007) proposed an $M^{[X]}/G/1$ queueing system with restricted admissibility of arriving batches and Bernoulli schedule server vacation.

In this chapter, we discuss $M^{[X]}/G/1$ queue with two types of service, multiple vacation, random breakdown and restricted admissibility. Here a single server provides two types of service and each arriving customer has the option of choosing either type of service. If there are no customer waiting in the system then the server goes for vacation with random duration. On returning from vacation, if the server again finds no customer waiting in the system, then the server continues to go for vacation until he finds at least one customer in the system. Here the server takes multiple vacation. The service time and the vacation time are generally (arbitrary) distributed. The system may break down at random. Further we assume that once the system breaks down, it enters a repair process immediately and the customer whose service is interrupted comes back to the head of the queue where the arrival follows Poisson. The breakdown and repair times are exponentially distributed. The customers arrive to the system in batches of variable size, but served one by one on a first come - first served basis. In addition, we assume that restricted admissibility of arriving batches in which not all batches are allowed to join the system at all times.

Here we derive time dependent probability generating functions in terms of Laplace transforms. We also derive the average queue size and average system size. Some particular cases and numerical results are also discussed.

The rest of this chapter is organized as follows. The mathematical description of our model is given in section 6.2. Definitions and Equations governing the system are given in section 6.3 and 6.4 respectively. The time dependent solution have been obtained in section 6.5. Steady state results have been derived explicitly in section 6.6. Average queue size and average system size are computed in section 6.7. Some particular cases and numerical results are discussed in section 6.8 and 6.9 respectively.

6.2 Description of the model

We assume the following to describe the queueing model of our study.

a) Customers arrive at the system in batches of variable size in a compound Poisson process and they are provided one by one service on a 'first come - first served basis'. Let $\lambda c_i dt$ $(i = 1, 2, \ldots)$ be the first order probability that a batch of i customers arrives at the system during a short interval of time $(t, t+dt]$, where $0 \leq c_i \leq 1$ and $\sum_{i=1}^{\infty} c_i = 1$ and $\lambda > 0$ is the arrival rate of batches.

b) The server provides two types of service. Just before the service, a customer may choose first type of service with probability p_1 or second type of service with probability p_2, where $p_1 + p_2 = 1$.

c) The service time follows a general (arbitrary) distribution with distribution function $B_i(s)$ and density function $b_i(s)$. Let $\mu_i(x)dx$ be the conditional probability density of service completion during the interval $(x, x+dx]$, given that the elapsed service time is x, so that

$$\mu_i(x) = \frac{b_i(x)}{1 - B_i(x)}, \quad i = 1, 2.$$

and therefore,

$$b_i(s) = \mu_i(s) e^{-\int\limits_0^s \mu_i(x)dx}, \quad i = 1, 2.$$

d) If the system becomes empty, then the server goes for vacation. On returning from vacation, if there are no customer waiting in the system, then the server continues vacation until he finds at least one customer in the system. Here the server takes multiple vacation.

e) The server's vacation time follows a general (arbitrary) distribution with distribution function $V(t)$ and density function $v(t)$. Let $\gamma(x)dx$ be the conditional probability density of vacation completion during the interval $(x, x + dx]$, given that the elapsed vacation time is x, so that

$$\gamma(x) = \frac{v(x)}{1 - V(x)}$$

and therefore,

$$v(t) = \gamma(t) e^{-\int\limits_0^t \gamma(x)dx}.$$

f) The system may breakdown at random, and breakdowns are assumed to occur according to a Poisson stream with mean breakdown rate $\eta > 0$. Further we assume that once the system breaks down, it enters a repair process immediately and the customer whose service is interrupted comes back to the head of the queue. The repair times are exponentially distributed with mean repair rate $\beta > 0$.

g) In addition, we assume that restricted admissibility of batches in which not all batches are allowed to join the system at all times. Let α $(0 \leq \alpha \leq 1)$ and ξ $(0 \leq \xi \leq 1)$ be the probability that an arriving batch will be allowed to join the system during the period of server's non-vacation period and vacation period respectively.

h) Various stochastic processes involved in the system are assumed to be independent of each other.

6.3 Definitions

We define

$P_n^{(1)}(x,t)$ = Probability that at time t, the server is active providing service and there are n $(n \geq 0)$ customers in the queue excluding the one customer in the first type of service being served and the elapsed service time is x. $P_n^{(1)}(t) = \int_0^\infty P_n^{(1)}(x,t)dx$ denotes the probability that at time t there are n customers in the queue excluding the one customer in the first type of service irrespective of the value of x.

$P_n^{(2)}(x,t)$ = Probability that at time t, the server is active providing service and there are n $(n \geq 0)$ customers in the queue excluding the one customer in the second type of service being served and the elapsed service time is x. $P_n^{(2)}(t) = \int_0^\infty P_n^{(2)}(x,t)dx$ denotes the probability that at time t there are n customers in the queue excluding the one customer in the second type of service irrespective of the value of x.

$V_n(x,t)$ = Probability that at time t, the server is under vacation with elapsed vacation time is x and there are n $(n \geq 0)$ customers in the queue. $V_n(t) = \int_0^\infty V_n(x,t)dx$ denotes the probability that at time t there are n customers in the queue and the server is under vacation irrespective of the value of x.

$R_n(t)$ = Probability that at time t, the server is inactive due to system breakdown and the system is under repair, while there are n $(n \geq 0)$ customers in the queue.

6.4 Equations governing the system

The model is then, governed by the following set of differential - difference equations:

$$\frac{\partial}{\partial x}P_0^{(1)}(x,t)+\frac{\partial}{\partial t}P_n^{(1)}(x,t)+[\lambda+\mu_1(x)+\eta]P_0^{(1)}(x,t)=\lambda(1-\alpha)P_0^{(1)}(x,t) \tag{6.1}$$

$$\begin{aligned}\frac{\partial}{\partial x}P_n^{(1)}(x,t)+\frac{\partial}{\partial t}P_n^{(1)}(x,t)+[\lambda+\mu_1(x)+\eta]P_n^{(1)}(x,t)&=\lambda(1-\alpha)P_n^{(1)}(x,t)\\&+\lambda\alpha\sum_{k=1}^{n}c_kP_{n-k}^{(1)}(x,t),\quad n\geq 1\end{aligned} \tag{6.2}$$

$$\frac{\partial}{\partial x}P_0^{(2)}(x,t)+\frac{\partial}{\partial t}P_0^{(2)}(x,t)+[\lambda+\mu_2(x)+\eta]P_0^{(2)}(x,t)=\lambda(1-\alpha)P_0^{(2)}(x,t) \tag{6.3}$$

$$\begin{aligned}\frac{\partial}{\partial x}P_n^{(2)}(x,t)+\frac{\partial}{\partial t}P_n^{(2)}(x,t)+[\lambda+\mu_2(x)+\eta]P_n^{(2)}(x,t)&=\lambda(1-\alpha)P_n^{(2)}(x,t)\\&+\lambda\alpha\sum_{k=1}^{n}c_kP_{n-k}^{(2)}(x,t),\quad n\geq 1\end{aligned} \tag{6.4}$$

$$\frac{\partial}{\partial x}V_0(x,t)+\frac{\partial}{\partial t}V_0(x,t)+[\lambda+\gamma(x)]V_0(x,t)=\lambda(1-\xi)V_0(x,t) \tag{6.5}$$

$$\begin{aligned}\frac{\partial}{\partial x}V_n(x,t)+\frac{\partial}{\partial t}V_n(x,t)+[\lambda+\gamma(x)]V_n(x,t)&=\lambda(1-\xi)V_n(x,t)\\&+\lambda\xi\sum_{k=1}^{n}c_kV_{n-k}(x,t),\quad n\geq 1\end{aligned} \tag{6.6}$$

$$\frac{d}{dt}R_0(t)+(\lambda+\beta)R_0(t)=0 \tag{6.7}$$

$$\begin{aligned}\frac{d}{dt}R_n(t)+(\lambda+\beta)R_n(t)=&\lambda\sum_{k=1}^{n}c_kR_{n-k}(t)+\eta\int_0^\infty P_{n-1}^{(1)}(x,t)dx\\&+\eta\int_0^\infty P_{n-1}^{(2)}(x,t)dx\end{aligned} \tag{6.8}$$

The above set of equations are to be solved subject to the following boundary conditions

$$P_n^{(1)}(0,t) = p_1 \int_0^{\infty} \gamma(x) V_{n+1}(x,t)dx + p_1 \int_0^{\infty} \mu_1(x) P_{n+1}^{(1)}(x,t)dx$$
$$+ p_1 \int_0^{\infty} \mu_2(x) P_{n+1}^{(2)}(x,t)dx + p_1 \beta R_{n+1}(t), \quad n \geq 0 \tag{6.9}$$
$$P_n^{(2)}(0,t) = p_2 \int_0^{\infty} \gamma(x) V_{n+1}(x)dx + p_2 \int_0^{\infty} \mu_1(x) P_{n+1}^{(1)}(x,t)dx$$
$$+ p_2 \int_0^{\infty} \mu_2(x) P_{n+1}^{(2)}(x,t)dx + p_2 \beta R_{n+1}(t), \quad n \geq 0 \tag{6.10}$$
$$V_0(0,t) = \int_0^{\infty} \gamma(x) V_0(x,t)dx + \int_0^{\infty} \mu_1(x) P_0^{(1)}(x,t)dx$$
$$+ \int_0^{\infty} \mu_2(x) P_0^{(2)}(x,t)dx + \beta R_0(t) \tag{6.11}$$
$$V_n(0,t) = 0, \quad n \geq 1 \tag{6.12}$$

We assume that initially there are no customers in the system and the server is idle. So the initial conditions are

$$V_n(0) = P_n^{(i)}(0) = R_n(0) = 0 \text{ for } n = 0,1,2,..., \quad i = 1,2. \tag{6.13}$$

6.5 Probability generating functions of the queue length: The time-dependent solution

In this section, we obtain the transient solution for the above set of differential-difference equations.

Theorem: *The system of differential-difference equations to describe an $M^{[X]}/G/1$ queue with two types of service subject to random breakdown and multiple vacation with restricted admissibility are given by equations (6.1) to (6.12) with initial conditions (6.13) and the generating functions of transient solution are given by equations (6.53), (6.57) to (6.59).*

Proof: We define the probability generating functions for i= 1, 2.

$$P^{(i)}(x,z,t) = \sum_{n=0}^{\infty} z^n P_n^{(i)}(x,t); \quad P^{(i)}(z,t) = \sum_{n=0}^{\infty} z^n P_n^{(i)}(t); \tag{6.14}$$

$$V(x,z,t) = \sum_{n=0}^{\infty} z^n V_n(x,t); \quad V(z,t) = \sum_{n=0}^{\infty} z^n V_n(t); \tag{6.15}$$

$$R(z,t) = \sum_{n=0}^{\infty} z^n R_n(t); \qquad C(z) = \sum_{n=1}^{\infty} c_n z^n; \tag{6.16}$$

which are convergent inside the circle given by $\mid z \mid \leq 1$ and define the Laplace transform of a function $f(t)$ as

$$\bar{f}(s) = \int_0^{\infty} e^{-st} f(t)dt, \quad \Re(s) > 0. \tag{6.17}$$

We take the Laplace transform of equations (6.1) to (6.12) and using equation (6.13) we get

$$\frac{\partial}{\partial x}\bar{P}_0^{(1)}(x,s) + (s + \lambda\alpha + \mu_1(x) + \eta)\bar{P}_0^{(1)}(x,s) = 0 \tag{6.18}$$

$$\frac{\partial}{\partial x}\bar{P}_n^{(1)}(x,s) + (s + \lambda\alpha + \mu_1(x) + \eta)\bar{P}_n^{(1)}(x,s) = \lambda\alpha \sum_{k=1}^{n} c_k \bar{P}_{n-k}^{(1)}(x,s), n \geq 1 \tag{6.19}$$

$$\frac{\partial}{\partial x}\bar{P}_0^{(2)}(x,s) + [s + \lambda\alpha + \mu_2(x) + \eta]\bar{P}_0^{(2)}(x,s) = 0 \tag{6.20}$$

$$\frac{\partial}{\partial x}\bar{P}_n^{(2)}(x,s) + [s + \lambda\alpha + \mu_2(x) + \eta]\bar{P}_n^{(2)}(x,s) = \lambda\alpha \sum_{k=1}^{n} c_k \bar{P}_{n-k}^{(2)}(x,s), n \geq 1 \tag{6.21}$$

$$\frac{\partial}{\partial x}\bar{V}_0(x,s) + [s + \lambda\xi + \gamma(x)]\bar{V}_0(x,s) = 0 \tag{6.22}$$

$$\frac{\partial}{\partial x}\bar{V}_n(x,s) + [s + \lambda\xi + \gamma(x)]\bar{V}_n(x,s) = \lambda\xi \sum_{k=1}^{n} c_k \bar{V}_{n-k}(x,s), n \geq 1 \tag{6.23}$$

$$(s+\lambda+\beta)\bar{R}_0(s)=0 \tag{6.24}$$

$$(s+\lambda+\beta)\bar{R}_n(s)=\lambda\sum_{k=1}^{n}c_k\bar{R}_{n-k}(s)+\eta\int_0^\infty\bar{P}_{n-1}^{(1)}(x,s)dx$$
$$+\eta\int_0^\infty\bar{P}_{n-1}^{(2)}(x,s)dx \tag{6.25}$$

$$\bar{P}_n^{(1)}(0,s)=p_1\beta\bar{R}_{n+1}(s)+p_1\int_0^\infty\gamma(x)\bar{V}_{n+1}(x,s)dx$$
$$+p_1\int_0^\infty\mu_1(x)\bar{P}_{n+1}^{(1)}(x,s)dx$$
$$+p_1\int_0^\infty\mu_2(x)\bar{P}_{n+1}^{(2)}(x,s)dx,\quad n\geq 0 \tag{6.26}$$

$$\bar{P}_n^{(2)}(0,s)=p_2\beta\bar{R}_{n+1}(s)+p_2\int_0^\infty\gamma(x)\bar{V}_{n+1}(x,s)dx$$
$$+p_2\int_0^\infty\mu_1(x)\bar{P}_{n+1}^{(1)}(x,s)dx$$
$$+p_2\int_0^\infty\mu_2(x)\bar{P}_{n+1}^{(2)}(x,s)dx,\quad n\geq 0 \tag{6.27}$$

$$\bar{V}_0(0,s)=\int_0^\infty\gamma(x)\bar{V}_0(x,s)dx+\int_0^\infty\mu_1(x)\bar{P}_0^{(1)}(x,s)dx$$
$$+\int_0^\infty\mu_2(x)\bar{P}_0^{(2)}(x,s)dx+\beta\bar{R}_0(s) \tag{6.28}$$

$$\bar{V}_n(0,s)=0,\quad n\geq 1 \tag{6.29}$$

Now multiplying equations (6.19), (6.21), (6.23) and (6.25) by suitable powers of z, adding to equations (6.18), (6.20), (6.22) and (6.24) and summing over n from 0 to ∞ and using the generating function defined in (6.14) to (6.16), we get

$$\frac{\partial}{\partial x}\bar{P}^{(1)}(x,z,s)+[s+\lambda\alpha-\lambda\alpha C(z)+\mu_1(x)+\eta]\bar{P}^{(1)}(x,z,s)=0 \tag{6.30}$$

$$\frac{\partial}{\partial x}\bar{P}^{(2)}(x,z,s)+[s+\lambda\alpha-\lambda\alpha C(z)+\mu_2(x)+\eta]\bar{P}^{(2)}(x,z,s)=0 \tag{6.31}$$

$$\frac{\partial}{\partial x}\bar{V}(x,z,s)+[s+\lambda\xi-\lambda\xi C(z)+\gamma(x)]\bar{V}(x,z,s)=0 \tag{6.32}$$

$$(s+\lambda-\lambda C(z)+\beta)\bar{R}(z,s)=\eta z\int_0^\infty \bar{P}^{(1)}(x,z,s)dx$$
$$+\eta z\int_0^\infty \bar{P}^{(2)}(x,z,s)dx \tag{6.33}$$

For the boundary condition, we multiply both sides of equation (6.26) by z^n summing over n from 0 to ∞ and use the equations (6.14) to (6.16), we get

$$\begin{aligned} zP^{(1)}(0,z,s)=\;& p_1\int_0^\infty \gamma(x)\bar{V}(x,z,s)dx-p_1\int_0^\infty \gamma(x)\bar{V}_0(x,s)dx\\ &+p_1\int_0^\infty \mu_1(x)\bar{P}^{(1)}(x,z,s)dx-p_1\int_0^\infty \mu_1(x)\bar{P}_0^{(1)}(x,s)dx\\ &+p_1\int_0^\infty \mu_2(x)\bar{P}^{(2)}(x,z,s)dx-p_1\int_0^\infty \mu_2(x)\bar{P}_0^{(2)}(x,s)dx\\ &+p_1\beta\bar{R}(z,s)-p_1\beta\bar{R}_0(s),\quad n\geq 0 \end{aligned} \tag{6.34}$$

Performing similar operation on equations (6.27) to (6.29), we get

$$\begin{aligned} zP^{(2)}(0,z,s)=\;& p_2\int_0^\infty \gamma(x)\bar{V}(x,z,s)dx-p_2\int_0^\infty \gamma(x)\bar{V}_0(x,s)dx\\ &+p_2\int_0^\infty \mu_1(x)\bar{P}^{(1)}(x,z,s)dx-p_2\int_0^\infty \mu_1(x)\bar{P}_0^{(1)}(x,s)dx\\ &+p_2\int_0^\infty \mu_2(x)\bar{P}^{(2)}(x,z,s)dx-p_2\int_0^\infty \mu_2(x)\bar{P}_0^{(2)}(x,s)dx\\ &+p_2\beta\bar{R}(z,s)-p_2\beta\bar{R}_0(s),\quad n\geq 0 \end{aligned} \tag{6.35}$$

$$\bar{V}(0,z,s)=\bar{V}_0(0,s) \tag{6.36}$$

Using equation (6.36) in (6.34) and (6.35), we get

$$\begin{aligned} zP^{(1)}(0,z,s)=\;& p_1\int_0^\infty \gamma(x)\bar{V}(x,z,s)dx+p_1\int_0^\infty \mu_1(x)\bar{P}^{(1)}(x,z,s)dx\\ &+p_1\int_0^\infty \mu_2(x)\bar{P}^{(2)}(x,z,s)dx\\ &+p_1\beta\bar{R}(z,s)-p_1\bar{V}_0(0,s) \end{aligned} \tag{6.37}$$

$$zP^{(2)}(0,z,s) = p_2\int_0^\infty \gamma(x)\bar{V}(x,z,s)dx + p_2\int_0^\infty \mu_1(x)\bar{P}^{(1)}(x,z,s)dx$$
$$+ p_2\int_0^\infty \mu_2(x)\bar{P}^{(2)}(x,z,s)dx$$
$$+ p_2\beta\bar{R}(z,s) - p_2\bar{V}_0(0,s) \tag{6.38}$$

Integrating equation (6.30) between 0 and x, we get

$$\bar{P}^{(1)}(x,z,s) = \bar{P}^{(1)}(0,z,s)e^{-[s+\lambda\alpha-\lambda\alpha C(z)+\eta]x-\int\limits_0^x \mu_1(t)dt} \tag{6.39}$$

where $P^{(1)}(0,z,s)$ is given by equation (6.37).

Again integrating equation (6.39) by parts with respect to x, yields

$$\bar{P}^{(1)}(z,s) = \bar{P}^{(1)}(0,z,s)\left[\frac{1-\bar{B}_1(s+\lambda\alpha-\lambda\alpha C(z)+\eta)}{s+\lambda\alpha-\lambda\alpha C(z)+\eta}\right] \tag{6.40}$$

where

$$\bar{B}_1(s+\lambda\alpha-\lambda\alpha C(z)+\eta) = \int\limits_0^\infty e^{-[s+\lambda\alpha-\lambda\alpha C(z)+\eta]x}dB_1(x)$$

is the Laplace-Stieltjes transform of the first type of service time $B_1(x)$.

Now multiplying both sides of equation (6.39) by $\mu_1(x)$ and integrating over x, we obtain

$$\int\limits_0^\infty \bar{P}^{(1)}(x,z,s)\mu_1(x) = \bar{P}^{(1)}(0,z,s)\bar{B}_1[s+\lambda\alpha-\lambda\alpha C(z)+\eta] \tag{6.41}$$

Similarly, on integrating equations (6.31) and (6.32) from 0 to x, we get

$$\bar{P}^{(2)}(x,z,s) = \bar{P}^{(2)}(0,z,s)e^{-[s+\lambda\alpha-\lambda\alpha C(z)+\eta]x-\int\limits_0^x \mu_2(t)dt} \tag{6.42}$$

$$\bar{V}(x,z,s) = \bar{V}(0,z,s)e^{-[s+\lambda\xi-\lambda\xi C(z)]x-\int\limits_0^x \gamma(t)dt} \tag{6.43}$$

where $\bar{V}(0,z,s)$ and $\bar{P}^{(2)}(0,z,s)$ are given by equation (6.36) and (6.38). Again integrating equation (6.42) and (6.43) by parts with respect to x, yields

$$\bar{P}^{(2)}(z,s) = \bar{P}^{(2)}(0,z,s)\left[\frac{1-\bar{B}_2(s+\lambda\alpha-\lambda\alpha C(z)+\eta)}{s+\lambda\alpha-\lambda\alpha C(z)+\eta}\right] \tag{6.44}$$

$$\bar{V}(z,s) = \bar{V}(0,z,s)\left[\frac{1-\bar{V}(s+\lambda\xi-\lambda\xi C(z))}{s+\lambda\xi-\lambda\xi C(z)}\right] \tag{6.45}$$

where

$$\bar{B}_2(s+\lambda\alpha-\lambda\alpha C(z)+\eta) = \int_0^\infty e^{-[s+\lambda\alpha-\lambda\alpha C(z)+\eta]x}dB_2(x)$$

$$\bar{V}(s+\lambda\xi-\lambda\xi C(z)) = \int_0^\infty e^{-[s+\lambda\xi-\lambda\xi C(z)]x}dV(x)$$

is the Laplace-Stieltjes transform of the second type of service time $B_2(x)$ and vacation time $V(x)$. Now multiplying both sides of equation (6.42) and (6.43) by $\mu_2(x)$ and $\gamma(x)$ and integrating over x, we obtain

$$\int_0^\infty \bar{P}^{(2)}(x,z,s)\mu_2(x)dx = \bar{P}^{(2)}(0,z,s)\bar{B}_2[s+\lambda\alpha-\lambda\alpha C(z)+\eta] \tag{6.46}$$

$$\int_0^\infty \bar{V}(x,z,s)\gamma(x)dx = \bar{V}(0,z,s)\bar{V}[s+\lambda\xi-\lambda\xi C(z)] \tag{6.47}$$

Using equations (6.36), (6.41), (6.46) and (6.47) in (6.37) and (6.38), we get

$$\begin{aligned}[z-p_1\bar{B}_1(a)]\bar{P}^{(1)}(0,z,s) =& p_1[\bar{V}(c)-1]\bar{V}_0(0,s)+p_1\beta\bar{R}(z,s)\\ &+p_1\bar{B}_2(a)\bar{P}^{(2)}(0,z,s) \qquad (6.48)\\ [z-p_2\bar{B}_2(a)]\bar{P}^{(2)}(0,z,s) =& p_2[\bar{V}(c)-1]\bar{V}_0(0,s)+p_2\beta\bar{R}(z,s)\\ &+p_2\bar{B}_1(a)\bar{P}^{(1)}(0,z,s) \qquad (6.49)\end{aligned}$$

Using equation (6.48) and (6.49), we get

$$[z-(p_1\bar{B}_1(a)+p_2\bar{B}_2(a))]\bar{P}^{(1)}(0,z,s) = p_1[\bar{V}(c)-1]\bar{V}_0(0,s)+p_1\beta\bar{R}(z,s) \quad (6.50)$$

$$[z-(p_1\bar{B}_1(a)+p_2\bar{B}_2(a))]\bar{P}^{(2)}(0,z,s) = p_2[\bar{V}(c)-1]\bar{V}_0(0,s)+p_2\beta\bar{R}(z,s) \quad (6.51)$$

where $a = s + \lambda\alpha - \lambda\alpha C(z) + \eta$ and $c = s + \lambda\xi - \lambda\xi C(z)$.

Substituting equations (6.39) and (6.42) in (6.33), we get

$$\bar{R}(z,s) = \frac{\alpha z}{ab}[\bar{P}^{(1)}(0,z,s)(1-\bar{B}_1(a)) + \bar{P}^{(2)}(0,z,s)(1-\bar{B}_2(a))] \quad (6.52)$$

Using equations (6.50) and (6.51) in (6.52), we get

$$\bar{R}(z,s) = \frac{\eta z[1-(p_1\bar{B}_1(a)+p_2\bar{B}_2(a))][\bar{V}(c)-1]\bar{V}_0(0,s)}{Dr} \quad (6.53)$$

where $b = s + \lambda - \lambda C(z) + \beta$.

$$Dr = ab[z-(p_1\bar{B}_1(a)+p_2\bar{B}_2(a))] - \eta\beta z[1-(p_1\bar{B}_1(a)+p_2\bar{B}_2(a))] \quad (6.54)$$

By substituting equation (6.53) in (6.50) and (6.51), we get,

$$\bar{P}^{(1)}(0,z,s) = \frac{p_1 ab[\bar{V}(c)-1]\bar{V}_0(0,s)}{Dr} \quad (6.55)$$

$$\bar{P}^{(2)}(0,z,s) = \frac{p_2 ab[\bar{V}(c)-1]\bar{V}_0(0,s)}{Dr} \quad (6.56)$$

Using equations (6.36), (6.55), (6.56) in (6.40), (6.44) and (6.45), we have

$$\bar{P}^{(1)}(z,s) = \frac{p_1 b[1-\bar{B}_1(a)][\bar{V}(c)-1]\bar{V}_0(0,s)}{Dr} \quad (6.57)$$

$$\bar{P}^{(2)}(z,s) = \frac{p_2 b[1-\bar{B}_2(a)][\bar{V}(c)-1]\bar{V}_0(0,s)}{Dr} \quad (6.58)$$

$$\bar{V}(z,s) = \frac{[1-\bar{V}(c)]}{c}\bar{V}_0(0,s) \tag{6.59}$$

Thus $\bar{R}(z,s), \bar{P}^{(1)}(z,s),\ \bar{P}^{(2)}(z,s)$ and $\bar{V}(z,s)$ are completely determined from equations (6.53), (6.57) to (6.59) which completes the proof of the theorem.

6.6 The steady state results

In this section, we shall derive the steady state probability distribution for our queueing model. To define the steady probabilities we suppress, the argument t wherever it appears in the time-dependent analysis. This can be obtained by applying the Tauberian property

$$\lim_{s\to 0} s\bar{f}(s) = \lim_{t\to\infty} f(t)$$

In order to determine $\bar{P}^{(1)}(z,s),\ \bar{P}^{(2)}(z,s),\ \bar{V}(z,s)$ and $\bar{R}(z,s)$ completely, we have yet to determine the unknown $\bar{V}_0(0,s)$ which appears in the numerators of the right hand sides of equations (6.53), (6.57), (6.58) and (6.59).

For that purpose, we shall use the normalizing condition

$$P^{(1)}(1) + P^{(2)}(1) + V(1) + R(1) = 1$$

The steady state probabilities for an $M^{[X]}/G/1$ queue with two types of service subject to random breakdown and multiple vacation with restricted admissibility are given by

$$P^{(1)}(1) = \frac{\lambda\beta\xi p_1 E(I)(1-\bar{B}_1(\eta))E(V)}{dr}V_0(0) \tag{6.60}$$

$$P^{(2)}(1) = \frac{\lambda\beta\xi p_2 E(I)(1-\bar{B}_2(\eta))E(V)}{dr}V_0(0) \tag{6.61}$$

$$R(1) = \frac{\lambda\eta\xi E(I)[1-(p_1\bar{B}_1(\eta)+p_2\bar{B}_2(\eta))]E(V)}{dr}V_0(0) \tag{6.62}$$

$$V(1) = E(V)V_0(0) \tag{6.63}$$

where $dr = \eta\beta - [1-(p_1\bar{B}_1(\eta)+p_2\bar{B}_2(\eta))][\lambda E(I)(\alpha\eta+\beta)+\eta\beta]$.

$P^{(1)}(1), P^{(2)}(1)$, $V(1)$, $R(1)$ denote the steady state probabilities that the server is providing first type of service, second type of service, server on vacation and server under repair without regard to the number of customers in the queue.

Multiplying both sides of equations (6.53), (6.57), (6.58) and (6.59) by s, taking limit as $s \to 0$, by applying Tauberian property and simplifying, we obtain

$$P^{(1)}(z) = \frac{p_1(\lambda-\lambda C(z)+\beta)[\bar{V}(a_2)-1][1-\bar{B}_1(a_1)]V_0(0)}{D(z)} \tag{6.64}$$

$$P^{(2)}(z) = \frac{p_2(\lambda-\lambda C(z)+\beta)[\bar{V}(a_2)-1][1-\bar{B}_2(a_1)]V_0(0)}{D(z)} \tag{6.65}$$

$$V(z) = \frac{[1-\bar{V}(a_2)]}{a_2}V_0(0) \tag{6.66}$$

$$R(z) = \frac{\eta z[\bar{V}(a_2)-1][1-(p_1\bar{B}_1(a_1)+p_2\bar{B}_2(a_1))]}{D(z)}V_0(0) \tag{6.67}$$

where

$$\begin{aligned} D(z) = {} & a_1(\lambda-\lambda C(z)+\beta)[z-(p_1\bar{B}_1(a_1)+p_2\bar{B}_2(a_1)] \\ & -\eta z\beta[1-(p_1\bar{B}_1(a_1)+p_2\bar{B}_2(a_1))], \end{aligned} \tag{6.68}$$

$a_1 = \lambda\alpha - \lambda\alpha C(z) + \eta$, $a_2 = \lambda\xi - \lambda\xi C(z)$.

Let $W_q(z)$ denote the probability generating function of the queue size irrespective of the server state. Then adding equations (6.64) to (6.67), we obtain

$$W_q(z) = P^{(1)}(z) + P^{(2)}(z) + V(z) + R(z)$$

$$W_q(z) = \frac{N(z)}{D(z)} V_0(0) + (\frac{1 - \bar{V}(a_2)}{a_2}) V_0(0) \tag{6.69}$$

where

$$N(z) = [\bar{V}(a_2) - 1][1 - (p_1 \bar{B}_1(a_1) + p_2 \bar{B}_2(a_1))] \times [\lambda(1 - C(z)) + \beta + \eta z] \tag{6.70}$$

and $D(z)$ is given in the equation (6.68).

we see that for z=1, $W_q(1)$ is indeterminate of the form 0/0. Therefore, we apply L'Hopital's rule and on simplifying, we obtain

$$V_0(0) = \frac{\eta\beta - [\lambda E(I)(\alpha\beta + \eta) + \eta\beta][1 - p_1 \bar{B}_1(\eta) - p_2 \bar{B}_2(\eta)]}{dr_1} \tag{6.71}$$

where

$$dr_1 = E(V)[(p_1 \bar{B}_1(\eta) + p_2 \bar{B}_2(\eta)) - 1] \times [\lambda E(I)(\beta(\alpha - \xi) + \eta(1 - \xi)) + \eta\beta] + \eta\beta E(V)$$

$C(1)$= 1, $C'(1) = E(I)$ is mean batch size of the arriving customers,
$E(V) = -\bar{V}'(0)$
and hence, the utilization factor ρ of the system is given by

$$\rho = \frac{\lambda\xi(\beta + \eta)E(I)[1 - p_1 \bar{B}_1(\eta) - p_2 \bar{B}_2(\eta)]}{\eta\beta - [1 - p_1 \bar{B}_1(\eta) - p_2 \bar{B}_2(\eta)][\lambda E(I)(\beta(\alpha - \xi) + \eta(1 - \xi)) + \eta\beta]} \tag{6.72}$$

where $\rho < 1$ is the stability condition under which the steady states exits.

Substituting for $V_0(0)$ from (6.71) into (6.69), we have completely and explicitly determined the probability generating function of the queue size.

6.7 The average queue size and the average system size

Let L_q the denote the mean number of customers in the queue under the steady state. Then we have

$$L_q = \frac{d}{dz} W_q(z) \quad \text{at } z = 1$$

since this formula gives 0/0 form, then we write
$W_q(z) = \dfrac{N(z)}{D(z)} V_0(0) + (\dfrac{1-\bar{V}(a_2)}{a_2}) V_0(0)$

$$\begin{aligned}
N'(z) =& \bar{V}'(a_2)a_2'[1-(p_1\bar{B}_1(a_1)+p_2\bar{B}_2(a_1))][\lambda(1-C(z))+\beta+\eta z] \\
&+(\bar{V}(a_2)-1)[-a_1'(p_1\bar{B}_1'(a_1)+p_2\bar{B}_2'(a_1))][\lambda(1-C(z))+\beta+\eta z] \\
&+(\bar{V}(a_2)-1)[1-(p_1\bar{B}_1(a_1)+p_2\bar{B}_2(a_1))][-\lambda C'(z)+\eta] \\
N''(z) =& [\bar{V}''(a_2)a_2'^2[1-(p_1\bar{B}_1(a_1)+p_2\bar{B}_2(a_1))] \\
&+\bar{V}'(a_2)a_2''[1-(p_1\bar{B}_1(a_1)+p_2\bar{B}_2(a_1))] \\
&+\bar{V}'(a_2)a_2'(-a_1'(p_1\bar{B}_1'(a_1)+p_2\bar{B}_2'(a_1))] \\
&\times[\lambda(1-C(z))+\beta+\eta z] \\
&+\bar{V}'(a_2)a_2'(1-(p_1\bar{B}_1(a_1)+p_2\bar{B}_2(a_1)))(-\lambda C'(z)+\eta) \\
&+[\bar{V}(a_2)a_2'(-a_1'(p_1\bar{B}_1'(a_1)+p_2\bar{B}_2'(a_1))) \\
&+(\bar{V}(a_2)-1)(-a_1^2((p_1\bar{B}_1''(a_1)+p_2\bar{B}_2''(a_1)) \\
&-a_1''(p_1\bar{B}_1'(a_1)+p_2\bar{B}_2'(a_1))))][\lambda(1-C(z))+\beta+\eta z] \\
&+(\bar{V}(a_2)-1)(-a_1'(p_1\bar{B}_1'(a_1)+p_2\bar{B}_2'(a_1)))[-\lambda C'(z)+\eta] \\
&+[\bar{V}'(a_2)a_2'[1-(p_1\bar{B}_1(a_1)+p_2\bar{B}_2(a_1))] \\
&+(\bar{V}(a_2)-1)(-a_1'(p_1\bar{B}_1'(a_1)+p_2\bar{B}_2'(a_1)))][-\lambda C'(z)+\eta] \\
&+(\bar{V}(a_2)-1)(1-(p_1\bar{B}_1(a_1)+p_2\bar{B}_2(a_1))][-\lambda C''(z)]
\end{aligned}$$

$$
\begin{aligned}
D'(z) =& [a_1'[z - (p_1\bar{B}_1(a_1) + p_2\bar{B}_2(a_1)) \\
& + a_1[1 - (p_1\bar{B}_1'(a_1) + p_2\bar{B}_2'(a_1))a_1']] \\
& \times (\lambda - \lambda C(z) + \beta) + a_1(z - (p_1\bar{B}_1(a_1) + p_2\bar{B}_2(a_1))(-\lambda C'(z)) \\
& - \eta\beta(1 - (p_1\bar{B}_1(a_1) + p_2\bar{B}_2(a_1))) \\
& - \eta\beta z(-p_1\bar{B}_1'(a_1) - p_2\bar{B}_2'(a_1))a_1') \\
D''(z) =& [a_1''(z - (p_1\bar{B}_1(a_1) + p_2\bar{B}_2(a_1)) \\
& + 2a_1'(1 - a_1'(p_1\bar{B}_1'(a_1) + p_2\bar{B}_2'(a_1))) \\
& - a_1(a_1^2(p_1\bar{B}_1''(a_1) + p_2\bar{B}_2''(a_1)) \\
& + a_1''(p_1\bar{B}_1'(a_1) + p_2\bar{B}_2'(a_1)))](\lambda - \lambda C(z) + \beta) \\
& + 2[a_1'(z - (p_1\bar{B}_1(a_1) + p_2\bar{B}_2(a_1))) \\
& + a_1(1 - (p_1\bar{B}_1'(a_1) + p_2\bar{B}_2'(a_1)))](-\lambda C'(z)) \\
& + a_1(z - (p_1\bar{B}_1(a_1) + p_2\bar{B}_2(a_1)))(-\lambda C''(z)) \\
& - 2\eta\beta(-a_1'(p_1\bar{B}_1'(a_1) + p_2\bar{B}_2'(a_1)) \\
& - \eta\beta z(-a_1^2(p_1\bar{B}_1''(a_1)) + p_2\bar{B}_2''(a_1)) \\
& - a_1''(p_1\bar{B}_1'(a_1) + p_2\bar{B}_2'(a_1)))
\end{aligned}
$$

$$
L_q = \frac{D'(1)N''(1) - N'(1)D''(1)}{2D'(1)^2}V_0(0) + \frac{\lambda\xi E(I)E(V^2)}{2}V_0(0) \tag{6.73}
$$

where primes and double primes in (6.73) denote first and second derivative at z =1 respectively. Carrying out the derivative at $z = 1$, we have

$$
N'(1) = \lambda\xi E(I)(\beta + \eta)E(V)[1 - (p_1\bar{B}_1(\eta) + p_2\bar{B}_2(\eta))] \tag{6.74}
$$

$$
\begin{aligned}
N''(1) =& [1 - (p_1\bar{B}_1(\eta) + p_2\bar{B}_2(\eta))][\lambda^2\xi^2(E(I))^2(\beta + \eta)E(V^2) \\
& + \lambda\xi E(V)((\beta + \eta)E(I(I-1)) + 2E(I)(\eta - \lambda E(I)))] \\
& + 2\lambda^2\alpha\xi(E(I))^2(\beta + \eta)E(V)[p_1\bar{B}_1'(\eta) + p_2\bar{B}_2'(\eta)]
\end{aligned} \tag{6.75}
$$

$$D'(1) = \eta\beta - [\lambda E(I)(\alpha\beta + \eta) + \eta\beta][1 - (p_1\bar{B}_1(\eta) + p_2\bar{B}_2(\eta))] \tag{6.76}$$

$$\begin{aligned} D''(1) =& [1 - (p_1\bar{B}_1(\eta) + p_2\bar{B}_2(\eta))][-\lambda E(I(I-1))(\alpha\beta + \eta) \\ & + 2\lambda^2\alpha(E(I))^2] - 2\lambda\alpha\beta\eta E(I)(p_1\bar{B}_1'(\eta) + p_2\bar{B}_2'(\eta)) \\ & - 2\lambda(\alpha\beta + \eta)E(I)[1 + \lambda\alpha E(I)(p_1\bar{B}_1'(\eta) + p_2\bar{B}_2'(\eta))] \end{aligned} \tag{6.77}$$

where $E(B_1^2)$, $E(B_2^2)$ and $E(V^2)$ are the second moment of the service times and vacation time respectively. $E(I(I-1))$ is the second factorial moment of the batch size of arriving customers. Then if we substitute the values from (6.74), (6.75), (6.76) and (6.77) into (6.73), we obtain L_q in the closed form.

Further, we find the average system size L by using Little's formula. Thus we have

$$L = L_q + \rho \tag{6.78}$$

where L_q has been found by equation (6.73) and ρ is obtained from equation (6.72).

6.8 Particular cases

Case 1: If there is no restricted admissibility i.e, $\alpha = \xi = 1$, then our model reduces to the $M^{[X]}/G/1$ queue with two types of service, random breakdown and multiple vacation.

Using this in the main result of (6.71), (6.72) and (6.73), we can find the idle probability $V_0(0)$, utilization factor ρ and the mean queue size L_q can be simplified to the following expressions.

$$V_0(0) = \frac{\eta\beta - [\lambda E(I)(\beta + \eta) + \eta\beta][1 - p_1\bar{B}_1(\eta) + p_2\bar{B}_2(\eta)]}{dr_1}$$

where

$$dr_1 = E(V)[(p_1\bar{B}_1(\eta) + p_2\bar{B}_2(\eta)) - 1]\eta\beta + \eta\beta E(V)$$

$$\rho = \frac{\lambda(\beta+\eta)E(I)[1 - p_1\bar{B}_1(\eta) - p_2\bar{B}_2(\eta)]}{\eta\beta - [1 - p_1\bar{B}_1(\eta) - p_2\bar{B}_2(\eta)]\eta\beta}$$

$$L_q = \frac{D'(1)N''(1) - N'(1)D''(1)}{2D'(1)^2}V_0(0) + \frac{\lambda E(I)E(V^2)}{2}$$

where

$$\begin{aligned}
N'(1) =& \lambda E(I)(\beta+\eta)E(V)[1 - (p_1\bar{B}_1(\eta) + p_2\bar{B}_2(\eta))] \\
N''(1) =& [1 - (p_1\bar{B}_1(\eta) + p_2\bar{B}_2(\eta))][\lambda^2(E(I))^2(\beta+\eta)E(V^2) \\
&+ \lambda E(V)(\beta+\eta)E(I(I-1)) + 2E(I)\lambda E(V)(\eta - \lambda E(I))] \\
&+ 2\lambda^2(E(I))^2(\beta+\eta)E(V)[p_1\bar{B}_1'(\eta) + p_2\bar{B}_2'(\eta)] \\
D'(1) =& \eta\beta - [\lambda E(I)(\beta+\eta) + \eta\beta][1 - (p_1\bar{B}_1(\eta) + p_2\bar{B}_2(\eta))] \\
D''(1) =& [1 - (p_1\bar{B}_1(\eta) + p_2\bar{B}_2(\eta))][-\lambda E(I(I-1))(\beta+\eta) \\
&+ 2\lambda^2(E(I))^2] - 2\lambda(\beta+\eta)E(I)[1 + \lambda E(I)(p_1\bar{B}_1'(\eta) + p_2\bar{B}_2'(\eta))] \\
&- 2\lambda\beta\eta E(I)(p_1\bar{B}_1'(\eta) + p_2\bar{B}_2'(\eta))
\end{aligned}$$

Case 2: If there is no type 2 service, no restricted admissibility and service and vacation times are exponentialy distributed. i.e, $p_2 = 0$ and $\alpha = \xi = 1$, then our model reduces to the $M^{[X]}/M/1$ queue with random breakdown and multiple vacation.

Using this in the main result of (6.71), (6.72) and (6.73), we can find the idle probability $V_0(0)$, utilization factor ρ and the mean queue size L_q can be simplified to the following expressions.

$$V_0(0) = \frac{\gamma\beta(\eta+\mu) - \gamma[\lambda E(I)(\beta+\eta) + \eta\beta]}{\beta\mu}$$

$$\rho = \frac{\lambda E(I)(\beta+\eta)}{\beta\mu}$$

$$L_q = \frac{D'(1)N''(1) - N'(1)D''(1)}{2D'(1)^2} V_0(0) + \frac{\lambda E(I)}{\gamma^2} V_0(0)$$

where

$$\begin{aligned}
N'(1) =& \gamma\lambda\eta E(I)(\beta+\eta) \\
N''(1) =& \eta(\eta+\mu)[2\lambda^2(E(I))^2(\beta+\eta) + \lambda\gamma((\beta+\eta)E(I(I-1)) \\
& + 2E(I)(\eta - \lambda E(I)))] - 2\lambda^2\mu\gamma(E(I))^2(\beta+\eta) \\
D'(1) =& \eta\beta(\eta+\mu) - \eta[\lambda E(I)(\beta+\eta) + \eta\beta] \\
D''(1) =& \eta(\eta+\mu)[-\lambda E(I(I-1))(\beta+\eta) + 2\lambda^2(E(I))^2] \\
& - 2\lambda(\beta+\eta)E(I)[(\eta+\mu)^2 - \lambda\mu E(I)] + 2\lambda\eta\beta\mu E(I)
\end{aligned}$$

6.9 Numerical results

In order to see the effect of various parameters on utilization factor ρ and various other queue characteristics such as L, L_q. We base our numerical example on the result found in case 2.

For this purpose in Table 6.1, we choose the following arbitrary values: $E(I)$=0.3, $E(I(I-1))$= 0.04, η= 3, β= 4 , $\mu = 5$, $\gamma = 2$ while λ varies from 0.1 to 1.0 such that the stability condition is satisfied.

The Table 6.1 clearly shows as long as increasing the arrival rate, the utilization factor, the average queue size and average system size of our queueing model are all increases.

In Table 6.2, we can choose the following arbitrary values: $\eta = 1$, $\beta = 9$, $\lambda = 1$, $E(I)$=0.1, $E(I(I-1))$= 0.02, γ =4, p =0.6 while μ varies from 1 to 10 such that the stability condition is satisfied.

Table 6.1: Computed values of various queue characteristics

λ	ρ	L_q	L
0.1	0.010500	0.033090	0.043590
0.2	0.021000	0.088705	0.109705
0.3	0.031000	0.167169	0.198669
0.4	0.042000	0.268822	0.310822
0.5	0.052500	0.394018	0.446518
0.6	0.063000	0.543128	0.606128
0.7	0.073500	0.716539	0.790039
0.8	0.084000	0.914656	0.998656
0.9	0.094500	1.137903	1.232403
1.0	0.105000	1.386723	1.491723

Table 6.2: Computed values of various queue characteristics

μ	ρ	L_q	L
1	0.112500	1.019793	1.132293
2	0.056250	0.482087	0.538337
3	0.037500	0.337050	0.374550
4	0.028125	0.270386	0.298511
5	0.022500	0.232197	0.254697
6	0.018750	0.207477	0.226227
7	0.016071	0.190178	0.206249
8	0.014063	0.177397	0.191460
9	0.012500	0.167571	0.180071
10	0.011250	0.159781	0.171031

The Table 6.2 clearly shows as long as increasing the service rate, the utilization factor, the average queue size and average system size of our queueing model are all decreases.

CHAPTER SEVEN

$M^{[X]}/G/1$ Queue with Two Stage Heterogeneous Service, Random Breakdown, Delayed Repairs and Extended Server Vacations with Bernoulli Schedule

CHAPTER

SEVEN

$M^{[X]}/G/1$ QUEUE WITH TWO STAGE HETEROGENEOUS SERVICE, RANDOM BREAKDOWN, DELAYED REPAIRS AND EXTENDED SERVER VACATIONS WITH BERNOULLI SCHEDULE

7.1 Introduction

Server vacation models are useful for the systems in which the server wants to utilize the idle time for different purposes. In fact, queueing systems with server breakdowns are very common in communication systems and manufacturing systems. The study on two phases queueing system with vacation have become an interesting area in queueing theory. Many researchers have put their efforts in this area by considering various aspects like two phase queueing system with Bernoulli feedback, random break downs, Bernoulli vacation etc.

Also queueing systems with breakdowns have been studied by several authors including Federgruen and Green (1986), Tang (1997), Li et al. (1997),

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Aissani and Artalejo (1998), Madan et al. (2003), Choudhury and Tadj (2009) and Deepak Gupta et al. (2011). Wang (2004) studied an M/G/1 queue with a second optional service and server breakdowns.

Recently Maraghi et al. (2009) have studied some queueing systems with vacations and breakdowns. Thangaraj and Vanitha (2010a) have obtained transient solution for M/G/1 queue with two-stage heterogeneous service with compulsory server vacation and random breakdowns. Khalaf et al. (2010) studied an $M^{[X]}/G/1$ queue with Bernoulli schedule general vacation times, random breakdowns, general delay times for repairs to start and general repair times. They have obtained steady state results in terms of the probability generating functions for the number of customers in the queue. Choudhury and Madan (2005) and Madan (2000a) have studied two stage service with server vacations.

In this chapter, we consider $M^{[X]}/G/1$ queue with two stage service, random breakdown, delayed repairs and extended server vacations. Customers arrive at the system in batches of variable size in a compound Poisson process and they are provided one by one service on a first come - first served basis. The server provides two stages of service which is essential for all customers with service times having general (arbitrary) distribution. As soon as the second stage of a customer's service is completed, the server will take a vacation with probability p or may continue to stay in the system with probability $1 - p$. On completion of first phase of vacation, the server has the further option of taking an extended vacation. We assume that with probability r the server takes an extended vacation and with probability $1 - r$ rejoins the system immediately after completion of phase one vacation. The system may break down at random and breakdowns are assumed to occur according to a Poisson. Further, we assume that once the system breaks down, its repairs do not start immediately and there is a delay time, the customer whose service is interrupted comes back to the head of the queue. Repair times, delay times and vacation times follow general (arbitrary) distribution.

Here we derive time dependent probability generating functions in terms of Laplace transforms. We also derive the average queue size, system size and average waiting time in the queue, the system. Some particular cases and numerical results are also discussed.

The rest of this chapter is organized as follows. The mathematical description of our model is given in section 7.2. Definitions and equations governing the system are given in section 7.3. The time dependent solution have been obtained in section 7.4 and corresponding steady state results have been derived explicitly in section 7.5. Average queue size, system size and average waiting time in the queue, system are computed in section 7.6. Some particular cases and numerical results are discussed in section 7.7 and 7.8 respectively.

7.2 Mathematical description of the model

We assume the following to describe the queueing model of our study.

a) Customers arrive at the system in batches of variable size in a compound Poisson process and they are provided one by one service on a first come - first served basis. Let $\lambda c_i dt$ $(i = 1, 2, . . .)$ be the first order probability that a batch of i customers arrives at the system during a short interval of time $(t, t + dt]$, where $0 \leq c_i \leq 1$, $\sum_{i=1}^{\infty} c_i = 1$ and $\lambda > 0$ is the arrival rate of batches.

b) The server provides two stages of service which is essential for all customers. The service time follows a general (arbitrary) distribution with distribution function $B_i(s)$ and density function $b_i(s)$. Let $\mu_i(x)dx$ be the conditional probability density of service completion during the interval

$(x, x+dx]$, given that the elapsed service time is x, so that

$$\mu_i(x) = \frac{b_i(x)}{1 - B_i(x)}, \quad i = 1, 2,$$

and therefore,

$$b_i(s) = \mu_i(s) e^{-\int_0^s \mu_i(x)dx}, \quad i = 1, 2.$$

c) As soon as the second stage of a customer's service is completed, the server will take a vacation with probability p or may continue to stay in the system with probability $1 - p$. On completion of first phase of vacation, the server has the further option of taking an extended vacation. We assume that with probability r, the server takes an extended vacation and with probability $1 - r$ rejoins the system immediately after completion of phase one vacation.

d) The server's vacation time follows a general (arbitrary) distribution with distribution function V(t) and density function v(t). Let $\beta_i(x)dx$ be the conditional probability of a completion of a vacation during the interval $(x, x+dx]$, given that the elapsed vacation time is x, so that

$$\beta_i(x) = \frac{v(x)}{1 - V(x)}, \quad i = 1, 2,$$

and therefore,

$$v(t) = \beta_i(t) e^{-\int_0^t \beta_i(x)dx}, \quad i = 1, 2.$$

e) The system may break down at random and breakdowns are assumed to occur according to a Poisson stream with mean breakdown rate $\alpha > 0$. Further we assume that once the system breaks down, its repairs do not start immediately and there is a delay time, the customer whose service is interrupted comes back to the head of the queue.

f) The delay times follow a general (arbitrary) distribution with distribution function $F(x)$ and density function $f(x)$. Let $\theta(x)dx$ be the conditional probability of a completion of a delay during the interval $(x, x+dx]$, given that the elapsed delay time is x, so that

$$\theta(x) = \frac{f(x)}{1 - F(x)}$$

and therefore,

$$f(t) = \theta(t)e^{-\int\limits_0^t \theta(x)dx}.$$

g) The duration of repairs follows a general (arbitrary) distribution with distribution function $G(x)$ and density function $g(x)$. Let $\gamma(x)dx$ be the conditional probability of a completion of repairs during the interval $(x, x+dx]$, given that the elapsed repair time is x, so that

$$\gamma(x) = \frac{g(x)}{1 - G(x)}$$

and therefore,

$$g(t) = \gamma(t)e^{-\int\limits_0^t \gamma(x)dx}.$$

i) Various stochastic processes involved in the system are assumed to be independent of each other.

7.3 Definitions and equations governing of the system

We define

$P_n^{(1)}(x,t)$ = Probability that at time t, the server is active providing first stage of service and there are n $(n \geq 0)$ customers in the queue excluding the one being served and the elapsed service time is x. Accordingly, $P_n^{(1)}(t) = \int_0^\infty P_n^{(1)}(x,t)dx$ denotes the probability that at time t there are n customers in the queue excluding the one customer in the first stage of service irrespective of the value of x.

$P_n^{(2)}(x,t)$ = Probability that at time t, the server is active providing second stage of service and there are n $(n \geq 0)$ customers in the queue excluding the one being served and the elapsed service time is x. Accordingly, $P_n^{(2)}(t) = \int_0^\infty P_n^{(2)}(x,t)dx$ denotes the probability that at time t there are n customers in the queue excluding the one customer in the second stage of service irrespective of the value of x.

$V_n^{(1)}(x,t)$ = Probability that at time t, the server is under phase one vacation with elapsed vacation time is x and there are n $(n \geq 0)$ customers in the queue. Accordingly $V_n^{(1)}(t)=\int_0^\infty V_n^{(1)}(x,t)dx$ denotes the probability that at time t there are n customers in the queue and the server is under phase one vacation irrespective of the value of x.

$V_n^{(2)}(x,t)$ = Probability that at time t, the server is under extended vacation with elapsed vacation time is x and there are n $(n \geq 0)$ customers in the queue. Accordingly $V_n^{(2)}(t)=\int_0^\infty V_n^{(2)}(x,t)dx$ denotes the probability that at time t there are n customers in the queue and the server is under extended vacation irrespective of the value of x.

$D_n(x,t)$ = Probability that at time t, there are n $(n \geq 0)$ customers in the queue and the server is inactive due to system breakdown and waiting for repairs to start with elapsed delay time is x. Accordingly $D_n(t)= \int_0^\infty D_n(x,t)dx$ denotes the probability that at time t, there are n customers in the queue and the server is waiting for repairs to start irrespective of the value of x.

$R_n(x,t)$ = probability that at time t, there are n $(n \geq 0)$ customers in the queue, and the server is under repair with elapsed repair time is x.

Accordingly $R_n(t)=\int_0^\infty R_n(x,t)dx$ denotes the probability that at time t, there are n customers in the queue and the server is under repair irrespective of the value of x.

$Q(t)$ is the probability that at time t, there are no customers in the system and the server is idle but available in the system.

The system is then governed by the following set of differential-difference equations:

$$\frac{\partial}{\partial x}P_0^{(1)}(x,t)+\frac{\partial}{\partial t}P_0^{(1)}(x,t)+[\lambda+\mu_1(x)+\alpha]P_0^{(1)}(x,t)=0 \tag{7.1}$$

$$\frac{\partial}{\partial x}P_n^{(1)}(x,t)+\frac{\partial}{\partial t}P_n^{(1)}(x,t)+[\lambda+\mu_1(x)+\alpha]P_n^{(1)}(x,t)=\lambda\sum_{k=1}^{n}c_kP_{n-k}^{(1)}(x,t),\quad n\geq 1 \tag{7.2}$$

$$\frac{\partial}{\partial x}P_0^{(2)}(x,t)+\frac{\partial}{\partial t}P_0^{(2)}(x,t)+[\lambda+\mu_2(x)+\alpha]P_0^{(2)}(x,t)=0 \tag{7.3}$$

$$\frac{\partial}{\partial x}P_n^{(2)}(x,t)+\frac{\partial}{\partial t}P_n^{(2)}(x,t)+[\lambda+\mu_2(x)+\alpha]P_n^{(2)}(x,t)=\lambda\sum_{k=1}^{n}c_kP_{n-k}^{(2)}(x,t),\quad n\geq 1 \tag{7.4}$$

$$\frac{\partial}{\partial x}V_0^{(1)}(x,t)+\frac{\partial}{\partial t}V_0^{(1)}(x,t)+[\lambda+\beta_1(x)]V_0^{(1)}(x,t)=0 \tag{7.5}$$

$$\frac{\partial}{\partial x}V_n^{(1)}(x,t)+\frac{\partial}{\partial t}V_n^{(1)}(x,t)+[\lambda+\beta_1(x)]V_n^{(1)}(x,t)=\lambda\sum_{k=1}^{n}c_kV_{n-k}^{(1)}(x,t),\quad n\geq 1 \tag{7.6}$$

$$\frac{\partial}{\partial x}V_0^{(2)}(x,t)+\frac{\partial}{\partial t}V_0^{(2)}(x,t)+[\lambda+\beta_2(x)]V_0^{(2)}(x,t)=0 \tag{7.7}$$

$$\frac{\partial}{\partial x}V_n^{(2)}(x,t)+\frac{\partial}{\partial t}V_n^{(2)}(x,t)+[\lambda+\beta_2(x)]V_n^{(2)}(x,t)=\lambda\sum_{k=1}^{n}c_kV_{n-k}^{(2)}(x,t),\quad n\geq 1 \tag{7.8}$$

$$\frac{\partial}{\partial x}D_0(x,t)+\frac{\partial}{\partial x}D_0(x,t)+[\lambda+\theta(x)]D_0(x,t)=0 \tag{7.9}$$

$$\frac{\partial}{\partial x}D_n(x,t)+\frac{\partial}{\partial x}D_n(x,t)+[\lambda+\theta(x)]D_n(x,t)=\lambda\sum_{k=1}^{n}c_kD_{n-k}(x,t),$$
$$n\geq 1 \qquad (7.10)$$

$$\frac{\partial}{\partial x}R_0(x,t)+\frac{\partial}{\partial x}R_0(x,t)+[\lambda+\gamma(x)]R_0(x,t)=0 \qquad (7.11)$$

$$\frac{\partial}{\partial x}R_n(x,t)+\frac{\partial}{\partial x}R_n(x,t)+[\lambda+\gamma(x)]R_n(x,t)=\lambda\sum_{k=1}^{n}c_kR_{n-k}(x,t),$$
$$n\geq 1 \qquad (7.12)$$

$$\begin{aligned}\frac{d}{dt}Q(t)=&-\lambda Q(t)+(1-p)\int_0^\infty P_0^{(2)}(x,t)\mu_2(x)dx\\&+\int_0^\infty R_0(x,t)\gamma(x)dx+(1-r)\int_0^\infty V_0^{(1)}(x,t)\beta_1(x)dx\\&+\int_0^\infty V_0^{(2)}(x,t)\beta_2(x)dx\end{aligned} \qquad (7.13)$$

The above set of equations are to be solved subject to the following boundary conditions:

$$\begin{aligned}P_n^{(1)}(0,t)=&\lambda c_{n+1}Q(t)+(1-p)\int_0^\infty P_{n+1}^{(2)}(x,t)\mu_2(x)dx\\&+(1-r)\int_0^\infty V_{n+1}^{(1)}(x,t)\beta_1(x)dx+\int_0^\infty V_{n+1}^{(2)}(x,t)\beta_2(x)dx\\&+\int_0^\infty R_{n+1}(x,t)\gamma(x)dx,\quad n\geq 0\end{aligned} \qquad (7.14)$$

$$P_n^{(2)}(0,t)=\int_0^\infty P_n^{(1)}(x,t)\mu_1(x)dx,\quad n\geq 0 \qquad (7.15)$$

$$V_n^{(1)}(0,t)=p\int_0^\infty P_n^{(2)}(x,t)\mu_2(x)dx,\quad n\geq 0 \qquad (7.16)$$

$$V_n^{(2)}(0,t)=r\int_0^\infty V_n^{(1)}(x,t)\beta_1(x)dx,\quad n\geq 0 \qquad (7.17)$$

$$D_0(0,t)=0 \qquad (7.18)$$

$$D_n(0,t)=\alpha\int_0^\infty P_{n-1}^{(1)}(x,t)dx+\alpha\int_0^\infty P_{n-1}^{(2)}(x,t)dx,\ n\geq 1 \qquad (7.19)$$

$$R_n(0,t)=\int_0^\infty D_n(x,t)\theta(x)dx,\quad n\geq 0 \qquad (7.20)$$

We assume that initially there are no customers in the system and the server is idle. So the initial conditions are

$$P_n^{(i)}(0) = V_n^{(i)}(0) = 0, \text{ for } i = 1, 2, \quad Q(0) = 1 \text{ and}$$
$$D_n(0) = 0, R_n(0) = 0 \text{ for } n = 0, 1, 2, ... \tag{7.21}$$

7.4 Probability generating functions of the queue length: The time - dependent solution

In this section, we obtain the transient solution for the above set of differential - difference equations.

Theorem: *The system of differential difference equations to describe an* $M^{[X]}/G/1$ *Queue with Two Stage Heterogeneous Service, Random Breakdown, Delayed Repairs and Extended Server Vacations with Bernoulli Schedule are given by equations (7.1) to (7.20) with initial conditions (7.21) and the generating functions of transient solution are given by equations (7.84) to (7.89).*

Proof : We define the probability generating functions, for i =1, 2.

$$P^{(i)}(x, z, t) = \sum_{n=0}^{\infty} z^n P_n^{(i)}(x, t); \quad P^{(i)}(z, t) = \sum_{n=0}^{\infty} z^n P_n^{(i)}(t);$$

$$V^{(i)}(x, z, t) = \sum_{n=0}^{\infty} z^n V_n^{(i)}(x, t); \quad V^{(i)}(z, t) = \sum_{n=0}^{\infty} z^n V_n^{(i)}(t);$$

$$D(x, z, t) = \sum_{n=0}^{\infty} z^n D_n(x, t); \;\; D(z, t) = \sum_{n=0}^{\infty} z^n D_n(t); \quad C(z) = \sum_{n=1}^{\infty} c_n z^n;$$

$$R(x, z, t) = \sum_{n=0}^{\infty} z^n R_n(x, t); R(z, t) = \sum_{n=0}^{\infty} z^n R_n(t); \tag{7.22}$$

which are convergent inside the circle given by $| z | \leq 1$ and define the Laplace transform of a function $f(t)$ as

$$\bar{f}(s) = \int_0^\infty e^{-st} f(t) dt, \quad \Re(s) > 0.$$

Taking the Laplace transform of equations (7.1) to (7.20) and using (7.21), we obtain

$$\frac{\partial}{\partial x}\bar{P}_0^{(1)}(x,s) + (s+\lambda+\mu_1(x)+\alpha)\bar{P}_0^{(1)}(x,s) = 0 \tag{7.23}$$

$$\frac{\partial}{\partial x}\bar{P}_n^{(1)}(x,s) + (s+\lambda+\mu_1(x)+\alpha)\bar{P}_n^{(1)}(x,s) = \lambda\sum_{k=1}^{n} c_k \bar{P}_{n-k}^{(1)}(x,s), \quad n \geq 1 \tag{7.24}$$

$$\frac{\partial}{\partial x}\bar{P}_0^{(2)}(x,s) + (s+\lambda+\mu_2(x)+\alpha)\bar{P}_0^{(2)}(x,s) = 0 \tag{7.25}$$

$$\frac{\partial}{\partial x}\bar{P}_n^{(2)}(x,s) + (s+\lambda+\mu_2(x)+\alpha)\bar{P}_n^{(2)}(x,s) = \lambda\sum_{k=1}^{n} c_k \bar{P}_{n-k}^{(2)}(x,s), \quad n \geq 1 \tag{7.26}$$

$$\frac{\partial}{\partial x}\bar{V}_0^{(1)}(x,s) + (s+\lambda+\beta_1(x))\bar{V}_0^{(1)}(x,s) = 0 \tag{7.27}$$

$$\frac{\partial}{\partial x}\bar{V}_n^{(1)}(x,s) + (s+\lambda+\beta_1(x))\bar{V}_n^{(1)}(x,s) = \lambda\sum_{k=1}^{n} c_k \bar{V}_{n-k}^{(1)}(x,s), \ n \geq 1 \tag{7.28}$$

$$\frac{\partial}{\partial x}\bar{V}_0^{(2)}(x,s) + (s+\lambda+\beta_2(x))\bar{V}_0^{(2)}(x,s) = 0 \tag{7.29}$$

$$\frac{\partial}{\partial x}\bar{V}_n^{(2)}(x,s) + (s+\lambda+\beta_2(x))\bar{V}_n^{(2)}(x,s) = \lambda\sum_{k=1}^{n} c_k \bar{V}_{n-k}^{(2)}(x,s), \ n \geq 1 \tag{7.30}$$

$$\frac{\partial}{\partial x}\bar{D}_0(x,s) + (\lambda+\theta(x))\bar{D}_0(x,s) = 0 \tag{7.31}$$

$$\frac{\partial}{\partial x}\bar{D}_n(x,s) + (\lambda+\theta(x))\bar{D}_n(x,s) = \lambda\sum_{k=1}^{n} c_k \bar{D}_{n-k}(x,s), n \geq 1 \tag{7.32}$$

$$\frac{\partial}{\partial x}\bar{R}_0(x,s) + (\lambda+\gamma(x))\bar{R}_0(x,s) = 0 \tag{7.33}$$

$$\frac{\partial}{\partial x}\bar{R}_n(x,s)+(\lambda+\gamma(x))\bar{R}_n(x,s)=\lambda\sum_{k=1}^{n}c_k\bar{R}_{n-k}(x,s), n\geq 1 \tag{7.34}$$

$$\begin{aligned}(s+\lambda)\bar{Q}(s)=&1+(1-p)\int_0^\infty \bar{P}_0^{(2)}(x,s)\mu_2(x)dx\\&+(1-r)\int_0^\infty \bar{V}_0^{(1)}(x,s)\beta_1(x)dx\\&+\int_0^\infty \bar{R}_0(x,s)\gamma(x)dx+\int_0^\infty \bar{V}_0^{(2)}(x,s)\beta_2(x)dx\end{aligned} \tag{7.35}$$

$$\begin{aligned}\bar{P}_n^{(1)}(0,s)=&\lambda c_{n+1}\bar{Q}(s)+(1-p)\int_0^\infty \bar{P}_{n+1}^{(2)}(x,s)\mu_2(x)dx\\&+(1-r)\int_0^\infty \bar{V}_{n+1}^{(1)}(x,s)\beta_1(x)dx+\int_0^\infty \bar{V}_{n+1}^{(2)}(x,s)\beta_2(x)dx\\&+\int_0^\infty \bar{R}_{n+1}(x,s)\gamma(x)dx,\quad n\geq 0\end{aligned} \tag{7.36}$$

$$\bar{P}_n^{(2)}(0,s)=\int_0^\infty \bar{P}_n^{(1)}(x,s)\mu_1(x)dx,\ n\geq 0 \tag{7.37}$$

$$\bar{V}_n^{(1)}(0,s)=p\int_0^\infty \bar{P}_n^{(2)}(x,s)\mu_2(x)dx,\ n\geq 0 \tag{7.38}$$

$$\bar{V}_n^{(2)}(0,s)=r\int_0^\infty \bar{V}_n^{(1)}(x,s)\beta_1(x)dx,\ n\geq 0 \tag{7.39}$$

$$\bar{D}_0(0,s)=0 \tag{7.40}$$

$$\bar{D}_n(0,s)=\alpha\int_0^\infty \bar{P}_{n-1}^{(1)}(x,s)dx+\alpha\int_0^\infty \bar{P}_{n-1}^{(2)}(x,s)dx,\quad n\geq 1 \tag{7.41}$$

$$\bar{R}_n(0,s)=\int_0^\infty \bar{D}_n(x,s)\theta(x)dx,\quad n\geq 0 \tag{7.42}$$

Now multiplying equations (7.24), (7.26), (7.28), (7.30), (7.32) and (7.34) by z^n and summing over n from 1 to ∞, adding to equations (7.23), (7.25), (7.27), (7.29), (7.31), (7.33) and using the generating functions defined in (7.22), we get

$$\frac{\partial}{\partial x}\bar{P}^{(1)}(x,z,s)+[s+\lambda-\lambda C(z)+\mu_1(x)+\alpha]\bar{P}^{(1)}(x,z,s)=0 \tag{7.43}$$

$$\frac{\partial}{\partial x}\bar{P}^{(2)}(x,z,s)+[s+\lambda-\lambda C(z)+\mu_2(x)+\alpha]\bar{P}^{(2)}(x,z,s)=0 \tag{7.44}$$

$$\frac{\partial}{\partial x}\bar{V}^{(1)}(x,z,s)+[s+\lambda-\lambda C(z)+\beta_1(x)]\bar{V}^{(1)}(x,z,s)=0 \tag{7.45}$$

$$\frac{\partial}{\partial x}\bar{V}^{(2)}(x,z,s)+[s+\lambda-\lambda C(z)+\beta_2(x)]\bar{V}^{(2)}(x,z,s)=0 \tag{7.46}$$

$$\frac{\partial}{\partial x}\bar{D}(x,z,s)+[s+\lambda-\lambda C(z)+\theta(x)]\bar{D}(x,z,s)=0 \tag{7.47}$$

$$\frac{\partial}{\partial x}\bar{R}(x,z,s)+[s+\lambda-\lambda C(z)+\gamma(x)]\bar{R}(x,z,s)=0 \tag{7.48}$$

For the boundary conditions, we multiply both sides of equation (7.36) by z^n summing over n from 0 to ∞ and use the equations (7.22), we get

$$\begin{aligned} z\bar{P}^{(1)}(0,z,s)=&\lambda C(z)\bar{Q}(s)+(1-p)\int_0^\infty \bar{P}^{(2)}(x,z,s)\mu_2(x)dx \\ &-(1-p)\int_0^\infty \bar{P}_0^{(2)}(x,s)\mu_1(x)dx \\ &+(1-r)\int_0^\infty \bar{V}^{(1)}(x,z,s)\beta_1(x)dx \\ &-(1-r)\int_0^\infty \bar{V}_0^{(1)}(x,s)\beta_1(x)dx \\ &+\int_0^\infty \bar{V}^{(2)}(x,z,s)\beta_2(x)dx-\int_0^\infty \bar{V}_0^{(2)}(x,s)\beta_2(x)dx \\ &+\int_0^\infty \bar{R}(x,z,s)\gamma(x)dx-\int_0^\infty \bar{R}_0(x,s)\gamma(x)dx \end{aligned}$$

Using equation (7.35), the above equation becomes

$$\begin{aligned} z\bar{P}^{(1)}(0,z,s)=&(1-s\bar{Q}(s))+\lambda(C(z)-1)\bar{Q}(s) \\ &+(1-p)\int_0^\infty \bar{P}^{(2)}(x,z,s)\mu_2(x)dx \\ &+(1-r)\int_0^\infty \bar{V}^{(1)}(x,z,s)\beta_1(x)dx \\ &+\int_0^\infty \bar{V}^{(2)}(x,z,s)\beta_2(x)dx+\int_0^\infty \bar{R}(x,z,s)\gamma(x)dx \end{aligned} \tag{7.49}$$

Performing similar operation on equations (7.37) to (7.42), we get

$$\bar{P}^{(2)}(0,z,s) = \int_0^\infty \bar{P}^{(1)}(x,z,s)\mu_1(x)dx \tag{7.50}$$

$$\bar{V}^{(1)}(0,z,s) = p\int_0^\infty \bar{P}^{(2)}(x,z,s)\mu_2(x)dx \tag{7.51}$$

$$\bar{V}^{(2)}(0,z,s) = r\int_0^\infty \bar{V}^{(1)}(x,z,s)\beta_1(x)dx \tag{7.52}$$

$$\bar{D}(0,z,s) = \alpha z\int_0^\infty \bar{P}^{(1)}(x,z,s)dx + \alpha z\int_0^\infty \bar{P}^{(2)}(x,z,s)dx \tag{7.53}$$

$$\bar{R}(0,z,s) = \int_0^\infty \bar{D}(x,z,s)\theta(x)dx \tag{7.54}$$

Integrating equation (7.43) between 0 and x, we get

$$\bar{P}^{(1)}(x,z,s) = \bar{P}^{(1)}(0,z,s)e^{-[s+\lambda-\lambda C(z)+\alpha]x-\int\limits_0^x \mu_1(t)dt} \tag{7.55}$$

where $P^{(1)}(0,z,s)$ is given by equation (7.49).

Again integrating equation (7.55) by parts with respect to x, yields

$$\bar{P}^{(1)}(z,s) = \bar{P}^{(1)}(0,z,s)\left[\frac{1-\bar{B}_1(s+\lambda-\lambda C(z)+\alpha)}{s+\lambda-\lambda C(z)+\alpha}\right] \tag{7.56}$$

where

$$\bar{B}_1(s+\lambda-\lambda C(z)+\alpha) = \int\limits_0^\infty e^{-[s+\lambda-\lambda C(z)+\alpha]x}dB_1(x)$$

is the Laplace-Stieltjes transform of the first stage of service time $B_1(x)$. Now multiplying both sides of equation (7.55) by $\mu_1(x)$ and integrating over x, we obtain

$$\int\limits_0^\infty \bar{P}^{(1)}(x,z,s)\mu_1(x)dx = \bar{P}^{(1)}(0,z,s)\bar{B}_1[s+\lambda-\lambda c(z)+\alpha] \tag{7.57}$$

Similarly, on integrating equations (7.44) to (7.48) from 0 to x, we get

$$\bar{P}^{(2)}(x,z,s) = \bar{P}^{(2)}(0,z,s)e^{-[s+\lambda-\lambda C(z)+\alpha]x-\int\limits_0^x \mu_2(t)dt} \tag{7.58}$$

$$\bar{V}^{(1)}(x,z,s) = \bar{V}^{(1)}(0,z,s)e^{-[s+\lambda-\lambda C(z)]x-\int\limits_0^x \beta_1(t)dt} \tag{7.59}$$

$$\bar{V}^{(2)}(x,z,s) = \bar{V}^{(2)}(0,z,s)e^{-[s+\lambda-\lambda C(z)]x-\int\limits_0^x \beta_2(t)dt} \tag{7.60}$$

$$\bar{D}(x,z,s) = \bar{D}(0,z,s)e^{-[s+\lambda-\lambda C(z)]x-\int\limits_0^x \theta(t)dt} \tag{7.61}$$

$$\bar{R}(x,z,s) = \bar{R}(0,z,s)e^{-[s+\lambda-\lambda C(z)]x-\int\limits_0^x \gamma(t)dt} \tag{7.62}$$

where $\bar{P}^{(2)}(0,z,s)$, $\bar{V}^{(1)}(0,z,s)$, $\bar{V}^{(2)}(0,z,s)$, $\bar{D}(0,z,s)$ and $\bar{R}(0,z,s)$ are given by equations (7.50) to (7.54).

Again integrating equations (7.58) to (7.62) by parts with respect to x, yields

$$\bar{P}^{(2)}(z,s) = \bar{P}^{(2)}(0,z,s)\left[\frac{1-\bar{B}_2(s+\lambda-\lambda C(z)+\alpha)}{s+\lambda-\lambda C(z)+\alpha}\right] \tag{7.63}$$

$$\bar{V}^{(1)}(z,s) = \bar{V}^{(1)}(0,z,s)\left[\frac{1-\bar{V}_1(s+\lambda-\lambda C(z)}{s+\lambda-\lambda C(z)}\right] \tag{7.64}$$

$$\bar{V}^{(2)}(z,s) = \bar{V}^{(2)}(0,z,s)\left[\frac{1-\bar{V}_2(s+\lambda-\lambda C(z)}{s+\lambda-\lambda C(z))}\right] \tag{7.65}$$

$$\bar{D}(z,s) = \bar{D}(0,z,s)\left[\frac{1-\bar{F}(s+\lambda-\lambda C(z)}{s+\lambda-\lambda C(z)}\right] \tag{7.66}$$

$$\bar{R}(z,s) = \bar{R}(0,z,s)\left[\frac{1-\bar{G}(s+\lambda-\lambda C(z)}{s+\lambda-\lambda C(z)}\right] \tag{7.67}$$

where

$$\bar{B}_2(s+\lambda-\lambda C(z)+\alpha) = \int_0^\infty e^{-[s+\lambda-\lambda C(z)+\alpha]x}dB_2(x)$$

$$\bar{V}_1(s+\lambda-\lambda C(z)) = \int_0^\infty e^{-[s+\lambda-\lambda C(z)]x}dV_1(x)$$

$$\bar{V}_2(s+\lambda-\lambda C(z)) = \int_0^\infty e^{-[s+\lambda-\lambda C(z)]x} dV_2(x)$$

$$\bar{F}(s+\lambda-\lambda C(z)) = \int_0^\infty e^{-[s+\lambda-\lambda C(z)]x} dF(x)$$

$$\bar{G}(s+\lambda-\lambda C(z)) = \int_0^\infty e^{-[s+\lambda-\lambda C(z)]x} dG(x)$$

are the Laplace-Stieltjes transform of the second stage of service time $B_2(x)$, phase one vacation time $V_1(x)$, extended vacation time $V_2(x)$, delay time $F(x)$ and repair time $G(x)$.

Now multiplying both sides of equations (7.58) to (7.62) by $\mu_2(x)$, $\beta_1(x)$, $\beta_2(x)$, $\theta(x)$ and $\gamma(x)$ and integrating over x, we obtain

$$\int_0^\infty \bar{P}^{(2)}(x,z,s)\mu_2(x)dx = \bar{P}^{(2)}(0,z,s)\bar{B}_2[s+\lambda-\lambda C(z)+\alpha] \tag{7.68}$$

$$\int_0^\infty \bar{V}^{(1)}(x,z,s)\beta_1(x)dx = \bar{V}^{(1)}(0,z,s)\bar{V}_1[s+\lambda-\lambda C(z))] \tag{7.69}$$

$$\int_0^\infty \bar{V}^{(2)}(x,z,s)\beta_2(x)dx = \bar{V}^{(2)}(0,z,s)\bar{V}_2[s+\lambda-\lambda C(z))] \tag{7.70}$$

$$\int_0^\infty \bar{D}(x,z,s)\theta(x)dx = \bar{D}(0,z,s)\bar{F}[s+\lambda-\lambda C(z))] \tag{7.71}$$

$$\int_0^\infty \bar{R}(x,z,s)\gamma(x)dx = \bar{R}(0,z,s)\bar{G}[s+\lambda-\lambda C(z))] \tag{7.72}$$

Now, using equation (7.57) in (7.50), we get

$$\bar{P}^{(2)}(0,z,s) = \bar{B}_1(a)\bar{P}^{(1)}(0,z,s) \tag{7.73}$$

By using equations (7.68) and (7.73) in (7.51), we get

$$\bar{V}^{(1)}(0,z,s) = p\bar{B}(a)\bar{P}^{(1)}(0,z,s) \tag{7.74}$$

Now using equations (7.69) and (7.74) in (7.52), we get

$$\bar{V}^{(2)}(0,z,s) = rp\bar{B}(a)\bar{V}_1(b)\bar{P}^{(1)}(0,z,s) \tag{7.75}$$

Similarly, using equations (7.55), (7.58) and (7.73) in (7.53), we get

$$\bar{D}(0,z,s) = \alpha z\left[\frac{1-\bar{B}(a)}{a}\right]\bar{P}^{(1)}(0,z,s) \tag{7.76}$$

Now using equations (7.71) and (7.76) in (7.54), we get

$$\bar{R}(0,z,s) = \alpha z\bar{F}(b)\left[\frac{1-\bar{B}(a)}{a}\right]\bar{P}^{(1)}(0,z,s) \tag{7.77}$$

Using equations (7.68), (7.69) (7.70), (7.72) in (7.49), we get

$$\begin{aligned} z\bar{P}^{(1)}(0,z,s) =& [1-s\bar{Q}(s)] + (1-p)\bar{B}_2(a)\bar{P}^{(2)}(0,z,s) \\ &+ \lambda(C(z)-1)\bar{Q}(s) + (1-r)\bar{V}_1(b)\bar{V}^{(1)}(0,z,s) \\ &+ \bar{V}_2(b)\bar{V}^{(2)}(0,z,s) + \bar{G}(b)\bar{R}(0,z,s) \end{aligned}$$

where

$a = s+\lambda-\lambda C(z)+\alpha$, $b = s+\lambda-\lambda C(z)$ and $\bar{B}(a) = \bar{B}_1(a)\bar{B}_2(a)$.

Similarly using equations (7.73) to (7.75) and (7.77) in the above equation, we get

$$\bar{P}^{(1)}(0,z,s) = \frac{[1-sQ\bar{(s)}] + \lambda[C(z)-1]\bar{Q}(s)}{Dr} \tag{7.78}$$

where

$$\begin{aligned} Dr =& z - \bar{B}(a)[1 - p + p\bar{V}_1(b)(1 - r + r\bar{V}_2(b))] \\ & - \frac{\alpha z}{a}\bar{F}(b)\bar{G}(b)[1 - \bar{B}(a)] \end{aligned}$$

Substituting the value of $\bar{P}^{(1)}(0, z, s)$ from equation (7.78) into equations (7.73) to (7.77), we get

$$\bar{P}^{(2)}(0, z, s) = \bar{B}_1(a)\frac{[(1 - s\bar{Q}(s)) + \lambda(C(z) - 1)\bar{Q}(s)]}{Dr} \quad (7.79)$$

$$\bar{V}^{(1)}(0, z, s) = p\bar{B}(a)\frac{[(1 - s\bar{Q}(s)) + \lambda(C(z) - 1)\bar{Q}(s)]}{Dr} \quad (7.80)$$

$$\bar{V}^{(2)}(0, z, s) = rp\bar{B}(a)\bar{V}_1(b)\frac{[(1 - s\bar{Q}(s)) + \lambda(C(z) - 1)\bar{Q}(s)]}{Dr} \quad (7.81)$$

$$\bar{D}(0, z, s) = \alpha z\left[\frac{1 - \bar{B}(a)}{a}\right]\frac{[(1 - s\bar{Q}(s)) + \lambda(C(z) - 1)\bar{Q}(s)]}{Dr} \quad (7.82)$$

$$\bar{R}(0, z, s) = \alpha z\bar{F}(b)\left[\frac{1 - \bar{B}(a)}{a}\right]\frac{[(1 - s\bar{Q}(s)) + \lambda(C(z) - 1)\bar{Q}(s)]}{Dr} \quad (7.83)$$

Using equations (7.78) to (7.83) in (7.56), (7.63) to (7.67), we get

$$\bar{P}^{(1)}(z, s) = \frac{[1 - \bar{B}_1(a)]}{a}\frac{[(1 - s\bar{Q}(s)) + \lambda(C(z) - 1)\bar{Q}(s)]}{Dr} \quad (7.84)$$

$$\bar{P}^{(2)}(z, s) = \bar{B}_1(a)\frac{[1 - \bar{B}_2(a)]}{a}\frac{[(1 - s\bar{Q}(s)) + \lambda(C(z) - 1)\bar{Q}(s)]}{Dr} \quad (7.85)$$

$$\bar{V}^{(1)}(z, s) = p\bar{B}(a)\frac{[1 - \bar{V}_1(b)]}{b}\frac{[(1 - s\bar{Q}(s)) + \lambda(C(z) - 1)\bar{Q}(s)]}{Dr} \quad (7.86)$$

$$\bar{V}^{(2)}(z, s) = pr\bar{B}(a)\bar{V}_1(b)\frac{[1 - \bar{V}_2(b)]}{b}\frac{[(1 - s\bar{Q}(s)) + \lambda(C(z) - 1)\bar{Q}(s)]}{Dr} \quad (7.87)$$

$$\bar{D}(z, s) = \alpha z\frac{[1 - \bar{F}(b)]}{b}\frac{[1 - \bar{B}(a)]}{a}\frac{[(1 - s\bar{Q}(s)) + \lambda(C(z) - 1)\bar{Q}(s)]}{Dr} \quad (7.88)$$

$$\bar{R}(z, s) = \alpha z\bar{F}(b)\frac{[1 - \bar{G}(b)]}{b}\frac{[1 - \bar{B}(a)]}{a}\frac{[(1 - s\bar{Q}(s)) + \lambda(C(z) - 1)\bar{Q}(s)]}{Dr} \quad (7.89)$$

Thus $\bar{P}^{(1)}(z, s)$, $\bar{P}^{(2)}(z, s)$, $\bar{V}^{(1)}(z, s)$, $\bar{V}^{(2)}(z, s)$ $\bar{D}(z, s)$ and $\bar{R}(z, s)$ are completely determined from equations (7.84) to (7.89) which completes the proof of the theorem.

7.5 The steady state results

In this section, we shall derive the steady state probability distribution for our queueing model. These probabilities are obtained by suppressing the argument t wherever it appears in the time-dependent analysis. This can be obtained by applying the Tauberian property,

$$\lim_{s\to 0} s\bar{f}(s) = \lim_{t\to\infty} f(t)$$

In order to determine $\bar{P}^{(1)}(z,s)$, $\bar{P}^{(2)}(z,s)$, $\bar{V}^{(1)}(z,s)$, $\bar{V}^{(2)}(z,s)$ $\bar{D}(z,s)$ and $\bar{R}(z,s)$ completely, we have yet to determine the unknown Q which appears in the numerators of the right hand sides of equations (7.84) to (7.89). For that purpose, we shall use the normalizing condition

$$P^{(1)}(1) + P^{(2)}(1) + V^{(1)}(1) + V^{(2)}(1) + D(1) + R(1) + Q = 1$$

The steady state probabilities for an $M^{[X]}/G/1$ queue with two stage heterogeneous service, random breakdown, delayed repair and extended server vacation with Bernoulli schedule are given by

$$\begin{aligned}
P^{(1)}(1) =& \frac{\lambda E(I)[1-\bar{B}_1(\alpha)]Q}{\alpha d r_1} \\
P^{(2)}(1) =& \frac{\lambda E(I)\bar{B}_1(\alpha)[1-\bar{B}_2(\alpha)]Q}{\alpha d r_1} \\
V^{(1)}(1) =& \frac{\lambda p E(I)\bar{B}_1(\alpha)\bar{B}_2(\alpha)E(V)Q}{d r_1} \\
V^{(2)}(1) =& \frac{\lambda p r E(I)\bar{B}_1(\alpha)\bar{B}_2(\alpha)E(eV)Q}{d r_1}
\end{aligned}$$

$$
\begin{aligned}
D(1) =& \frac{\lambda E(I)(1-\bar{B}_1(\alpha)\bar{B}_2(\alpha))E(D)Q}{dr_1} \\
R(1) =& \frac{\lambda E(I)(1-\bar{B}_1(\alpha)\bar{B}_2(\alpha))E(R)Q}{dr_1}
\end{aligned}
$$

where

$$
\begin{aligned}
dr_1 =& -\lambda p\bar{B}_1(\alpha)\bar{B}_2(\alpha)E(I)[E(V)+rE(eV)] \\
&-\frac{\lambda E(I)}{\alpha}(1-\bar{B}_1(\alpha)\bar{B}_2(\alpha))(\alpha+\lambda E(I)) \\
&-\lambda E(I)(1-\bar{B}_1(\alpha)\bar{B}_2(\alpha))(E(D)+E(R))
\end{aligned}
$$

$P^{(1)}(1),\ P^{(2)}(1),\ V^{(1)}(1), V^{(2)}(1), D(1), R(1)$ and Q are the steady state probabilities that the server is providing first stage of service, second stage of service, server under phase one vacation, extended vacation, delay time, repair time and server under idle respectively without regard to the number of customers in the system.

Thus multiplying both sides of equations (7.84) to (7.89) by s, taking limit as $s \to 0$, applying Tauberian property and simplifying, we obtain

$$
P^{(1)}(z) = \frac{[\bar{B}_1(f_1(z))-1](f_2(z))Q}{f_1(z)dr} \tag{7.90}
$$

$$
P^{(2)}(z) = \frac{f_2(z)\bar{B}_1(f_1(z))[\bar{B}_2(f_1(z))-1]Q}{f_1(z)dr} \tag{7.91}
$$

$$
V^{(1)}(z) = \frac{p\bar{B}_1(f_1(z))\bar{B}_2(f_1(z))[\bar{V}_1(f_2(z))-1]Q}{dr} \tag{7.92}
$$

$$
V^{(2)}(z) = \frac{pr\bar{B}_1(f_1(z))\bar{B}_2(f_1(z))\bar{V}_1(f_2(z))[\bar{V}_2(f_2(z))-1]Q}{dr} \tag{7.93}
$$

$$
D(z) = \frac{\alpha z[\bar{F}(f_2(z))-1][1-\bar{B}_1(f_1(z))\bar{B}_2(f_1(z))]Q}{f_1(z)dr} \tag{7.94}
$$

$$
R(z) = \frac{\alpha z\bar{F}(f_2(z))[\bar{G}(f_2(z))-1][1-\bar{B}_1(f_1(z))\bar{B}_2(f_1(z))]Q}{f_1(z)dr} \tag{7.95}
$$

where

$$dr = z - \bar{B}_1(f_1(z))\bar{B}_2(f_1(z))[1 - p + p\bar{V}_1(f_2(z))(1 - r + r\bar{V}_2(f_2(z))] \\ - \frac{\alpha z}{f_1(z)}\bar{F}(f_2(z))\bar{G}(f_2(z))[1 - \bar{B}_1(f_1(z))\bar{B}_2(f_1(z))],$$

$f_1(z) = \lambda - \lambda C(z) + \alpha$ and $f_2(z) = \lambda - \lambda C(z)$.

Let $W_q(z)$ denote the probability generating function of the queue size irrespective of the state of the system. Then adding equations (7.90) to (7.95), we obtain

$$W_q(z) = P^{(1)}(z) + P^{(2)}(z) + V^{(1)}(z) + V^{(2)}(z) + D(z) + R(z)$$

$$\begin{aligned} W_q(z) =& \frac{[\bar{B}_1(f_1(z)) - 1](f_2(z))Q}{f_1(z)dr} \\ &+ \frac{(f_2(z))\bar{B}_1(f_1(z))[\bar{B}_2(f_1(z)) - 1]Q}{f_1(z)dr} \\ &+ \frac{p\bar{B}_1(f_1(z))\bar{B}_2(f_2(z))[\bar{V}_1(f_2(z)) - 1]Q}{dr} \\ &+ \frac{pr\bar{B}_1(f_1(z))\bar{B}_2(f_1(z))\bar{V}_1(f_2(z))[\bar{V}_2(f_2(z)) - 1]Q}{dr} \\ &+ \frac{\alpha z[\bar{F}(f_2(z)) - 1][1 - \bar{B}_1(f_1(z))\bar{B}_2(f_1(z))]Q}{f_1(z)dr} \\ &+ \frac{\alpha z\bar{F}(f_2(z))[\bar{G}(f_2(z)) - 1][1 - \bar{B}_1(f_1(z))\bar{B}_2(f_1(z))]Q}{f_1(z)dr} \end{aligned} \tag{7.96}$$

we see that for z=1, $W_q(z)$ is indeterminate of the form 0/0. Therefore, we apply L'Hopital's rule and on simplifying, we obtain

$$W_q(1) = \frac{\lambda E(I)Q[(1 - \bar{B}_1(\alpha)\bar{B}_2(\alpha))N + p\alpha\bar{B}_1(\alpha)\bar{B}_2(\alpha)M]}{-\lambda E(I)[1 - \bar{B}_1(\alpha)\bar{B}_2(\alpha)]N - \lambda p\alpha E(I)\bar{B}_1(\alpha)\bar{B}_2(\alpha)M + \alpha\bar{B}_1(\alpha)\bar{B}_2(\alpha)}$$

where

$N = 1 + \alpha(E(D) + E(R))$ and $M = E(V) + rE(eV)$, C(1)= 1, $C'(1) = E(I)$ is mean batch size of the arriving customers, $-\bar{V}_1'(0) = E(V)$ the mean first phase of vacation time, $-\bar{V}_2'(0) = E(eV)$ the mean extended vacation time, $-\bar{F}'(0) = E(D)$ the mean delay time and $-\bar{G}'(0) = E(R)$ the mean repair time.

Therefore adding Q to the above equation and equating to 1, simplifying, we get

$$Q = 1 - \rho \tag{7.97}$$

and hence the utilization factor ρ of the system is given by

$$\rho = \lambda E(I) \left[\frac{1}{\alpha \bar{B}_1(\alpha)\bar{B}_2(\alpha)} + \frac{(E(D) + (ER))}{\bar{B}_1(\alpha)\bar{B}_2(\alpha)} - \frac{1}{\alpha} - E(D) - E(R) + pM \right] \tag{7.98}$$

where $\rho < 1$ is the stability condition under which the steady state exists. Equation (7.97) gives the probability that the server is idle.

Substituting Q from (7.97) into (7.96), we have completely and explicitly determined $W_q(z)$, the probability generating function of the queue size.

7.6 The average queue size and the average waiting time

Let L_q denote the mean number of customers in the queue under the steady state. Then

$$L_q = \frac{d}{dz} W_q(z) \ \text{ at } z = 1$$

since this formula gives indeterminate of the form 0/0, then we write $W_q(z)$ given in (7.96) as $W_q(z) = \dfrac{N(z)}{D(z)}Q$ where

$$
\begin{aligned}
N(z) =& - [1 - \bar{B}_1(f_1(z))\bar{B}_2(f_1(z))][f_2(z) \\
& + \alpha z(1 - \bar{F}(f_2(z))\bar{G}(f_2(z)))] - pf_1(z))\bar{B}_1(f_1(z))\bar{B}_2(f_1(z)) \\
& \times [1 - \bar{V}_1(f_2(z))(1 - r + r\bar{V}_2(f_2(z))] \\
D(z) =& f_1(z)[z - \bar{B}_1(f_1(z))\bar{B}_2(f_1(z))(1 - p + p\bar{V}_1(f_2(z))(1 - r + r\bar{V}_2(f_2(z)))] \\
& - \alpha z\bar{F}(f_2(z))\bar{G}(f_2(z))[1 - \bar{B}_1(f_1(z))\bar{B}_2(f_1(z))] \\
N'(z) =& [\bar{B}_1'(f_1(z))f_1'(z)\bar{B}_2(f_1(z)) + \bar{B}_1(f_1(z))\bar{B}_2'(f_1(z))f_1'(z)] \\
& \times [f_2(z) + \alpha z(1 - \bar{F}(f_2(z))\bar{G}(f_2(z)))] - [1 - \bar{B}_1(f_1(z))\bar{B}_2(f_1(z))] \\
& \times [f_2'(z) + \alpha(1 - \bar{F}(f_2(z))\bar{G}(f_2(z))) \\
& - \alpha z(\bar{F}'(f_2(z))f_2'(z)\bar{G}(f_2(z)) + \bar{F}(f_2(z))\bar{G}'(f_2(z))f_2'(z))] \\
& - p[1 - \bar{V}_1(f_2(z))(1 - r + r\bar{V}_2(f_2(z))] \\
& \times [f_1'(z))\bar{B}_1(f_1(z))\bar{B}_2(f_1(z)) \\
& + f_1(z)\bar{B}_1'(f_1(z))f_1'(z)\bar{B}_2(f_1(z)) + f_1(z)\bar{B}_1(f_1(z))\bar{B}_2'(f_1(z))f_1'(z)] \\
& + pf_1(z)\bar{B}_1(f_1(z))\bar{B}_2(f_1(z))[\bar{V}_1'(f_2(z))f_2'(z)(1 - r + r\bar{V}_2(f_2(z)) \\
& + \bar{V}_1(f_2(z))r\bar{V}_2'(f_2(z))f_2'(z)] \\
D'(z) =& f_1'(z)[z - \bar{B}_1(f_1(z))\bar{B}_2(f_1(z))(1 - p + p\bar{V}_1(f_2(z)) \\
& (1 - r + r\bar{V}_2(f_2(z)))] + f_1(z)[1 - (\bar{B}_1'(f_1(z))f_1'(z)\bar{B}_2(f_1(z)) \\
& + \bar{B}_1(f_1(z))\bar{B}_2'(f_1(z))f_1'(z))(1 - p + p\bar{V}_1(f_2(z)) \\
& \times (1 - r + r\bar{V}_2(f_2(z))) - \bar{B}_1(f_1(z))\bar{B}_2(f_1(z)) \\
& \times (p\bar{V}_1'(f_2(z))f_2'(z)(1 - r + r\bar{V}_2(f_2(z))) \\
& + p\bar{V}_1(f_2(z))r\bar{V}_2'(f_2(z))f_2'(z))] \\
& - \alpha\bar{F}(f_2(z))\bar{G}(f_2(z))[1 - \bar{B}_1(f_1(z))\bar{B}_2(f_1(z))] \\
& - \alpha z[\bar{F}'(f_2(z))f_2'(z)\bar{G}(f_2(z)) + \bar{F}(f_2(z))\bar{G}'(f_2(z))f_2'(z))]
\end{aligned}
$$

$$\times [1 - \bar{B}_1(f_1(z))\bar{B}_2(f_1(z))] + \alpha z\bar{F}(f_2(z))\bar{G}(f_2(z))$$
$$\times [\bar{B}_1'(f_1(z))f_1'(z)\bar{B}_2(f_1(z)) + \bar{B}_1(f_1(z))\bar{B}_2'(f_1(z))f_1'(z)]$$

Then, we use

$$\begin{aligned} L_q &= \lim_{z\to 1} \frac{d}{dz} W_q(z) \\ &= \lim_{z\to 1} \left[\frac{D'(z)N''(z) - N'(z)D''(z)}{2(D'(z))^2}\right] Q \\ &= \left[\frac{D'(1)N''(1) - N'(1)D''(1)}{2(D'(1))^2}\right] Q \end{aligned} \tag{7.99}$$

where primes and double primes in the above equation denote the first and second derivatives at $z = 1$ respectively. Carrying out the derivative at $z = 1$, we have

$$N'(1) = \lambda E(I)[(1 - \bar{B}_1(\alpha)\bar{B}_2(\alpha))N + p\alpha\bar{B}_1(\alpha)\bar{B}_2(\alpha)M] \tag{7.100}$$

$$\begin{aligned} N''(1) =& (\lambda E(I))^2[\alpha(1 - \bar{B}_1(\alpha)\bar{B}_2(\alpha))[E(D^2) + E(R^2) \\ &+ 2E(D)E(R)] + 2[\bar{B}_1'(\alpha)\bar{B}_2(\alpha) + \bar{B}_1(\alpha)\bar{B}_2'(\alpha)]N \\ &+ p\alpha\bar{B}_1(\alpha)\bar{B}_2(\alpha)[E(V^2) + 2rE(V)E(eV) + rE(eV^2)] \\ &- 2pM(\bar{B}_1(\alpha)\bar{B}_2(\alpha) + \alpha\bar{B}_1'(\alpha)\bar{B}_2(\alpha) + \alpha\bar{B}_1(\alpha)\bar{B}_2'(\alpha))] \\ &+ \lambda E(I(I-1))[(1 - \bar{B}_1(\alpha)\bar{B}_2(\alpha))N + p\alpha\bar{B}_1(\alpha)\bar{B}_2(\alpha)M] \\ &+ 2\lambda\alpha E(I)[1 - \bar{B}_1(\alpha)\bar{B}_2(\alpha)](E(D) + E(R)) \end{aligned} \tag{7.101}$$

$$\begin{aligned} D'(1) =& -\lambda E(I)[1 - \bar{B}_1(\alpha)\bar{B}_2(\alpha)]N \\ &- \lambda p\alpha E(I)\bar{B}_1(\alpha)\bar{B}_2(\alpha)M + \alpha\bar{B}_1(\alpha)\bar{B}_2(\alpha) \end{aligned} \tag{7.102}$$

$$\begin{aligned} D''(1) =& -(\lambda E(I))^2\bar{B}_1(\alpha)\bar{B}_2(\alpha)[-2pM + \alpha p(E(V^2) \\ &+ 2rE(V)E(eV) + rE(eV^2)) \\ &- \alpha(E(D^2) + E(R^2) + 2E(D)E(R))] \\ &- \alpha(\lambda E(I))^2[E(D^2) + E(R^2) + 2E(D)E(R)] \end{aligned}$$

$$- \lambda E(I(I-1))[(1-\bar{B}_1(\alpha)\bar{B}_2(\alpha))N + \alpha p\bar{B}_1(\alpha)\bar{B}_2(\alpha)M]$$
$$- 2(\lambda E(I))^2[\bar{B}_1'(\alpha)\bar{B}_2(\alpha) + \bar{B}_1(\alpha)\bar{B}_2'(\alpha)](N - \alpha pM)$$
$$- 2\lambda E(I)[1 + \alpha(E(D) + E(R))(1 - \bar{B}_1(\alpha)\bar{B}_2(\alpha))$$
$$+ \alpha(\bar{B}_1'(\alpha)\bar{B}_2(\alpha) + \bar{B}_1(\alpha)\bar{B}_2'(\alpha))] \quad (7.103)$$

where $E(V^2), E(R^2), E(D^2), E(eV^2)$ are the second moment of phase one vacation time, repair time, delay time and the extended vacation time respectively. $E(I(I-1))$ is the second factorial moment of the batch size of arriving customers.

Then if we substitute the values $N'(1), N''(1), D'(1), D''(1)$ from equations (7.100) to (7.103) into equation (7.99), we obtain L_q in the closed form.

Further, we find the average system size L by using Little's formula. Thus we have

$$L = L_q + \rho \quad (7.104)$$

where L_q has been found by equation (7.99) and ρ is obtained from equation (7.98).

Let W_q and W denote the average waiting time in the queue and in the system respectively. Then by using Little's formula, we obtain

$$W_q = \frac{L_q}{\lambda}$$

$$W = \frac{L}{\lambda}$$

where L_q and L have been found in equations (7.99) and (7.104).

7.7 Particular cases

Case 1: If there is no delay for repairs to start, no extended vacation and no second stage service i.e, E(D)=0, $\bar{F}(b) = 1$, r =0 and $\bar{B}_2(\alpha) = 1$. Then our model reduces to a single server $M^{[X]}/G/1$ queue with random breakdown, phase one vacation.

In this case, we find the idle probability Q, utilization factor ρ and the average queue size L_q can be simplified to the following expressions.

$$
\begin{aligned}
Q =& 1-\rho \\
\rho =& \lambda E(I)[\frac{1}{\alpha\bar{B}_1(\alpha)} + \frac{E(R)}{\bar{B}_1(\alpha)} - \frac{1}{\alpha} - E(R) + pE(V)] \\
L_q =& \left[\frac{D'(1)N''(1) - N'(1)D''(1)}{2(D'(1))^2}\right] Q
\end{aligned}
$$

where

$$
\begin{aligned}
N'(1) =& \lambda E(I)[(1-\bar{B}_1(\alpha))(1+\alpha E(R)) + p\alpha\bar{B}_1(\alpha)E(V)] \\
N''(1) =& (\lambda E(I))^2[\alpha(1-\bar{B}_1(\alpha))E(R^2) + 2\bar{B}_1'(\alpha)E(R) \\
&+ \alpha p\bar{B}_1(\alpha)E(V^2) - 2pE(V)(\bar{B}_1(\alpha) + \alpha\bar{B}_1'(\alpha))] \\
&+ \lambda E(I(I-1))[(1-\bar{B}_1(\alpha))(1+\alpha E(R)) \\
&+ p\alpha\bar{B}_1(\alpha)E(V)] + 2\lambda\alpha E(I)E(R)[1-\bar{B}_1(\alpha)] \\
D'(1) =& -\lambda E(I)[1-\bar{B}_1(\alpha)][1+\alpha E(R)] \\
&- \lambda p\alpha E(I)\bar{B}_1(\alpha)E(V) + \alpha\bar{B}_1(\alpha) \\
D''(1) =& -(\lambda E(I))^2\bar{B}_1(\alpha)[\alpha pE(V^2) - 2pE(V) - \alpha E(R^2)] \\
&- \alpha(\lambda E(I))^2E(R^2) - \lambda E(I(I-1)) \\
&\times [(1-\bar{B}_1(\alpha))(1+\alpha E(R)) + \alpha p\bar{B}_1(\alpha)(E(V))] \\
&- 2(\lambda E(I))^2\bar{B}_1'(\alpha)[1-p\alpha(E(V)) + \alpha E(R)] \\
&- 2\lambda E(I)[1+\alpha E(R)(1-\bar{B}_1(\alpha)) + \alpha\bar{B}_1'(\alpha)]
\end{aligned}
$$

In the above equations if repair time is exponentially distributed then the result coincide with the result given by Maraghi et al. (2009).

Case 2: If there is no extended vacation and no second stage service i.e, r =0 and $\bar{B}_2(\alpha) = 1$. Then our model reduces to a single server $M^{[X]}/G/1$ queue with random breakdown, delayed repairs and phase one vacation.

In this case we find the idle probability Q, utilization factor ρ and the average queue size L_q can be simplified to the following expressions.

$$
\begin{aligned}
Q =& 1-\rho \\
\rho =& \lambda E(I)\left[\frac{1}{\alpha \bar{B}_1(\alpha)}+\frac{(E(D)+(ER))}{\bar{B}_1(\alpha)}-\frac{1}{\alpha}-E(D)-E(R)+pE(V)\right] \\
L_q =& \left[\frac{D'(1)N''(1)-N'(1)D''(1)}{2(D'(1))^2}\right]Q
\end{aligned}
$$

where

$$
\begin{aligned}
N'(1) =& \lambda E(I)[(1-\bar{B}_1(\alpha))(1+\alpha(E(D)+E(R))) \\
&+p\alpha \bar{B}_1(\alpha)E(V)] \\
N''(1) =& (\lambda E(I))^2[\alpha(1-\bar{B}_1(\alpha))(E(D^2)+E(R^2)+2E(D)E(R))] \\
&+2\bar{B}_1'(\alpha)[1+\alpha(E(D)+E(R))] \\
&+p\alpha \bar{B}_1(\alpha)E(V^2)-2pE(V)(\bar{B}_1(\alpha)+\alpha \bar{B}_1'(\alpha)) \\
&+\lambda E(I(I-1))[(1-\bar{B}_1(\alpha))(1+\alpha(E(D)+E(R))) \\
&+p\alpha \bar{B}_1(\alpha)E(V)] \\
&+2\lambda\alpha E(I)[1-\bar{B}_1(\alpha)][E(D)+E(R)] \\
D'(1) =& -\lambda E(I)[1-\bar{B}_1(\alpha)][1+\alpha(E(D)+E(R))] \\
&-\lambda p\alpha E(I)\bar{B}_1(\alpha)E(V)+\alpha \bar{B}_1(\alpha) \\
D''(1) =& -(\lambda E(I))^2\bar{B}_1(\alpha)[-2pE(V)+\alpha pE(V^2) \\
&-\alpha(E(D^2)+E(R^2)+2E(D)E(R))] \\
&-\alpha(\lambda E(I))^2[E(D^2)+E(R^2)+2E(D)E(R)] \\
&-\lambda E(I(I-1))[(1-\bar{B}_1(\alpha))(1+\alpha(E(D)+E(R)))
\end{aligned}
$$

$$
\begin{aligned}
&+ \alpha p\bar{B}_1(\alpha)E(V)] - 2(\lambda E(I))^2\bar{B}_1'(\alpha) \\
&\times [1 - p\alpha E(V) + \alpha(E(D) + E(R))] \\
&- 2\lambda E(I)[1 + \alpha(E(D) + E(R))(1 - \bar{B}_1(\alpha)) + \alpha(\bar{B}_1'(\alpha))]
\end{aligned}
$$

The above equations coincide with result given by Khalaf et al. (2010).

Case 3: If there is no delay for repairs to start, no extended vacation. Once the system breakdown, if its repairs start immediately and there is no delay time i.e, E(D)=0, $\bar{F}(b) = 1$. Once the first phase of vacation finish, the server is ready to start the service and there is no extended vacation time i.e, r=0.

If $E(I) = 1, E(I(I-1)) = 0$ then our model reduces to a single server $M/G/1$ queue with two stage service with random breakdown, delayed repairs and phase one vacation.

In this case we find the idle probability Q, utilization factor ρ and the average queue size L_q can be simplified to the following expressions.

$$
\begin{aligned}
Q =& 1 - \rho \\
\rho =& \lambda[\frac{1}{\alpha\bar{B}_1(\alpha)\bar{B}_2(\alpha)} + \frac{E(R)}{\bar{B}_1(\alpha)\bar{B}_2(\alpha)} - \frac{1}{\alpha} - E(R) + pE(V)] \\
L_q =& \left[\frac{D'(1)N''(1) - N'(1)D''(1)}{2(D'(1))^2}\right] Q
\end{aligned}
$$

where

$$
\begin{aligned}
N'(1) =& \lambda[(1 - \bar{B}_1(\alpha)\bar{B}_2(\alpha))(1 + \alpha E(R)) \\
&+ p\alpha\bar{B}_1(\alpha)\bar{B}_2(\alpha)E(V)] \\
N''(1) =& \lambda^2[\alpha(1 - \bar{B}_1(\alpha)\bar{B}_2(\alpha))E(R^2) \\
&+ 2(\bar{B}_1'(\alpha)\bar{B}_2(\alpha) + \bar{B}_1(\alpha)\bar{B}_2'(\alpha))(1 + \alpha E(R))
\end{aligned}
$$

$$
\begin{aligned}
&+ \alpha p\bar{B}_1(\alpha)\bar{B}_2(\alpha)E(V^2) - 2pE(V)(\bar{B}_1(\alpha)\bar{B}_2(\alpha) \\
&+ \alpha\bar{B}_1'(\alpha)\bar{B}_2(\alpha) + \alpha\bar{B}_1(\alpha)\bar{B}_2'(\alpha))] \\
&+ 2\lambda\alpha(1 - \bar{B}_1(\alpha)\bar{B}_2(\alpha))E(R) \\
D'(1) =& - \lambda[1 - \bar{B}_1(\alpha)\bar{B}_2(\alpha)](1 + \alpha E(R)) \\
&+ \alpha\bar{B}_1(\alpha)\bar{B}_2(\alpha) - \lambda p\alpha\bar{B}_1(\alpha)\bar{B}_2(\alpha)E(V) \\
D''(1) =& - \lambda^2\bar{B}_1(\alpha)\bar{B}_2(\alpha)[\alpha pE(V^2) - 2pE(V) - \alpha E(R^2)] - \lambda^2\alpha E(R^2) \\
&- 2\lambda^2[\bar{B}_1'(\alpha)\bar{B}_2(\alpha) + \bar{B}_1(\alpha)\bar{B}_2'(\alpha)][1 - p\alpha E(V) + \alpha E(R)] \\
&- 2\lambda[1 + \alpha E(R)(1 - \bar{B}_1(\alpha)\bar{B}_2(\alpha)) \\
&+ \alpha\bar{B}_1'(\alpha)\bar{B}_2(\alpha) + \alpha\bar{B}_1(\alpha)\bar{B}_2'(\alpha)]
\end{aligned}
$$

If repair times are exponentially distributed and p =1, then the above results coincide with Thangaraj and Vanitha (2010a).

7.8 Numerical results

To numerically illustrate the results obtained in this work, we consider that the service times, vacation times, delay times, extended vacation times and repair times are exponentially distributed. .

In order to see the effect of various parameters on server's idle time Q, utilization factor ρ and various other queue characteristics such as L, W, L_q, W_q. We base our numerical example on the result found in case 3.

For this purpose in Table 7.1, we can choose the following values: $\mu_1 = 9,\ \mu_2 = 8, \alpha = 1,\ \beta = 4,\ \gamma = 7$ and $p = 0.2$ while λ varies from 0.1 to 10 such that the stability condition is satisfied.

It clearly shows as long as increasing the arrival rate, the server's idle time decreases while the utilization factor, the average queue size, system size, the average waiting time in the queue and the system of our queueing model are all increases.

Table 7.1: Computed values of various queue characteristics

λ	Q	ρ	L_q	L	W_q	W
0.1	0.966400	0.033600	0.002305	0.035876	0.023045	0.358760
0.2	0.932900	0.067100	0.014198	0.081341	0.070990	0.406705
0.3	0.899300	0.100700	0.032922	0.133637	0.109741	0.445456
0.4	0.865700	0.134300	0.056262	0.190548	0.140655	0.476370
0.5	0.832100	0.167900	0.082436	0.250293	0.164872	0.500586
0.6	0.798600	0.201400	0.110012	0.311441	0.183354	0.519068
0.7	0.765000	0.235000	0.137841	0.372841	0.196916	0.532630
0.8	0.731400	0.268600	0.165002	0.433573	0.206252	0.541960
0.9	0.697900	0.302100	0.190758	0.492900	0.211953	0.547667
1.0	0.664300	0.335700	0.214524	0.550239	0.214524	0.550239

Table 7.2: Computed values of various queue characteristics

β	Q	ρ	L_q	L	W_q	W
1	0.888000	0.112000	0.079153	0.191184	0.263843	0.637280
2	0.903000	0.097000	0.074950	0.171981	0.249833	0.573270
3	0.908000	0.092000	0.072672	0.164703	0.242239	0.549009
4	0.910500	0.087500	0.071358	0.160889	0.237859	0.536297
5	0.912000	0.088000	0.070512	0.158543	0.235040	0.528478
6	0.913000	0.087000	0.069924	0.156955	0.233081	0.523185
7	0.913700	0.086300	0.069492	0.155809	0.231641	0.519364
8	0.914200	0.085800	0.069162	0.154943	0.230540	0.516477
9	0.914600	0.085400	0.068901	0.154265	0.229670	0.514218
10	0.915000	0.085000	0.068690	0.153721	0.228966	0.512403

In Table 7.2, we can choose the following arbitrary values: $\mu_1 = 9$, $\mu_2 = 8, \alpha = 0.5,\ \lambda = 0.3,\ \gamma = 4$ and $p = 0.1$ while β varies from 1 to 10 such that the stability condition is satisfied.

It clearly shows as long as increasing the vacation rate, the server's idle time increases while the utilization factor, average queue size, system size, average waiting time in the queue and system of our queueing model are all decreases.

CHAPTER EIGHT

$M^{[X]}/G/1$ Queue with Three Stage Service, Server Vacations and Service Interruptions

CHAPTER

EIGHT

$M^{[X]}/G/1$ QUEUE WITH THREE STAGE SERVICE, SERVER VACATIONS AND SERVICE INTERRUPTIONS

8.1 Introduction

Queueing system are powerful tool for modeling communication networks, transportation networks, production lines, operating systems, etc. In recent years, computer networks and data communication systems are the fastest growing technologies, which lead to glorious development in many applications. For example, the swift advance in internet, audio data traffic, video data traffic, etc. Many authors have discussed about two stages of services. In this chapter we have developed a three stages of services which will be more advantageous in large scale industries.

Queueing models with vacations have been investigated by many authors including Fuhrmann and Cooper (1985), Scholl and Kleinrock (1983), Shanthikumar (1988), Rosenberg and Yechiali (1993) and Arivudainambi and Godhandaraman (2012). Service interruptions are considered by Avi -Itzhak

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and Naor (1963), Thiruvengadan (1963), Federgruen and Green (1986), Basker et al. (2011) and Balamani (2012). Triple stages of service with service interruptions, have studied by Maragatha Sundari and Srinivasan (2012b).

In this chapter, we consider a $M^{[X]}/G/1$ queue with three stage of heterogeneous service provided by a single server with general (arbitrary) service time distribution, subject to random interruption and server vacation. Each customer undergoes three stages of heterogeneous service. However at the completion of third stage of service, the server will take compulsory vacation. After compulsory vacation the server may take optional vacation with probability p or return back to the system with probability $(1-p)$ for next service. While serving the customer, we assume interruptions arrive at random and assumed to occur according to a Poisson process with mean rate α. Let β be the server rate of attending interruption which are exponentially distributed. Also we assume, the customer whose service is interrupted goes back to the head of the queue where the arrivals are Poisson. We assume that the customers arrive to the system in batches of variable size, but are served one by one on a first come - first served basis.

Here we derive time dependent probability generating functions in terms of Laplace transforms. We also derive the mean queue size, system size and mean waiting time in the queue and the system. Some particular cases and numerical results are also discussed.

The rest of the chapter is organized as follows. Model description is given in section 8.2. Definitions and equations governing the system are given in section 8.3 and 8.4 respectively. The time dependent solution have been obtained in section 8.5 and corresponding steady state results have been derived explicitly in section 8.6. Mean queue size and mean waiting time are computed in section 8.7. Some particular cases and numerical results are discussed in section 8.8 and 8.9 respectively.

8.2 Model description

We assume the following to describe the queueing model of our study.

a) Customers arrive at the system in batches of variable size in a compound Poisson process and they are provided one by one service on a first come - first served basis. Let $\lambda c_i dt$ ($i = 1, 2, \ldots$) be the first order probability that a batch of i customers arrives at the system during a short interval of time $(t, t+dt]$, where $0 \leq c_i \leq 1$, $\sum_{i=1}^{\infty} c_i = 1$ and $\lambda > 0$ is the arrival rate of batches.

b) A single server provides three stages of different service for each customer, with the service times having general (arbitrary) distribution. Let $B_i(v)$ and $b_i(v)$ (i =1, 2, 3) be the distribution and the density function of the first stage, second stage and third stage service respectively.

c) Let $\mu_i(x)dx$ be the conditional probability density of service completion during the interval $(x, x+dx]$, given that the elapsed service time is x, so that

$$\mu_i(x) = \frac{b_i(x)}{1 - B_i(x)}, \quad i = 1, 2, 3,$$

and therefore,

$$b_i(s) = \mu_i(s) e^{-\int_0^s \mu_i(x)dx}, \quad i = 1, 2, 3.$$

d) As soon as the completion of each third stage of service, the server will take compulsory vacation. After completion of compulsory vacation the server may take optional vacation with probability p or return back to the system with probability $1-p$. On returning from vacation the server starts instantly serving the customer at the head of the queue, if any.

e) The server's vacation time follows a general (arbitrary) distribution with distribution function $V_j(t)$ and density function $v_j(t)$. Let $\gamma_j(x)dx$ be the conditional probability density of vacation completion during the interval $(x, x+dx]$, given that the elapsed vacation time is x, so that

$$\gamma_j(x) = \frac{v_j(x)}{1 - V_j(x)}, \quad j = 1, 2,$$

and therefore,

$$v_j(t) = \gamma_j(t) e^{-\int_0^t \gamma_j(x)dx}, \quad j = 1, 2.$$

h) While serving the customers, we assume interruptions arrive at random with rate $\alpha > 0$. Let β be the server rate of attending interruption which are exponentially distributed. Once the interruption arrives, the customer whose service is interrupted comes back to the head of the queue.

i) Various stochastic processes involved in the system are assumed to be independent of each other.

8.3 Definitions

We define

$P_n^{(1)}(x,t)$ = Probability that at time t, the server is active providing first stage of service and there are n $(n \geq 0)$ customers in the queue excluding the one being served and the elapsed service time is x. Consequently $P_n^{(1)}(t) = \int_0^\infty P_n^{(1)}(x,t)dx$ denotes the probability that at time t there are n customers in the queue excluding the one customer in the first stage of service irrespective of the value of x.

$P_n^{(2)}(x,t)$ = Probability that at time t, the server is active providing second stage of service and there are n $(n \geq 0)$ customers in the queue excluding the one being served and the elapsed service time is x. Consequently $P_n^{(2)}(t) = \int_0^\infty P_n^{(2)}(x,t)dx$ denotes the probability that at time t there are n customers in the queue excluding the one customer in the second stage of service irrespective of the value of x.

$P_n^{(3)}(x,t)$ = Probability that at time t, the server is active providing third stage of service and there are n $(n \geq 0)$ customers in the queue excluding the one being served and the elapsed service time is x. Consequently $P_n^{(3)}(t) = \int_0^\infty P_n^{(3)}(x,t)dx$ denotes the probability that at time t there are n customers in the queue excluding one customer in the third stage of service irrespective of the value of x.

$V_n^{(1)}(x,t)$ = Probability that at time t, the server is under compulsory vacation with elapsed vacation time is x and there are n $(n \geq 0)$ customers in the queue. Consequently $V_n^{(1)}(t)= \int_0^\infty V_n^{(1)}(x,t)dx$ denotes the probability that at time t there are n customers in the queue and the server is under compulsory vacation irrespective of the value of x.

$V_n^{(2)}(x,t)$ = Probability that at time t, the server is under optional vacation with elapsed vacation time is x and there are n $(n \geq 0)$ customers in the queue. Consequently $V_n^{(2)}(t)= \int_0^\infty V_n^{(2)}(x,t)dx$ denotes the probability that at time t there are n customers in the queue and the server is under optional vacation irrespective of the value of x.

$R_n(t)$= Probability that at time t, the server is inactive due to the arrival of interruption while there are n $(n \geq 0)$ customers in the queue.

$Q(t)$ = Probability that at time t, there are no customers in the system and the server is idle but available in the system.

8.4 Equations governing the system

The model is then, governed by the following set of differential - difference equations:

$$\frac{\partial}{\partial x}P_0^{(1)}(x,t)+\frac{\partial}{\partial t}P_0^{(1)}(x,t)+[\lambda+\mu_1(x)+\alpha]P_0^{(1)}(x,t)=0 \tag{8.1}$$

$$\frac{\partial}{\partial x}P_n^{(1)}(x,t)+\frac{\partial}{\partial t}P_n^{(1)}(x,t)+[\lambda+\mu_1(x)+\alpha]P_n^{(1)}(x,t)=\lambda\sum_{k=1}^{n}c_kP_{n-k}^{(1)}(x,t),\quad n\geq 1 \tag{8.2}$$

$$\frac{\partial}{\partial x}P_0^{(2)}(x,t)+\frac{\partial}{\partial t}P_0^{(2)}(x,t)+[\lambda+\mu_2(x)+\alpha]P_0^{(2)}(x,t)=0 \tag{8.3}$$

$$\frac{\partial}{\partial x}P_n^{(2)}(x,t)+\frac{\partial}{\partial t}P_n^{(2)}(x,t)+[\lambda+\mu_2(x)+\alpha]P_n^{(2)}(x,t)=\lambda\sum_{k=1}^{n}c_kP_{n-k}^{(2)}(x,t),\quad n\geq 1 \tag{8.4}$$

$$\frac{\partial}{\partial x}P_0^{(3)}(x,t)+\frac{\partial}{\partial t}P_0^{(3)}(x,t)+[\lambda+\mu_3(x)+\alpha]P_0^{(3)}(x,t)=0 \tag{8.5}$$

$$\frac{\partial}{\partial x}P_n^{(3)}(x,t)+\frac{\partial}{\partial t}P_n^{(3)}(x,t)+[\lambda+\mu_3(x)+\alpha]P_n^{(3)}(x,t)=\lambda\sum_{k=1}^{n}c_kP_{n-k}^{(3)}(x,t),\quad n\geq 1 \tag{8.6}$$

$$\frac{\partial}{\partial x}V_0^{(1)}(x,t)+\frac{\partial}{\partial t}V_0^{(1)}(x,t)+[\lambda+\gamma_1(x)]V_0^{(1)}(x,t)=0 \tag{8.7}$$

$$\frac{\partial}{\partial x}V_n^{(1)}(x,t)+\frac{\partial}{\partial t}V_n^{(1)}(x,t)+[\lambda+\gamma_1(x)]V_n^{(1)}(x,t)=\lambda\sum_{k=1}^{n}c_kV_{n-k}^{(1)}(x,t),\quad n\geq 1 \tag{8.8}$$

$$\frac{\partial}{\partial x}V_0^{(2)}(x,t)+\frac{\partial}{\partial t}V_0^{(2)}(x,t)+[\lambda+\gamma_2(x)]V_0^{(2)}(x,t)=0 \tag{8.9}$$

$$\frac{\partial}{\partial x}V_n^{(2)}(x,t)+\frac{\partial}{\partial t}V_n^{(2)}(x,t)+[\lambda+\gamma_2(x)]V_n^{(2)}(x,t)=\lambda\sum_{k=1}^{n}c_kV_{n-k}^{(2)}(x,t),\quad n\geq 1 \tag{8.10}$$

$$\frac{d}{dt}R_0(t) = -(\lambda+\beta)R_0(t) \tag{8.11}$$

$$\begin{aligned}\frac{d}{dt}R_n(t) = &-(\lambda+\beta)R_n(t) + \lambda\sum_{k=1}^{n} c_k R_{n-k}(t)\\ &+\alpha\int_0^\infty P_{n-1}^{(1)}(x,t)dx + \alpha\int_0^\infty P_{n-1}^{(2)}(x,t)dx\\ &+\alpha\int_0^\infty P_{n-1}^{(3)}(x,t)dx\end{aligned} \tag{8.12}$$

$$\begin{aligned}\frac{d}{dt}Q(t) = &-\lambda Q(t) + (1-p)\int_0^\infty V_0^{(1)}(x,t)\gamma_1(x)dx\\ &+\beta R_0(t) + \int_0^\infty V_0^{(2)}(x,t)\gamma_2(x)dx\end{aligned} \tag{8.13}$$

The above set of equations are to be solved subject to the following boundary conditions:

$$\begin{aligned}P_n^{(1)}(0,t) = &\lambda c_{n+1}Q(t) + \beta R_{n+1}(t) + (1-p)\int_0^\infty V_{n+1}^{(1)}(x,t)\gamma_1(x)dx\\ &+\int_0^\infty V_{n+1}^{(2)}(x,t)\gamma_2(x)dx, \quad n\geq 0\end{aligned} \tag{8.14}$$

$$P_n^{(2)}(0,t) = \int_0^\infty P_n^{(1)}(x,t)\mu_1(x)dx, \quad n\geq 0 \tag{8.15}$$

$$P_n^{(3)}(0,t) = \int_0^\infty P_n^{(2)}(x,t)\mu_2(x)dx, \quad n\geq 0 \tag{8.16}$$

$$V_n^{(1)}(0,t) = \int_0^\infty P_n^{(3)}(x,t)\mu_3(x)dx, \quad n\geq 0 \tag{8.17}$$

$$V_n^{(2)}(0,t) = p\int_0^\infty V_n^{(1)}(x,t)\gamma_1(x)dx, \quad n\geq 0 \tag{8.18}$$

We assume that initially there are no customers in the system and the server is idle. So the initial conditions are

$$\begin{aligned}&P_n^{(i)}(0) = V_n^{(j)}(0) = R_n(0) = 0, \text{ for } \ n\geq 0,\\ &i=1,2,3; \quad j=1,2 \ \text{ and } Q(0)=1.\end{aligned} \tag{8.19}$$

8.5 Generating functions of the queue length: The time-dependent solution

In this section, we obtain the transient solution for the above set of differential-difference equations.

Theorem: *The system of differential difference equations to describe a batch arrival queue with three stages of heterogeneous service, server vacations and service interruptions are given by equations (8.1) to (8.18) with initial condition (8.19) and the generating functions of transient solution are given by equations (8.85) to (8.90).*

Proof: We define the probability generating functions, for i =1, 2, 3;

$$P^{(i)}(x,z,t)=\sum_{n=0}^{\infty}z^nP_n^{(i)}(x,t);\quad P^{(i)}(z,t)=\sum_{n=0}^{\infty}z^nP_n^{(i)}(t),\ C(z)=\sum_{n=1}^{\infty}c_nz^n; \tag{8.20}$$

$$V^{(j)}(x,z,t)=\sum_{n=0}^{\infty}z^nV_n^{(j)}(x,t);\ V^{(j)}(z,t)=\sum_{n=0}^{\infty}z^nV_n^{(j)}(t);\ j=1,2x>0 \tag{8.21}$$

which are convergent inside the circle given by $\mid z\mid\leq 1$ and define the Laplace transform of a function $f(t)$ as

$$\bar{f}(s)=\int_0^{\infty}e^{-st}f(t)dt,\quad \Re(s)>0. \tag{8.22}$$

Taking the Laplace transform of equations (8.1) to (8.18) and using (8.19), we obtain

$$\frac{\partial}{\partial x}\bar{P}_0^{(1)}(x,s)+[s+\lambda+\alpha+\mu_1(x)]\bar{P}_0^{(1)}(x,s)=0 \tag{8.23}$$

$$\frac{\partial}{\partial x}\bar{P}_n^{(1)}(x,s)+[s+\lambda+\alpha+\mu_1(x)]\bar{P}_n^{(1)}(x,s)=\lambda\sum_{k=1}^{n}c_k\bar{P}_{n-k}^{(1)}(x,s),n\geq 1 \tag{8.24}$$

$$\frac{\partial}{\partial x}\bar{P}_0^{(2)}(x,s)+[s+\lambda+\alpha+\mu_2(x)]\bar{P}_0^{(2)}(x,s)=0 \tag{8.25}$$

$$\frac{\partial}{\partial x}\bar{P}_n^{(2)}(x,s) + [s+\lambda+\alpha+\mu_2(x)]\bar{P}_n^{(2)}(x,s) = \lambda\sum_{k=1}^{n} c_k \bar{P}_{n-k}^{(2)}(x,s), n \geq 1 \tag{8.26}$$

$$\frac{\partial}{\partial x}\bar{P}_0^{(3)}(x,s) + [s+\lambda+\alpha+\mu_3(x)]\bar{P}_0^{(3)}(x,s) = 0 \tag{8.27}$$

$$\frac{\partial}{\partial x}\bar{P}_n^{(3)}(x,s) + [s+\lambda+\alpha+\mu_3(x)]\bar{P}_n^{(3)}(x,s) = \lambda\sum_{k=1}^{n} c_k \bar{P}_{n-k}^{(3)}(x,s), n \geq 1 \tag{8.28}$$

$$\frac{\partial}{\partial x}\bar{V}_0^{(1)}(x,s) + [s+\lambda+\gamma_1(x)]\bar{V}_0^{(1)}(x,s) = 0 \tag{8.29}$$

$$\frac{\partial}{\partial x}\bar{V}_n^{(1)}(x,s) + [s+\lambda+\gamma_1(x)]\bar{V}_n^{(1)}(x,s) = \lambda\sum_{k=1}^{n} c_k \bar{V}_{n-k}^{(1)}(x,s), n \geq 1 \tag{8.30}$$

$$\frac{\partial}{\partial x}\bar{V}_0^{(2)}(x,s) + [s+\lambda+\gamma_2(x)]\bar{V}_0^{(2)}(x,s) = 0 \tag{8.31}$$

$$\frac{\partial}{\partial x}\bar{V}_n^{(2)}(x,s) + [s+\lambda+\gamma_2(x)]\bar{V}_n^{(2)}(x,s) = \lambda\sum_{k=1}^{n} c_k \bar{V}_{n-k}^{(2)}(x,s), n \geq 1 \tag{8.32}$$

$$(s+\lambda+\beta)\bar{R}_0(s) = 0 \tag{8.33}$$

$$\begin{aligned}(s+\lambda+\beta)\bar{R}_n(s) = {} & \lambda\sum_{k=1}^{n} c_k \bar{R}_{n-k}(s) + \alpha\int_0^\infty \bar{P}_{n-1}^{(1)}(x,s)dx \\ & + \alpha\int_0^\infty \bar{P}_{n-1}^{(2)}(x,s)dx + \alpha\int_0^\infty \bar{P}_{n-1}^{(3)}(x,s)dx,\ n \geq 1\end{aligned} \tag{8.34}$$

$$\begin{aligned}[s+\lambda]\bar{Q}(s) = {} & 1 + \beta\bar{R}_0(s) + (1-p)\int_0^\infty \bar{V}_0^{(1)}(x,s)\gamma_1(x)dx \\ & + \int_0^\infty \bar{V}_0^{(2)}(x,s)\gamma_2(x)dx\end{aligned} \tag{8.35}$$

$$\begin{aligned}\bar{P}_n^{(1)}(0,s) = {} & \lambda c_{n+1}\bar{Q}(s) + \beta\bar{R}_{n+1}(s) \\ & + (1-p)\int_0^\infty \gamma_1(x)\bar{V}_{n+1}^{(1)}(x,s)dx \\ & + \int_0^\infty \gamma_2(x)\bar{V}_{n+1}^{(2)}(x,s)dx,\ n \geq 0\end{aligned} \tag{8.36}$$

$$\bar{P}_n^{(2)}(0,s) = \int_0^\infty \mu_1(x)\bar{P}_n^{(1)}(x,s)dx, \quad n \geq 0 \tag{8.37}$$

$$\bar{P}_n^{(3)}(0,s) = \int_0^\infty \mu_2(x)\bar{P}_n^{(2)}(x,s)dx, \quad n \geq 0 \tag{8.38}$$

$$\bar{V}_n^{(1)}(0,s) = \int_0^\infty \mu_3(x)\bar{P}_n^{(3)}(x,s)dx, \quad n \geq 0 \tag{8.39}$$

$$\bar{V}_n^{(2)}(0,s) = p\int_0^\infty \gamma_1(x)\bar{V}_n^{(1)}(x,s)dx, \quad n \geq 0 \tag{8.40}$$

Now multiplying equations (8.24), (8.26), (8.28), (8.30), (8.32) and (8.34) by z^n and summing over n from 1 to ∞, adding to equations (8.23), (8.25), (8.27), (8.29), (8.31), (8.33) and using the generating functions defined in (8.20) and (8.21), we get

$$\frac{\partial}{\partial x}\bar{P}^{(1)}(x,z,s) + [s+\lambda-\lambda C(z)+\alpha+\mu_1(x)]\bar{P}^{(1)}(x,z,s) = 0 \tag{8.41}$$

$$\frac{\partial}{\partial x}\bar{P}^{(2)}(x,z,s) + [s+\lambda-\lambda C(z)+\alpha+\mu_2(x)]\bar{P}^{(2)}(x,z,s) = 0 \tag{8.42}$$

$$\frac{\partial}{\partial x}\bar{P}^{(3)}(x,z,s) + [s+\lambda-\lambda C(z)+\alpha+\mu_3(x)]\bar{P}^{(3)}(x,z,s) = 0 \tag{8.43}$$

$$\frac{\partial}{\partial x}\bar{V}^{(1)}(x,z,s) + [s+\lambda-\lambda C(z)+\gamma_1(x)]\bar{V}^{(1)}(x,z,s) = 0 \tag{8.44}$$

$$\frac{\partial}{\partial x}\bar{V}^{(2)}(x,z,s) + [s+\lambda-\lambda C(z)+\gamma_2(x)]\bar{V}^{(2)}(x,z,s) = 0 \tag{8.45}$$

$$[s+\lambda-\lambda C(z)+\beta]\bar{R}(z,s) = \alpha z\int_0^\infty \bar{P}^{(1)}(x,z,s)dx + \alpha z\int_0^\infty \bar{P}^{(2)}(x,z,s)dx + \alpha z\int_0^\infty \bar{P}^{(3)}(x,z,s)dx, \tag{8.46}$$

For the boundary conditions, we multiply both sides of equation (8.36) by z^n summing over n from 0 to ∞, and use the equations (8.20) and (8.21), we get

$$\begin{aligned} z\bar{P}^{(1)}(0,z,s) =& \lambda C(z)\bar{Q}(s) + \beta\bar{R}(z,s) - \beta\bar{R}_0(s) \\ &+ (1-p)\int_0^\infty \gamma_1(x)\bar{V}^{(1)}(x,z,s)dx \\ &+ \int_0^\infty \gamma_2(x)\bar{V}^{(2)}(x,z,s)dx \\ &- (1-p)\int_0^\infty \gamma_1(x)\bar{V}_0^{(1)}(x,s)dx \\ &- \int_0^\infty \gamma_2(x)\bar{V}_0^{(2)}(x,s)\,\mathrm{d}x \end{aligned} \tag{8.47}$$

Using equation (8.35) in (8.47), we get

$$z\bar{P}^{(1)}(0,z,s) = [1 - s\bar{Q}(s)] + \lambda(C(z)-1)\bar{Q}(s) + \beta\bar{R}(z,s) \\ +(1-p)\int_0^\infty \gamma_1(x)\bar{V}^{(1)}(x,z,s)dx + \int_0^\infty \gamma_2(x)\bar{V}^{(2)}(x,z,s)dx \quad (8.48)$$

Performing similar operation on equations (8.37) to (8.40), we get

$$\bar{P}^{(2)}(0,z,s) = \int_0^\infty \mu_1(x)\bar{P}^{(1)}(x,z,s)dx \quad (8.49)$$

$$\bar{P}^{(3)}(0,z,s) = \int_0^\infty \mu_2(x)\bar{P}^{(2)}(x,z,s)dx \quad (8.50)$$

$$\bar{V}^{(1)}(0,z,s) = \int_0^\infty \mu_3(x)\bar{P}^{(3)}(x,z,s)dx \quad (8.51)$$

$$\bar{V}^{(2)}(0,z,s) = p\int_0^\infty \gamma_2(x)\bar{V}^{(1)}(x,z,s)dx \quad (8.52)$$

Integrating equation (8.41) between 0 and x, we get

$$\bar{P}^{(1)}(x,z,s) = \bar{P}^{(1)}(0,z,s)e^{-[s+\lambda-\lambda C(z)+\alpha]x-\int_0^x \mu_1(t)dt} \quad (8.53)$$

where $P^{(1)}(0,z,s)$ is given by equation (8.48).

Again integrating equation (8.53) by parts with respect to x, yields

$$\bar{P}^{(1)}(z,s) = \bar{P}^{(1)}(0,z,s)\left[\frac{1-\bar{B}_1(s+\lambda-\lambda C(z)+\alpha)}{s+\lambda-\lambda C(z)+\alpha}\right] \quad (8.54)$$

where

$$\bar{B}_1(s+\lambda-\lambda C(z)+\alpha) = \int_0^\infty e^{-[s+\lambda-\lambda C(z)+\alpha]x}dB_1(x) \quad (8.55)$$

is the Laplace-Stieltjes transform of the first stage of service time $B_1(x)$.

Now multiplying both sides of equation (8.53) by $\mu_1(x)$ and integrating over x, we obtain

$$\int_0^\infty \bar{P}^{(1)}(x,z,s)\mu_1(x)dx = \bar{P}^{(1)}(0,z,s)\bar{B}_1[s+\lambda-\lambda c(z)+\alpha] \quad (8.56)$$

Similarly, on integrating equations (8.42) to (8.45) from 0 to x, we get

$$\bar{P}^{(2)}(x,z,s) = \bar{P}^{(2)}(0,z,s)e^{-[s+\lambda-\lambda C(z)+\alpha]x-\int\limits_0^x \mu_2(t)dt} \tag{8.57}$$

$$\bar{P}^{(3)}(x,z,s) = \bar{P}^{(3)}(0,z,s)e^{-[s+\lambda-\lambda C(z)+\alpha]x-\int\limits_0^x \mu_3(t)dt} \tag{8.58}$$

$$\bar{V}^{(1)}(x,z,s) = \bar{V}^{(1)}(0,z,s)e^{-[s+\lambda-\lambda C(z)]x-\int\limits_0^x \gamma_1(t)dt} \tag{8.59}$$

$$\bar{V}^{(2)}(x,z,s) = \bar{V}^{(2)}(0,z,s)e^{-[s+\lambda-\lambda C(z)]x-\int\limits_0^x \gamma_2(t)dt} \tag{8.60}$$

where $\bar{P}^{(2)}(0,z,s)$, $\bar{P}^{(3)}(0,z,s)$, $\bar{V}^{(1)}(0,z,s)$ and $\bar{V}^{(2)}(0,z,s)$ are given by equations (8.49) to (8.52).

Again integrating equations (8.57) to (8.60) by parts with respect to x, yields

$$\bar{P}^{(2)}(z,s) = \bar{P}^{(2)}(0,z,s)\left[\frac{1-\bar{B}_2(s+\lambda-\lambda C(z)+\alpha)}{s+\lambda-\lambda C(z)+\alpha}\right] \tag{8.61}$$

$$\bar{P}^{(3)}(z,s) = \bar{P}^{(3)}(0,z,s)\left[\frac{1-\bar{B}_3(s+\lambda-\lambda C(z)+\alpha)}{s+\lambda-\lambda C(z)+\alpha}\right] \tag{8.62}$$

$$\bar{V}^{(1)}(z,s) = \bar{V}^{(1)}(0,z,s)\left[\frac{1-\bar{V}_1(s+\lambda-\lambda C(z))}{s+\lambda-\lambda C(z)}\right] \tag{8.63}$$

$$\bar{V}^{(2)}(z,s) = \bar{V}^{(2)}(0,z,s)\left[\frac{1-\bar{V}_2(s+\lambda-\lambda C(z))}{s+\lambda-\lambda C(z)}\right] \tag{8.64}$$

where

$$\bar{B}_2(s+\lambda-\lambda C(z)+\alpha) = \int_0^\infty e^{-[s+\lambda-\lambda C(z)+\alpha]x}dB_2(x) \tag{8.65}$$

$$\bar{B}_3(s+\lambda-\lambda C(z)+\alpha) = \int_0^\infty e^{-[s+\lambda-\lambda C(z)+\alpha]x}dB_3(x) \tag{8.66}$$

$$\bar{V}_1(s+\lambda-\lambda C(z)) = \int_0^\infty e^{-[s+\lambda-\lambda C(z)]x}dV_1(x) \tag{8.67}$$

$$\bar{V}_2(s+\lambda-\lambda C(z)) = \int_0^\infty e^{-[s+\lambda-\lambda C(z)]x}dV_2(x) \tag{8.68}$$

are the Laplace-Stieltjes transform of the second stage of service time $B_2(x)$, third stage of service time $B_3(x)$, compulsory vacation time $V_1(x)$ and optional vacation time $V_2(x)$ respectively.

Now multiplying both sides of equations (8.57) to (8.60) by $\mu_2(x)$, $\mu_3(x)$, $\gamma_1(x)$ and $\gamma_2(x)$ and integrating over x, we obtain

$$\int_0^\infty \bar{P}^{(2)}(x,z,s)\mu_2(x)dx = \bar{P}^{(2)}(0,z,s)\bar{B}_2[s+\lambda-\lambda C(z)+\alpha] \tag{8.69}$$

$$\int_0^\infty \bar{P}^{(3)}(x,z,s)\mu_3(x)dx = \bar{P}^{(3)}(0,z,s)\bar{B}_3[s+\lambda-\lambda C(z)+\alpha] \tag{8.70}$$

$$\int_0^\infty \bar{V}^{(1)}(x,z,s)\gamma_1(x)dx = \bar{V}^{(1)}(0,z,s)\bar{V}_1[s+\lambda-\lambda C(z)] \tag{8.71}$$

$$\int_0^\infty \bar{V}^{(2)}(x,z,s)\gamma_2(x)dx = \bar{V}^{(2)}(0,z,s)\bar{V}_2[s+\lambda-\lambda C(z)] \tag{8.72}$$

Using equation (8.56), equation (8.49) reduces to

$$\bar{P}^{(2)}(0,z,s) = \bar{P}^{(1)}(0,z,s)\bar{B}_1(a) \tag{8.73}$$

Now using equations (8.69) and (8.73) in (8.50), we get

$$\bar{P}^{(3)}(0,z,s) = \bar{P}^{(1)}(0,z,s)\bar{B}_1(a)\bar{B}_2(a) \tag{8.74}$$

By using equations (8.70) and (8.74) in (8.51), we get

$$\bar{V}^{(1)}(0,z,s) = \bar{B}_1(a)\bar{B}_2(a)\bar{B}_3(a)\bar{P}^{(1)}(0,z,s) \tag{8.75}$$

Using equations (8.71) and (8.75), we can write equation (8.52) as

$$\bar{V}^{(2)}(0,z,s) = p\bar{B}_1(a)\bar{B}_2(a)\bar{B}_3(a)\bar{V}_1(b)\bar{P}^{(1)}(0,z,s) \tag{8.76}$$

where $a = s+\lambda-\lambda C(z)+\alpha$ and $b = s+\lambda-\lambda C(z)$.

Now using equations (8.71), (8.72), (8.75) and (8.76) in (8.48), we get

$$[z-(1-p+p\bar{V}_2(b))\bar{B}_1(a)\bar{B}_2(a)\bar{B}_3(a)\bar{V}_1(b)]\bar{P}^{(1)}(0,z,s)=$$
$$(1-s\bar{Q}(s))+\lambda(C(z)-1)\bar{Q}(s)+\beta\bar{R}(z,s) \quad (8.77)$$

Using equations (8.53), (8.57) and (8.58) in (8.46), we get

$$\bar{R}(z,s)=\frac{\alpha z}{ac}[1-\bar{B}_1(a)\bar{B}_2(a)\bar{B}_3(a)]\bar{P}^{(1)}(0,z,s) \quad (8.78)$$

Similarly, using equations (8.78) in (8.77), we get

$$\bar{P}^{(1)}(0,z,s)=\frac{[1-s\bar{Q}(s)]+\lambda(C(z)-1)\bar{Q}(s)}{Dr} \quad (8.79)$$

where

$$Dr=z-(1-p+p\bar{V}_2(b))\bar{B}_1(a)\bar{B}_2(a)\bar{B}_3(a)\bar{V}_1(b)$$
$$-\frac{\alpha\beta z}{ac}[1-\bar{B}_1(a)\bar{B}_2(a)\bar{B}_3(a)], \quad (8.80)$$

where $c=s+\lambda-\lambda C(z)+\beta$.

Substituting the value of $\bar{P}^{(1)}(0,z,s)$ from equation (8.79) into equations (8.73), (8.74), (8.75), (8.76) and (8.78) we get

$$\bar{P}^{(2)}(0,z,s)=\frac{\bar{B}_1(a)[(1-s\bar{Q}(s))+\lambda(C(z)-1)\bar{Q}(s)]}{Dr} \quad (8.81)$$

$$\bar{P}^{(3)}(0,z,s)=\frac{\bar{B}_1(a)\bar{B}_2(a)[(1-s\bar{Q}(s))+\lambda(C(z)-1)\bar{Q}(s)]}{Dr} \quad (8.82)$$

$$\bar{V}^{(1)}(0,z,s)=\frac{\bar{B}_1(a)\bar{B}_2(a)\bar{B}_3(a)[(1-s\bar{Q}(s))+\lambda(C(z)-1)\bar{Q}(s)]}{Dr} \quad (8.83)$$

$$\bar{V}^{(2)}(0,z,s)=\frac{p\bar{B}_1(a)\bar{B}_2(a)\bar{B}_3(a)\bar{V}_1(b)[(1-s\bar{Q}(s))+\lambda(C(z)-1)\bar{Q}(s)]}{Dr} \quad (8.84)$$

$$\bar{R}(z,s)=\frac{\alpha z}{ac}\frac{(1-\bar{B}_1(a)\bar{B}_2(a)\bar{B}_3(a))}{Dr}[(1-s\bar{Q}(s))+\lambda(C(z)-1)\bar{Q}(s)] \quad (8.85)$$

Using equations (8.79), (8.81) to (8.84) in (8.54), (8.61) to (8.64), we get

$$\bar{P}^{(1)}(z,s) = \frac{[(1-s\bar{Q}(s)) + \lambda(C(z)-1)\bar{Q}(s)]}{Dr}\frac{[1-\bar{B}_1(a)]}{a} \tag{8.86}$$

$$\bar{P}^{(2)}(z,s) = \frac{\bar{B}_1(a)[(1-s\bar{Q}(s)) + \lambda(C(z)-1)\bar{Q}(s)]}{Dr}\frac{[1-\bar{B}_2(a)]}{a} \tag{8.87}$$

$$\bar{P}^{(3)}(z,s) = \frac{\bar{B}_1(a)\bar{B}_2(a)[(1-s\bar{Q}(s)) + \lambda(C(z)-1)\bar{Q}(s)]}{Dr}\frac{[1-\bar{B}_3(a)]}{a} \tag{8.88}$$

$$\bar{V}^{(1)}(z,s) = \frac{\bar{B}_1(a)\bar{B}_2(a)\bar{B}_3(a)}{Dr}\frac{[1-\bar{V}_1(b)]}{b} \times [(1-s\bar{Q}(s)) + \lambda(C(z)-1)\bar{Q}(s)] \tag{8.89}$$

$$\bar{V}^{(2)}(z,s) = \frac{p\bar{B}_1(a)\bar{B}_2(a)\bar{B}_3(a)\bar{V}_1(b)}{Dr}\frac{[1-\bar{V}_2(b)]}{b} \times [(1-s\bar{Q}(s)) + \lambda(C(z)-1)\bar{Q}(s)] \tag{8.90}$$

where Dr is given by equation (8.80). Thus $\bar{R}(z,s), \bar{P}^{(1)}(z,s),\ \bar{P}^{(2)}(z,s),\ \bar{P}^{(3)}(z,s)$, $\bar{V}^{(1)}(z,s)$ and $\bar{V}^{(2)}(z,s)$ are completely determined from equations (8.85) to (8.90) which completes the proof of the theorem.

8.6 The steady state results

In this section, we shall derive the steady state probability distribution for our queueing model. To define the steady state probabilities, we suppress the argument t wherever it appears in the time-dependent analysis. This can be obtained by applying the well-known Tauberian property,

$$\lim_{s\to 0} s\bar{f}(s) = \lim_{t\to\infty} f(t) \tag{8.91}$$

In order to determine $\bar{R}(z,s), \bar{P}^{(1)}(z,s),\ \bar{P}^{(2)}(z,s),\ \bar{P}^{(3)}(z,s),\ \bar{V}^{(1)}(z,s)$ and $\bar{V}^{(2)}(z,s)$ completely, we have yet to determine the unknown Q which appears in the numerators of the right hand sides of equations (8.85) to (8.90).

For that purpose, we shall use the normalizing condition

$$P^{(1)}(1) + P^{(2)}(1) + P^{(3)}(1) + V^{(1)}(1) + V^{(2)}(1) + R(1) + Q = 1$$

The steady state probabilities for $M^{[X]}/G/1$ queue with three stage heterogeneous service, server vacations and service interruption are given by

$$P^{(1)}(1) = \frac{\lambda\beta E(I)[1-\bar{B}_1(\alpha)]Q}{dr}$$
$$P^{(2)}(1) = \frac{\lambda\beta E(I)\bar{B}_1(\alpha)[1-\bar{B}_2(\alpha)]Q}{dr}$$
$$P^{(3)}(1) = \frac{\lambda\beta E(I)\bar{B}_1(\alpha)\bar{B}_2(\alpha)[1-\bar{B}_3(\alpha)]Q}{dr}$$
$$V^{(1)}(1) = \frac{\lambda\alpha\beta E(I)\bar{B}_1(\alpha)\bar{B}_2(\alpha)\bar{B}_3(\alpha)E(V_1)Q}{dr}$$
$$V^{(2)}(1) = \frac{p\lambda\alpha\beta E(I)\bar{B}_1(\alpha)\bar{B}_2(\alpha)\bar{B}_3(\alpha)E(V_2)Q}{dr}$$
$$R^{(1)}(1) = \frac{\lambda\alpha E(I)(1-\bar{B}_1(\alpha)\bar{B}_2(\alpha)\bar{B}_3(\alpha))Q}{dr}$$

where

$$dr = \lambda(\alpha+\beta)E(I)(\bar{B}(\alpha)-1) + \alpha\beta\bar{B}(\alpha)[1-\lambda E(I)(E(V_1)+pE(V_2))] \quad (8.92)$$

and $\bar{B}(\alpha) = \bar{B}_1(\alpha)\bar{B}_2(\alpha)\bar{B}_3(\alpha)$.

$P^{(1)}(1),\ P^{(2)}(1),\ P^{(3)}(1),\ V^{(1)}(1),\ V^{(2)}(1),\ R(1)$ and Q are the steady state probabilities that the server is providing first stage of service, second stage of service, third stage of service, server under compulsory vacation, optional vacation, service interruption and server under idle respectively without regard to the number of customers in the system.

Thus multiplying both sides of equations (8.85) to (8.90) by s, taking limit as $s \to 0$, applying property (8.91) and simplifying, we obtain

$$P^{(1)}(z) = \frac{f_2(z)f_3(z)[\bar{B}_1(f_1(z)) - 1]Q}{D(z)} \tag{8.93}$$

$$P^{(2)}(z) = \frac{f_2(z)f_3(z)\bar{B}_1(f_1(z))[\bar{B}_2(f_1(z)) - 1]Q}{D(z)} \tag{8.94}$$

$$P^{(3)}(z) = \frac{f_2(z)f_3(z)\bar{B}_1(f_1(z))\bar{B}_2(f_1(z))[\bar{B}_3(f_1(z)) - 1]Q}{D(z)} \tag{8.95}$$

$$V^{(1)}(z) = \frac{f_1(z)f_2(z)\bar{B}(f(z))[\bar{V}_1(f_3(z)) - 1]Q}{D(z)} \tag{8.96}$$

$$V^{(2)}(z) = \frac{pf_1(z)f_2(z)\bar{B}(f(z))\bar{V}_1(f_3(z))[\bar{V}_2(f_3(z)) - 1]Q}{D(z)} \tag{8.97}$$

$$R(z) = \frac{\alpha z f_3(z)[\bar{B}(f_1(z)) - 1]Q}{D(z)} \tag{8.98}$$

where

$$D(z) = f_1(z)f_2(z)[z-(1-p+p\bar{V}_2(f_3(z)))\bar{B}(f(z))\bar{V}_1(f_3(z))]-\alpha\beta z[1-\bar{B}(f(z))], \tag{8.99}$$

$\bar{B}(f(z)) = \bar{B}_1(f_1(z))\bar{B}_2(f_1(z))\bar{B}_3(f_1(z))$, $f_1(z) = \lambda - \lambda C(z) + \alpha$,
$f_2(z) = \lambda - \lambda C(z) + \beta$ and $f_3(z) = \lambda - \lambda C(z)$.

Let $W_q(z)$ denote the probability generating function of the queue size irrespective of the state of the system. Then adding equations (8.93) to (8.98), we obtain

$$W_q(z) = P^{(1)}(z) + P^{(2)}(z) + P^{(3)}(z) + V^{(1)}(z) + V^{(2)}(z) + R(z)$$

$$W_q(z) = \frac{f_2(z)f_3(z)[\bar{B}_1(f_1(z)) - 1]Q}{D(z)} + \frac{f_2(z)f_3(z)\bar{B}_1(f_1(z))[\bar{B}_2(f_1(z)) - 1]Q}{D(z)} + \frac{f_2(z)f_3(z)\bar{B}_1(f_1(z))\bar{B}_2(f_1(z))[\bar{B}_3(f_1(z)) - 1]Q}{D(z)}$$

$$+\frac{f_1(z)f_2(z)\bar{B}(f(z))[\bar{V}_1(f_3(z))-1]Q}{D(z)}$$
$$+\frac{pf_1(z)f_2(z)\bar{B}(f(z))\bar{V}_1(f_3(z))[\bar{V}_2(f_3(z))-1]Q}{D(z)}$$
$$+\frac{\alpha z f_3(z)[\bar{B}(f_1(z))-1]Q}{D(z)} \tag{8.100}$$

we see that for z=1, $W_q(z)$ is indeterminate of the form 0/0. Therefore, we apply L'Hopital's rule and on simplifying, we get

$$W_q(1)=\frac{\lambda(\alpha+\beta)E(I)[1-\bar{B}(\alpha)]+\lambda\alpha\beta E(I)\bar{B}(\alpha)(E(V_1)+pE(V_2))Q}{dr} \tag{8.101}$$

where C(1)= 1, $C'(1)=E(I)$ is mean batch size of the arriving customers, $E(V_j)=-\bar{V}_j'(0)\ ,\ \ j=1,2.$

since $W_q(1)+Q=1$, we have

$$Q=\frac{\lambda(\alpha+\beta)E(I)(\bar{B}(\alpha)-1)+\alpha\beta\bar{B}(\alpha)[1-\lambda E(I)(E(V_1)+pE(V_2))]}{\alpha\beta\bar{B}(\alpha)} \tag{8.102}$$

and hence the utilization factor ρ of the system is given by

$$\rho=1-Q \tag{8.103}$$

where $\rho<1$ is the stability condition under which the steady state exists. Equation (8.102) gives the probability that the server is idle.

Substituting Q from (8.102) into (8.100), we have completely and explicitly determined $W_q(z)$, the probability generating function of the queue size.

8.7 The mean queue size and the mean waiting time

Let L_q denote the mean number of customers in the queue under the steady state. Then

$$L_q = \frac{d}{dz} W_q(z) \text{ at } z = 1$$

since this formula gives 0/0 form, then we write $W_q(z)$ given in (8.100) as $W_q(z) = \frac{N(z)}{D(z)}\mathrm{Q}$
where

$$\begin{aligned} N(z) =& f_3(z)(f_2(z) + \alpha z)[\bar{B}(f(z)) - 1] + \bar{B}(f(z)) f_1(z) f_2(z) \\ & \times [\bar{V}_1(f_3(z))(1 - p + p\bar{V}_2(f_3(z))) - 1] \end{aligned}$$

and $D(z)$ is given by equation (8.99).

$$\begin{aligned} N'(z) =& [f_3'(z)(f_2(z) + \alpha z) + f_3(z)(f_2'(z) + \alpha)](\bar{B}(f(z)) - 1) \\ & + f_3(z)(f_2(z) + \alpha z)[\bar{B}_1'(f_1(z)) f_1'(z) \bar{B}_2(f_1(z)) \bar{B}_3(f_1(z)) \\ & + \bar{B}_1(f_1(z)) \bar{B}_2'(f_1(z)) f_1'(z) \bar{B}_3(f_1(z)) \\ & + \bar{B}_1(f_1(z)) \bar{B}_2(f_1(z)) \bar{B}_3'(f_1(z)) f_1'(z)] \\ & + [[\bar{B}_1'(f_1(z)) f_1'(z) \bar{B}_2(f_1(z)) \bar{B}_3(f_1(z)) \\ & + \bar{B}_1(f_1(z)) \bar{B}_2'(f_1(z)) f_1'(z) \bar{B}_3(f_1(z)) \\ & + \bar{B}_1(f_1(z)) \bar{B}_2(f_1(z)) \bar{B}_3'(f_1(z)) f_1'(z)] \\ & \times (\bar{V}_1(f_3(z))(1 - p + p\bar{V}_2(f_3(z))) - 1) \\ & + \bar{B}(f(z))(\bar{V}_1'(f_3(z)) f_3'(z)(1 - p + p\bar{V}_2(f_3(z))) \\ & + \bar{V}_1(f_3(z)) p\bar{V}_2'(f_3(z)) f_3'(z))] f_1(z) f_2(z) \\ & + \bar{B}(f(z))((\bar{V}_1(f_3(z))(1 - p + p\bar{V}_2(f_3(z))) - 1) \\ & \times [f_1'(z) f_2(z) + f_1(z) f_2'(z)] \end{aligned}$$

$$
\begin{aligned}
D'(z) =& [f_1'(z)f_2(z) + f_1(z)f_2'(z)][z - (1-p+p\bar{V}_2(f_3(z))) \\
& \times \bar{B}(f(z))\bar{V}_1(f_3(z))] + f_1(z)f_2(z) \\
& \times [1 - p\bar{V}_2'(f_3(z))f_3'(z)\bar{B}(f(z))\bar{V}_1(f_3(z)) \\
& - (1-p+p\bar{V}_2(f_3(z)))[\bar{B}_1'(f_1(z))f_1'(z)\bar{B}_2(f_1(z))\bar{B}_3(f_1(z))\bar{V}_1(f_3(z)) \\
& + \bar{B}_1(f_1(z))\bar{B}_2'(f_1(z))f_1'(z)\bar{B}_3(f_1(z))\bar{V}_1(f_3(z)) \\
& + \bar{B}_1(f_1(z))f_1(z)\bar{B}_2(f_1(z))\bar{B}_3'(f_1(z))f_1'(z))\bar{V}_1(f_3(z)) \\
& + \bar{B}(f(z))\bar{V}_1'(f_3(z))f_3'(z)]] - \alpha\beta(1 - \bar{B}f(z)) \\
& + \alpha\beta z[\bar{B}_1'(f_1(z))f_1'(z)\bar{B}_2(f_1(z))\bar{B}_3(f_1(z)) \\
& + \bar{B}_1(f_1(z))\bar{B}_2'(f_1(z))f_1'(z)\bar{B}_3(f_1(z)) \\
& + \bar{B}_1(f_1(z))\bar{B}_2(f_1(z))\bar{B}_3'(f_1(z))f_1'(z)]
\end{aligned}
$$

$$
\begin{aligned}
L_q &= \lim_{z\to 1} \frac{d}{dz} W_q(z) \\
&= \lim_{z\to 1} \left[\frac{D'(z)N''(z) - N'(z)D''(z)}{2(D'(z))^2} \right] Q \\
&= \left[\frac{D'(1)N''(1) - N'(1)D''(1)}{2(D'(1))^2} \right] Q
\end{aligned} \tag{8.104}
$$

where primes and double primes in (8.104) denote first and second derivative at $z = 1$ respectively. Carrying out the derivative at $z = 1$, we have

$$
\begin{aligned}
N'(1) =& \lambda(\alpha+\beta)E(I)(1 - \bar{B}(\alpha)) + \lambda\alpha\beta E(I)(E(V_1) + pE(V_2))\bar{B}(\alpha) \\
N''(1) =& [-\lambda(\alpha+\beta)E(I(I-1)) - 2\lambda E(I)(-\lambda E(I) + \alpha)][\bar{B}(\alpha) - 1] \\
& + 2\lambda^2(E(I))^2[\bar{B}_1'(\alpha)\bar{B}_2(\alpha)\bar{B}_3(\alpha) + \bar{B}_1(\alpha)\bar{B}_2'(\alpha)\bar{B}_3(\alpha) \\
& + \bar{B}_1(\alpha)\bar{B}_2(\alpha)\bar{B}_3'(\alpha)][\alpha + \beta - \alpha\beta(E(V_1) + pE(V_2))] \\
& + \bar{B}(\alpha)[\alpha\beta(\lambda^2(E(I))^2(E(V_1^2) + pE(V_2^2)) \\
& + \lambda E(I(I-1))(E(V_1) + pE(V_2)) \\
& + 2\lambda^2(E(I))^2 pE(V_1)E(V_2)) \\
& - 2\lambda^2(E(I))^2(\alpha+\beta)(E(V_1) + pE(V_2))]
\end{aligned}
$$

$$
\begin{aligned}
D'(1) =& \lambda(\alpha+\beta)E(I)(\bar{B}(\alpha)-1)+\alpha\beta\bar{B}(\alpha) \\
& \times [1-\lambda E(I)(E(V_1)+pE(V_2))] \\
D''(1) =& [-\lambda(\alpha+\beta)E(I(I-1))+2\lambda^2(E(I))^2][1-\bar{B}(\alpha)] \\
& -2\lambda(\alpha+\beta)E(I)[1-\lambda E(I)(E(V_1)+pE(V_2))]\bar{B}(\alpha) \\
& +[\bar{B}_1'(\alpha)\bar{B}_2(\alpha)\bar{B}_3(\alpha)+\bar{B}_1(\alpha)\bar{B}_2'(\alpha)\bar{B}_3(\alpha) \\
& +\bar{B}_1(\alpha)\bar{B}_2(\alpha)\bar{B}_3'(\alpha)][-2\lambda^2(E(I))^2(\alpha+\beta) \\
& +2\lambda^2(E(I))^2\alpha\beta(E(V_1)+pE(V_2))-2\lambda E(I)\alpha\beta] \\
& -2\lambda^2(E(I))^2\alpha\beta pE(V_1)E(V_2)\bar{B}(\alpha) \\
& -\alpha\beta\bar{B}(\alpha)[\lambda^2(E(I))^2(E(V_1^2)+pE(V_2^2)) \\
& +\lambda E(I(I-1))(E(V_1)+pE(V_2))]
\end{aligned}
$$

where $E(B_1^2)$, $E(B_2^2)$, $E(B_3^2)$, $E(V_1^2)$ and $E(V_2^2)$ are the second moment of the service times and vacation times respectively. $E(I(I-1))$ is the second factorial moment of the batch size of arriving customers. Then if we substitute the values $N'(1), N''(1), D'(1), D''(1)$ in (8.104), we obtain L_q in the closed form.

Further, we find the mean system size L by using Little's formula. Thus we have

$$L = L_q + \rho \tag{8.105}$$

where L_q has been found by equation (8.104) and ρ is obtained from equation (8.103).

Let W_q and W denote the mean waiting time in the queue and in the system respectively. Then by using Little's formula, we obtain

$$W_q = \frac{L_q}{\lambda}$$

$$W = \frac{L}{\lambda}$$

where L_q and L have been found in equations (8.104) and (8.105).

8.8 Particular cases

Case 1: When the server has no optional vacation, i.e, p=0.

Then our model reduces to the $M^{[X]}/G/1$ queue with three stage heterogeneous service, compulsory vacation and service interruption. Using this in the main result of (8.102), (8.103) and (8.104), we can find the idle probability Q, utilization factor ρ, and the mean queue size L_q can be simplified to the following expressions.

$$\begin{aligned}
Q =& \frac{\lambda(\alpha+\beta)E(I)(\bar{B}(\alpha)-1)+\alpha\beta\bar{B}(\alpha)[1-\lambda E(I)E(V_1)]}{\alpha\beta\bar{B}(\alpha)} \\
\rho =& 1-\frac{\lambda(\alpha+\beta)E(I)(\bar{B}(\alpha)-1)+\alpha\beta\bar{B}(\alpha)[1-\lambda E(I)E(V_1)]}{\alpha\beta\bar{B}(\alpha)} \\
L_q =& \left[\frac{D'(1)N''(1)-N'(1)D''(1)}{2(D'(1))^2}\right]Q
\end{aligned}$$

where

$$\begin{aligned}
N'(1) =& \lambda(\alpha+\beta)E(I)(1-\bar{B}(\alpha))+\lambda\alpha\beta E(I)E(V_1)\bar{B}(\alpha) \\
N''(1) =& [-\lambda(\alpha+\beta)E(I(I-1))-2\lambda E(I)(-\lambda E(I)+\alpha)][\bar{B}(\alpha)-1] \\
& +2\lambda^2(E(I))^2[\bar{B}_1'(\alpha)\bar{B}_2(\alpha)\bar{B}_3(\alpha)+\bar{B}_1(\alpha)\bar{B}_2'(\alpha)\bar{B}_3(\alpha) \\
& +\bar{B}_1(\alpha)\bar{B}_2(\alpha)\bar{B}_3'(\alpha)][\alpha+\beta-\alpha\beta E(V_1)] \\
& +\bar{B}(\alpha)[\alpha\beta(\lambda^2(E(I))^2E(V_1^2)+\lambda E(I(I-1))E(V_1)) \\
& -2\lambda^2(E(I))^2(\alpha+\beta)E(V_1)] \\
D'(1) =& \lambda(\alpha+\beta)E(I)(\bar{B}(\alpha)-1)+\alpha\beta\bar{B}(\alpha)[1-\lambda E(I)E(V_1)] \\
D''(1) =& [-\lambda(\alpha+\beta)E(I(I-1))+2\lambda^2(E(I))^2][1-\bar{B}(\alpha)] \\
& -2\lambda(\alpha+\beta)E(I)[1-\lambda E(I)E(V_1)]\bar{B}(\alpha) \\
& +[\bar{B}_1'(\alpha)\bar{B}_2(\alpha)\bar{B}_3(\alpha)+\bar{B}_1(\alpha)\bar{B}_2'(\alpha)\bar{B}_3(\alpha)+\bar{B}_1(\alpha)\bar{B}_2(\alpha)\bar{B}_3'(\alpha)]
\end{aligned}$$

$$\times [-2\lambda^2(E(I))^2(\alpha+\beta) + 2\lambda^2(E(I))^2\alpha\beta E(V_1) - 2\lambda E(I)\alpha\beta]$$
$$- \alpha\beta\bar{B}(\alpha)[\lambda^2(E(I))^2E(V_1^2) + \lambda E(I(I-1))E(V_1)]$$

Case 2: When the server has no optional vacation and $C(z) = z$ i.e, p=0, $E(I)$= 1 and $E(I(I-1)) = 0$, then our model reduces to the $M/G/1$ queue with three stage heterogeneous service, service interruption and compulsory vacation. Using this in the main result of (8.102), (8.103) and (8.104), we can find the idle probability Q, utilization factor ρ, and the mean queue size L_q can be simplified to the following expressions.

$$Q = \frac{\lambda(\alpha+\beta)(\bar{B}(\alpha)-1) + \alpha\beta\bar{B}(\alpha)[1-\lambda E(V_1)]}{\alpha\beta\bar{B}(\alpha)}$$
$$\rho = 1 - \frac{\lambda(\alpha+\beta)(\bar{B}(\alpha)-1) + \alpha\beta\bar{B}(\alpha)[1-\lambda E(V_1)]}{\alpha\beta\bar{B}(\alpha)}$$
$$L_q = \left[\frac{D'(1)N''(1) - N'(1)D''(1)}{2(D'(1))^2}\right] Q$$

where

$$N'(1) = \lambda(\alpha+\beta)(1-\bar{B}(\alpha)) + \lambda\alpha\beta\bar{B}(\alpha)E(V_1)$$
$$\begin{aligned} N''(1) = & -2\lambda(-\lambda+\alpha)][\bar{B}(\alpha)-1] \\ & + 2\lambda^2[\bar{B}_1'(\alpha)\bar{B}_2(\alpha)\bar{B}_3(\alpha) + \bar{B}_1(\alpha)\bar{B}_2'(\alpha)\bar{B}_3(\alpha) \\ & + \bar{B}_1(\alpha)\bar{B}_2(\alpha)\bar{B}_3'(\alpha)][\alpha+\beta-\alpha\beta E(V_1)] \\ & + \bar{B}(\alpha)[\lambda^2\alpha\beta E(V_1^2) - 2\lambda^2(\alpha+\beta)E(V_1)] \end{aligned}$$
$$D'(1) = \lambda(\alpha+\beta)(\bar{B}(\alpha)-1) + \alpha\beta\bar{B}(\alpha)(1-\lambda E(V_1))$$
$$\begin{aligned} D''(1) = & 2\lambda^2[1-\bar{B}(\alpha)] - 2\lambda(\alpha+\beta)[1-\lambda E(V_1)]\bar{B}(\alpha) \\ & + [\bar{B}_1'(\alpha)\bar{B}_2(\alpha)\bar{B}_3(\alpha) + \bar{B}_1(\alpha)\bar{B}_2'(\alpha)\bar{B}_3(\alpha) \\ & + \bar{B}_1(\alpha)\bar{B}_2(\alpha)\bar{B}_3'(\alpha)][-2\lambda^2(\alpha+\beta) \\ & + 2\lambda^2\alpha\beta E(V_1) - 2\lambda\alpha\beta] \\ & - \lambda^2\alpha\beta\bar{B}(\alpha)E(V_1^2) \end{aligned}$$

Case 3: If $C(z) = z$ and $p = 0$ then equations (8.85) to (8.89) coincide with results of Maragatha Sundari and Srinivasan (2012b).

Case 4: If there are no second and third stages of service and server has no compulsory vacation and $C(z) = z$ i.e, $E(I)$= 1 and $E(I(I-1)) = 0$

Then our model reduces to the $M/G/1$ queue with service interruption, Bernoulli vacation. Using this in the main result of (8.102), (8.103) and (8.104), we can find the idle probability Q, utilization factor ρ, and the mean queue size L_q can be simplified to the following expressions.

$$\begin{aligned}
Q =& \frac{\lambda(\alpha+\beta)(\bar{B}_1(\alpha)-1)+\alpha\beta\bar{B}_1(\alpha)[1-\lambda pE(V_2)]}{\alpha\beta\bar{B}_1(\alpha)} \\
\rho =& 1-\frac{\lambda(\alpha+\beta)(\bar{B}_1(\alpha)-1)+\alpha\beta\bar{B}_1(\alpha)[1-\lambda E(I)pE(V_2)]}{\alpha\beta\bar{B}_1(\alpha)} \\
L_q =& \left[\frac{D'(1)N''(1)-N'(1)D''(1)}{2(D'(1))^2}\right]Q
\end{aligned}$$

where

$$\begin{aligned}
N'(1) =& \lambda(\alpha+\beta)(1-\bar{B}_1(\alpha))+\lambda\alpha\beta pE(V_2)\bar{B}_1(\alpha) \\
N''(1) =& -2\lambda(-\lambda+\alpha)][\bar{B}_1(\alpha)-1] \\
& +2\lambda^2\bar{B}_1'(\alpha)[\alpha+\beta-\alpha\beta pE(V_2)] \\
& +\bar{B}_1(\alpha)\lambda^2 pE(V_2^2)-2\lambda^2 p(\alpha+\beta)E(V_2) \\
D'(1) =& \lambda(\alpha+\beta)(\bar{B}_1(\alpha)-1) \\
& +\alpha\beta\bar{B}_1(\alpha)[1-\lambda pE(V_2)] \\
D''(1) =& 2\lambda^2[1-\bar{B}_1(\alpha)]-2\lambda(\alpha+\beta)[1-\lambda pE(V_2)]\bar{B}_1(\alpha) \\
& +\bar{B}_1'(\alpha)[-2\lambda^2(\alpha+\beta)+2\lambda^2\alpha\beta pE(V_2)-2\lambda\alpha\beta] \\
& -\alpha\beta\bar{B}_1(\alpha)\lambda^2 pE(V_2^2)
\end{aligned}$$

The above equations coincides with results of Balamani (2012).

Case 5: When the vacation follows exponential distribution in case 4 then the results coincide with Baskar et al. (2011).

8.9 Numerical results

To numerically illustrate the results obtained in this work, we consider that the service times and vacation times are exponentially distributed with rates μ_1, μ_2, μ_3 and γ.

To see the effect of various parameters on server's idle time Q, utilization factor ρ and various other queue characteristics such as L, W, L_q, W_q, we base our numerical example on the result found in case 1.

In Table 8.1, we can choose the following arbitrary values: μ_1 =4, μ_2 =3, μ_3 =2, $E(I)$= 0.4, $E(I(I-1))$= 0.05, γ= 3, α= 2, β= 4 while λ varies from 0.1 to 1.0 such that the stability condition is satisfied.

It clearly shows as long as increasing the arrival rate, the server's idle time decreases while the utilization factor, the mean queue size, system size and mean waiting time in the queue, the system of our queueing model are all increases.

Table 8.1: Computed values of various queue characteristics

λ	Q	ρ	L_q	L	W_q	W
0.1	0.928333	0.071667	0.032789	0.104456	0.327893	1.044560
0.2	0.856667	0.143333	0.077539	0.220872	0.387695	1.104361
0.3	0.785000	0.215000	0.137586	0.352586	0.458620	1.175287
0.4	0.713333	0.286667	0.217609	0.504275	0.544021	1.260688
0.5	0.641667	0.358333	0.324374	0.682700	0.648749	1.365416
0.6	0.570000	0.430000	0.468056	0.898056	0.780093	1.496759
0.7	0.498333	0.501667	0.664675	1.166342	0.949536	1.666202
0.8	0.426667	0.573333	0.941022	1.514355	1.176277	1.892944
0.9	0.355000	0.645000	1.345516	1.990516	1.495017	2.211684
1.0	0.283333	0.716667	1.975569	2.692235	1.975569	2.692235

Table 8.2: Computed values of various queue characteristics

γ	Q	ρ	L_q	L	W_q	W
1	0.320000	0.680000	1.468438	2.148438	1.468438	2.148438
2	0.420000	0.580000	0.965074	1.545074	0.965074	1.545074
3	0.453333	0.546667	0.856197	1.402864	0.856197	1.402864
4	0.470000	0.530000	0.809040	1.339046	0.809046	1.339046
5	0.480000	0.520000	0.782767	1.302767	0.782767	1.302767
6	0.486667	0.513333	0.766021	1.279355	0.766021	1.279355
7	0.491429	0.508571	0.754421	1.262992	0.754421	1.262992
8	0.495000	0.505000	0.745911	1.250911	0.745911	1.250911
9	0.497778	0.502222	0.739402	1.241624	0.739402	1.241624
10	0.500000	0.500000	0.734263	1.234263	0.734263	1.234263

In Table 8.2, we can choose the following arbitrary values: μ_1 =5, μ_2 =4, μ_3 =2, $E(I)$= 0.4, $E(I(I-1))$= 0.05, $\lambda = 3$, α= 2, β= 4 while γ varies from 1 to 10 such that the stability condition is satisfied.

It clearly shows as long as increasing the vacation rate, the server's idle time increases while the utilization factor, the mean queue size, system size and mean waiting time in the queue, the system of our queueing model are all decreases.

CHAPTER NINE

$M^{[X]}/G/1$ Feedback Queue with Three Stage Heterogeneous Service, Server Vacations and Restricted Admissibility

CHAPTER

NINE

$M^{[X]}/G/1$ FEEDBACK QUEUE WITH THREE STAGE HETEROGENEOUS SERVICE, SERVER VACATIONS AND RESTRICTED ADMISSIBILITY

9.1 Introduction

Levy and Yechilai (1976), Madan (1991), Takagi (1992), Rosenberg and Yechiali (1993), Borthakur and Chaudhury (1997), Madan and Al-Rawwash (2005) and many others have studied vacation queues with different vacation policies. In some queueing systems with batch arrival there is a restriction such that not all batches are allowed to join the system at all time. This policy is named restricted admissibility. Madan and Abu-Dayyeh (2002), Madan and Choudhury (2004) and Badamchi Zadeh (2009, 2012) proposed an queueing system with restricted admissibility of arriving batches and Bernoulli schedule server vacation.

In this chapter, we consider a $M^{[X]}/G/1$ feedback queue with three stage service, server vacations and restricted admissibility. Each customer undergoes three stage of heterogeneous service with general (arbitrary) service time

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distributions. As soon as the completion of third stage of service, if the customer is dissatisfied with his service, he can immediately join the tail of the original queue as a feedback customer with probability p to repeat the same service or may depart the system with probability $1 - p$ if service happens to be successful. The vacation period has two heterogeneous phases with general (arbitrary) distributions. Further, after service completion of a customer the server may take phase one vacation with probability r or may continue to stay in the system with probability $1 - r$. After the completion of phase one vacation the server may take phase two optional vacation with probability θ or return back to the system with probability 1 - θ. Arrival to the system follows Poisson distribution. In addition, we assume restricted admissibility of arriving batches in which not all batches are allowed to join the system at all times.

Here we derive time dependent probability generating functions in terms of Laplace transforms. We also derive the mean queue size and mean system size. Some particular cases and numerical results are also discussed.

The rest of this chapter is organized as follows. Model description is given in section 9.2. Definitions and Equations governing the system are given in section 9.3 and 9.4 respectively. The time dependent solution have been obtained in section 9.5 and corresponding steady state results have been derived explicitly in section 9.6. Mean queue size and mean system size are computed in section 9.7. Some particular cases and numerical results are discussed in section 9.8 and 9.9 respectively.

9.2 Model description

We assume the following to describe the queueing model of our study.

a) Customers arrive at the system in batches of variable size in a compound

Poisson process and they are provided one by one service on a first come - first served basis. Let $\lambda c_i dt$ $(i = 1, 2, \ldots)$ be the first order probability that a batch of i customers arrives at the system during a short interval of time $(t, t + dt]$, where $0 \leq c_i \leq 1$, $\sum_{i=1}^{\infty} c_i = 1$ and $\lambda > 0$ is the arrival rate of batches.

b) A single server provides three stages of service for each customer, with the service time follows general (arbitrary) distribution. Let $B_i(v)$ and $b_i(v)$ (i =1, 2, 3) be the distribution and the density function of the first stage, second stage and third stage of service respectively.

c) Let $\mu_i(x)dx$ be the conditional probability density of service completion during the interval $(x, x + dx]$, given that the elapsed service time is x, so that

$$\mu_i(x) = \frac{b_i(x)}{1 - B_i(x)}, \quad i = 1, 2, 3,$$

and therefore,

$$b_i(s) = \mu_i(s) e^{-\int_0^s \mu_i(x)dx}, \quad i = 1, 2, 3.$$

d) Moreover, after the completion of third stage of service, if the customer is dissatisfied with his service, he can immediately join the tail of the original queue as a feedback customer for receiving the same service with probability p. Otherwise the customer may depart forever from the system with probability $(1 - p)$. Further, we do not distinguish the new arrival with feedback.

e) As soon as the completion of third stage of service, the server may take phase one vacation with probability r or may continue to stay in the system with probability $1 - r$. After completion of phase one vacation the server may take phase two vacation with probability θ or return back to the system with probability $1-\theta$. On returning from vacation the server starts instantly serving the customer at the head of the queue, if any.

f) The server's vacation time follows a general (arbitrary) distribution with distribution function $V_i(t)$ and density function $v_i(t)$. Let $\gamma_i(x)dx$ be the conditional probability density of vacation completion during the interval $(x, x+dx]$, given that the elapsed vacation time is x, so that

$$\gamma_i(x) = \frac{v_i(x)}{1-V_i(x)}, \quad i = 1, 2,$$

and therefore,

$$v_i(t) = \gamma_i(t)e^{-\int\limits_0^t \gamma_i(x)dx}, \quad i = 1, 2.$$

h) In addition, we assume that the restricted admissibility of batches in which not all batches are allowed to join the system at all times. Let α $(0 \le \alpha \le 1)$ and β $(0 \le \beta \le 1)$ be the probability that an arriving batch will be allowed to join the system during the period of server's non-vacation period and vacation period respectively.

g) Various stochastic processes involved in the system are assumed to be independent of each other.

9.3 Definitions

We define

$P_n^{(1)}(x,t)$ = Probability that at time t, the server is active providing first stage of service and there are n $(n \ge 0)$ customers in the queue excluding the one being served and the elapsed service time is x. Consequently $P_n^{(1)}(t) =$

$\int_0^\infty P_n^{(1)}(x,t)dx$ denotes the probability that at time t there are n customers in the queue excluding the one customer in the first stage of service irrespective of the value of x.

$P_n^{(2)}(x,t)$ = Probability that at time t, the server is active providing second stage of service and there are n $(n \geq 0)$ customers in the queue excluding the one being served and the elapsed service time is x. Consequently $P_n^{(2)}(t) = \int_0^\infty P_n^{(2)}(x,t)dx$ denotes the probability that at time t there are n customers in the queue excluding the one customer in the second stage of service irrespective of the value of x.

$P_n^{(3)}(x,t)$ = Probability that at time t, the server is active providing third stage of service and there are n $(n \geq 0)$ customers in the queue excluding the one being served and the elapsed service time is x. Consequently $P_n^{(3)}(t) = \int_0^\infty P_n^{(3)}(x,t)dx$ denotes the probability that at time t there are n customers in the queue excluding the one customer in the third stage of service irrespective of the value of x.

$V_n^{(1)}(x,t)$ = Probability that at time t, the server is under phase one vacation with elapsed vacation time is x and there are n $(n \geq 0)$ customers in the queue. Consequently $V_n^{(1)}(t) = \int_0^\infty V_n^{(1)}(x,t)dx$ denotes the probability that at time t there are n customers in the queue and the server is under phase one vacation irrespective of the value of x.

$V_n^{(2)}(x,t)$ = Probability that at time t, the server is under phase two vacation with elapsed vacation time is x and there are n $(n \geq 0)$ customers in the queue. Consequently $V_n^{(2)}(t)=\int_0^\infty V_n^{(2)}(x,t)dx$ denotes the probability that at time t there are n customers in the queue and the server is under phase two vacation irrespective of the value of x.

$Q(t)$ = Probability that at time t, there are no customers in the system and the server is idle but available in the system.

9.4 Equations governing the system

The model is then, governed by the following set of differential-difference equations:

$$\frac{\partial}{\partial x}P_0^{(1)}(x,t)+\frac{\partial}{\partial t}P_0^{(1)}(x,t)+[\lambda+\mu_1(x)]P_0^{(1)}(x,t)=\lambda(1-\alpha)P_0^{(1)}(x,t) \tag{9.1}$$

$$\frac{\partial}{\partial x}P_n^{(1)}(x,t)+\frac{\partial}{\partial t}P_n^{(1)}(x,t)+[\lambda+\mu_1(x)]P_n^{(1)}(x,t)=\lambda(1-\alpha)P_n^{(1)}(x,t)+\lambda\alpha\sum_{k=1}^{n}c_kP_{n-k}^{(1)}(x,t),\quad n\geq 1 \tag{9.2}$$

$$\frac{\partial}{\partial x}P_0^{(2)}(x,t)+\frac{\partial}{\partial t}P_0^{(2)}(x,t)+[\lambda+\mu_2(x)]P_0^{(2)}(x,t)=\lambda(1-\alpha)P_0^{(2)}(x,t) \tag{9.3}$$

$$\frac{\partial}{\partial x}P_n^{(2)}(x,t)+\frac{\partial}{\partial t}P_n^{(2)}(x,t)+[\lambda+\mu_2(x)]P_n^{(2)}(x,t)=\lambda(1-\alpha)P_n^{(2)}(x,t)+\lambda\alpha\sum_{k=1}^{n}c_kP_{n-k}^{(2)}(x,t),\ n\geq 1 \tag{9.4}$$

$$\frac{\partial}{\partial x}P_0^{(3)}(x,t)+\frac{\partial}{\partial t}P_0^{(3)}(x,t)+[\lambda+\mu_3(x)]P_0^{(2)}(x,t)=\lambda(1-\alpha)P_0^{(3)}(x,t) \tag{9.5}$$

$$\frac{\partial}{\partial x}P_n^{(3)}(x,t)+\frac{\partial}{\partial t}P_n^{(3)}(x,t)+[\lambda+\mu_3(x)]P_n^{(3)}(x,t)=\lambda(1-\alpha)P_n^{(3)}(x,t)+\lambda\alpha\sum_{k=1}^{n}c_kP_{n-k}^{(3)}(x,t),\ n\geq 1 \tag{9.6}$$

$$\frac{\partial}{\partial x}V_0^{(1)}(x,t)+\frac{\partial}{\partial t}V_0^{(1)}(x,t)+[\lambda+\gamma_1(x)]V_0^{(1)}(x,t)=\lambda(1-\beta)V_0^{(1)}(x,t) \tag{9.7}$$

$$\frac{\partial}{\partial x}V_n^{(1)}(x,t)+\frac{\partial}{\partial t}V_n^{(1)}(x,t)+[\lambda+\gamma_1(x)]V_n^{(1)}(x,t)=\lambda(1-\beta)V_n^{(1)}(x,t)+\lambda\beta\sum_{k=1}^{n}c_kV_{n-k}^{(1)}(x,t),\quad n\geq 1 \tag{9.8}$$

$$\frac{\partial}{\partial x}V_0^{(2)}(x,t)+\frac{\partial}{\partial t}V_0^{(2)}(x,t)+[\lambda+\gamma_2(x)]V_0^{(2)}(x,t)=\lambda(1-\beta)V_0^{(2)}(x,t) \tag{9.9}$$

$$\frac{\partial}{\partial x}V_n^{(2)}(x,t)+\frac{\partial}{\partial t}V_n^{(2)}(x,t)+[\lambda+\gamma_2(x)]V_n^{(2)}(x,t)=\lambda(1-\beta)V_n^{(2)}(x,t)$$
$$+\lambda\beta\sum_{k=1}^{n}c_kV_{n-k}^{(2)}(x,t),\quad n\geq 1 \quad (9.10)$$

$$\begin{aligned}\frac{d}{dt}Q(t)=&-\lambda Q(t)+(1-\theta)\int_0^\infty \gamma_1(x)V_0^{(1)}(x,t)dx\\&+\lambda(1-\alpha)Q(t)+\int_0^\infty \gamma_2(x)V_0^{(2)}(x,t)dx\\&+(1-p)(1-r)\int_0^\infty \mu_3(x)P_0^{(3)}(x,t)dx \qquad (9.11)\end{aligned}$$

The above set of equations are to be solved subject to the following boundary conditions:

$$\begin{aligned}P_n^{(1)}(0,t)=&\ \alpha\lambda C_{n+1}Q(t)+(1-\theta)\int_0^\infty \gamma_1(x)V_{n+1}^{(1)}(x,t)dx\\&+\int_0^\infty \gamma_2(x)V_{n+1}^{(2)}(x,t)dx\\&+p(1-r)\int_0^\infty \mu_3(x)P_n^{(3)}(x,t)dx\\&+(1-p)(1-r)\int_0^\infty \mu_3(x)P_{n+1}^{(3)}(x,t)dx,\quad n\geq 0 \qquad (9.12)\end{aligned}$$

$$P_n^{(2)}(0,t)=\int_0^\infty \mu_1(x)P_n^{(1)}(x,t)dx,\quad n\geq 0 \qquad (9.13)$$

$$P_n^{(3)}(0,t)=\int_0^\infty \mu_2(x)P_n^{(2)}(x,t)dx,\quad n\geq 0 \qquad (9.14)$$

$$V_n^{(1)}(0,t)=r(1-p)\int_0^\infty \mu_3(x)P_n^{(3)}(x,t)dx+rp\int_0^\infty \mu_3(x)P_{n-1}^{(3)}(x,t)dx,\quad n\geq 0 \qquad (9.15)$$

$$V_n^{(2)}(0,t)=\theta\int_0^\infty \gamma_1(x)V_n^{(1)}(x,t)dx,\quad n\geq 0 \qquad (9.16)$$

We assume that initially there are no customers in the system and the server is idle. So the initial conditions are

$$V_0^{(j)}(0) = V_n^{(j)}(0) = P_n^{(i)}(0) = 0 \text{ for } n = 0, 1, 2, ...;$$
$$j = 1, 2\ ; \quad i = 1, 2, 3 \text{ and } Q(0) = 1 \tag{9.17}$$

9.5 Generating functions of the queue length: The time-dependent solution

In this section, we obtain the transient solution for the above set of differential-difference equations.

Theorem : *The system of differential-difference equations to describe an $M^{[X]}/G/1$ feedback queue with three stages of heterogeneous service, Bernoulli vacation and optional server vacation with restricted admissibility are given by equations (9.1) to (9.16) with initial conditions (9.17) and the generating functions of transient solution are given by equations (9.70) to (9.74).*

Proof: We define the probability generating functions, for i =1, 2, 3

$$P^{(i)}(x,z,t) = \sum_{n=0}^{\infty} z^n P_n^{(i)}(x,t); \quad P^{(i)}(z,t) = \sum_{n=0}^{\infty} z^n P_n^{(i)}(t), C(z) = \sum_{n=1}^{\infty} c_n z^n,$$

$$V^{(j)}(x,z,t) = \sum_{n=0}^{\infty} z^n V_n^{(j)}(x,t); \quad V^{(j)}(z,t) = \sum_{n=0}^{\infty} z^n V_n^{(j)}(t), \ j = 1, 2. \tag{9.18}$$

which are convergent inside the circle given by $|\ z\ | \le 1$ and define the Laplace transform of a function $f(t)$ as

$$\bar{f}(s) = \int_0^{\infty} e^{-st} f(t) dt, \quad \Re(s) > 0.$$

Taking the Laplace transform of equations (9.1) to (9.16) and using (9.17), we obtain

$$\frac{\partial}{\partial x}\bar{P}_0^{(1)}(x,s) + (s+\lambda\alpha+\mu_1(x))\bar{P}_0^{(1)}(x,s) = 0 \tag{9.19}$$

$$\frac{\partial}{\partial x}\bar{P}_n^{(1)}(x,s) + (s+\lambda\alpha+\mu_1(x))\bar{P}_n^{(1)}(x,s) = \lambda\alpha\sum_{k=1}^{n} c_k\bar{P}_{n-k}^{(1)}(x,s), n\geq 1 \tag{9.20}$$

$$\frac{\partial}{\partial x}\bar{P}_0^{(2)}(x,s) + (s+\lambda\alpha+\mu_2(x))\bar{P}_0^{(2)}(x,s) = 0 \tag{9.21}$$

$$\frac{\partial}{\partial x}\bar{P}_n^{(2)}(x,s) + (s+\lambda\alpha+\mu_2(x))\bar{P}_n^{(2)}(x,s) = \lambda\alpha\sum_{k=1}^{n} c_k\bar{P}_{n-k}^{(2)}(x,s), n\geq 1 \tag{9.22}$$

$$\frac{\partial}{\partial x}\bar{P}_0^{(3)}(x,s) + (s+\lambda\alpha+\mu_3(x))\bar{P}_0^{(3)}(x,s) = 0 \tag{9.23}$$

$$\frac{\partial}{\partial x}\bar{P}_n^{(3)}(x,s) + (s+\lambda\alpha+\mu_3(x))\bar{P}_n^{(3)}(x,s) = \lambda\alpha\sum_{k=1}^{n} c_k\bar{P}_{n-k}^{(3)}(x,s), n\geq 1 \tag{9.24}$$

$$\frac{\partial}{\partial x}\bar{V}_0^{(1)}(x,s) + (s+\lambda\beta+\gamma_1(x))\bar{V}_0^{(1)}(x,s) = 0 \tag{9.25}$$

$$\frac{\partial}{\partial x}\bar{V}_n^{(1)}(x,s) + (s+\lambda\beta+\gamma_1(x))\bar{V}_n^{(1)}(x,s) = \lambda\beta\sum_{k=1}^{n} c_k\bar{V}_{n-k}^{(1)}(x,s), n\geq 1 \tag{9.26}$$

$$\frac{\partial}{\partial x}\bar{V}_0^{(2)}(x,s) + (s+\lambda\beta+\gamma_2(x))\bar{V}_0^{(2)}(x,s) = 0 \tag{9.27}$$

$$\frac{\partial}{\partial x}\bar{V}_n^{(2)}(x,s) + (s+\lambda\beta+\gamma_2(x))\bar{V}_n^{(2)}(x,s) = \lambda\beta\sum_{k=1}^{n} c_k\bar{V}_{n-k}^{(2)}(x,s), n\geq 1 \tag{9.28}$$

$$\begin{aligned}[s+\lambda\alpha]\bar{Q}(s) =& 1 + (1-\theta)\int_0^\infty \gamma_1(x)\bar{V}_0^{(1)}(x,s)dx \\ &+ \int_0^\infty \gamma_2(x)\bar{V}_0^{(2)}(x,s)dx \\ &+ (1-p)(1-r)\int_0^\infty \mu_3(x)\bar{P}_0^{(3)}(x,s)dx\end{aligned} \tag{9.29}$$

$$\bar{P}_n^{(1)}(0,s) = \alpha\lambda c_{n+1}\bar{Q}(s) + (1-\theta)\int_0^\infty \gamma_1(x)\bar{V}_{n+1}^{(1)}(x,s)dx$$
$$+ \int_0^\infty \gamma_2(x)\bar{V}_{n+1}^{(2)}(x,s)dx$$
$$+ p(1-r)\int_0^\infty \mu_3(x)\bar{P}_n^{(3)}(x,s)dx$$
$$+ (1-p)(1-r)\int_0^\infty \bar{P}_{n+1}^{(3)}(x,s)\mu_3(x)dx, \quad n \geq 0 \qquad (9.30)$$

$$\bar{P}_n^{(2)}(0,s) = \int_0^\infty \mu_1(x)\bar{P}_n^{(1)}(x,s)dx, \quad n \geq 0 \qquad (9.31)$$

$$\bar{P}_n^{(3)}(0,s) = \int_0^\infty \mu_2(x)\bar{P}_n^{(2)}(x,s)dx, \quad n \geq 0 \qquad (9.32)$$

$$\bar{V}_n^{(1)}(0,s) = r(1-p)\int_0^\infty \mu_3(x)\bar{P}_n^{(3)}(x,s)dx$$
$$+ rp\int_0^\infty \mu_3(x)\bar{P}_{n-1}^{(3)}(x,s)dx, \; n \geq 0 \qquad (9.33)$$

$$\bar{V}_n^{(2)}(0,s) = \theta\int_0^\infty \gamma_1(x)\bar{V}_n^{(1)}(x,s)dx, \quad n \geq 0 \qquad (9.34)$$

Now multiplying equations (9.20), (9.22), (9.24), (9.26) and (9.28) by z^n and summing over n from 1 to ∞, adding to equation (9.19), (9.21), (9.23), (9.25), (9.27) and using the generating functions defined in (9.18), we get

$$\frac{\partial}{\partial x}\bar{P}^{(1)}(x,z,s) + [s + \lambda\alpha(1-C(z)) + \mu_1(x)]\bar{P}^{(1)}(x,z,s) = 0 \qquad (9.35)$$

$$\frac{\partial}{\partial x}\bar{P}^{(2)}(x,z,s) + [s + \lambda\alpha(1-C(z)) + \mu_2(x)]\bar{P}^{(2)}(x,z,s) = 0 \qquad (9.36)$$

$$\frac{\partial}{\partial x}\bar{P}^{(3)}(x,z,s) + [s + \lambda\alpha(1-C(z)) + \mu_3(x)]\bar{P}^{(3)}(x,z,s) = 0 \qquad (9.37)$$

$$\frac{\partial}{\partial x}\bar{V}^{(1)}(x,z,s) + [s + \lambda\beta(1-C(z)) + \gamma_1(x)]\bar{V}^{(1)}(x,z,s) = 0 \qquad (9.38)$$

$$\frac{\partial}{\partial x}\bar{V}^{(2)}(x,z,s) + [s + \lambda\beta(1-C(z)) + \gamma_2(x)]\bar{V}^{(2)}(x,z,s) = 0 \qquad (9.39)$$

For the boundary conditions, we multiply both sides of equation (9.30) by z^n

summing over n from 0 to ∞ and use the equation (9.18), we get

$$
\begin{aligned}
z\bar{P}^{(1)}(0,z,s) = {} & \alpha\lambda C(z)\bar{Q}(s) + (1-\theta)\int_0^\infty \gamma_1(x)\bar{V}^{(1)}(x,z,s)dx \\
& + \int_0^\infty \gamma_2(x)\bar{V}^{(2)}(x,z,s)dx - \int_0^\infty \gamma_2(x)\bar{V}_0^{(2)}(x,s)dx \\
& + pz(1-r)\int_0^\infty \mu_3(x)\bar{P}^{(3)}(x,z,s)dx \\
& - (1-p)(1-r)\int_0^\infty \mu_3(x)\bar{P}^{(3)}(x,z,s)dx \\
& - (1-\theta)\int_0^\infty \gamma_1(x)\bar{V}_0^{(1)}(x,s)dx
\end{aligned}
$$

Using equation (9.29), the above equation becomes

$$
\begin{aligned}
z\bar{P}^{(1)}(0,z,s) = {} & 1 + [\lambda\alpha(C(z)-1)-s]\bar{Q}(s) \\
& + (1-\theta)\int_0^\infty \gamma_1(x)\bar{V}^{(1)}(x,z,s)dx \\
& + \int_0^\infty \gamma_2(x)\bar{V}^{(2)}(x,z,s)dx \\
& + (pz+1-p)(1-r)\int_0^\infty \mu_3(x)\bar{P}^{(3)}(x,z,s)dx \qquad (9.40)
\end{aligned}
$$

Performing similar operation on equations (9.31) to (9.34), we get

$$\bar{P}^{(2)}(0,z,s) = \int_0^\infty \mu_1(x)\bar{P}^{(1)}(x,z,s)dx \qquad (9.41)$$

$$\bar{P}^{(3)}(0,z,s) = \int_0^\infty \mu_2(x)\bar{P}^{(2)}(x,z,s)dx \qquad (9.42)$$

$$\bar{V}^{(1)}(0,z,s) = r(1-p+pz)\int_0^\infty \mu_3(x)\bar{P}^{(3)}(x,z,s)dx \qquad (9.43)$$

$$\bar{V}^{(2)}(0,z,s) = \theta\int_0^\infty \gamma_1(x)\bar{V}^{(1)}(x,z,s)dx \qquad (9.44)$$

Integrating equation (9.35) between 0 and x, we get

$$\bar{P}^{(1)}(x,z,s) = \bar{P}^{(1)}(0,z,s)e^{-[s+\lambda\alpha(1-C(z))]x-\int\limits_0^x \mu_1(t)dt} \tag{9.45}$$

where $\bar{P}^{(1)}(0,z,s)$ is given by equation (9.40).

Again integrating equation (9.45) by parts with respect to x, yields

$$\bar{P}^{(1)}(z,s) = \bar{P}^{(1)}(0,z,s)\left[\frac{1-\bar{B}_1(s+\lambda\alpha(1-C(z)))}{s+\lambda\alpha(1-C(z))}\right] \tag{9.46}$$

where

$$\bar{B}_1(s+\lambda\alpha(1-C(z))) = \int\limits_0^\infty e^{-[s+\lambda\alpha(1-C(z))]x}dB_1(x)$$

is the Laplace-Stieltjes transform of the first stage of service time $B_1(x)$. Now multiplying both sides of equation (9.45) by $\mu_1(x)$ and integrating over x, we obtain

$$\int\limits_0^\infty \bar{P}^{(1)}(x,z,s)\mu_1(x)dx = \bar{P}^{(1)}(0,z,s)\bar{B}_1[s+\lambda\alpha(1-c(z))] \tag{9.47}$$

Similarly, on integrating equations (9.36) to (9.39) from 0 to x, we get

$$\bar{P}^{(2)}(x,z,s) = \bar{P}^{(2)}(0,z,s)e^{-[s+\lambda\alpha(1-C(z))]x-\int\limits_0^x \mu_2(t)dt} \tag{9.48}$$

$$\bar{P}^{(3)}(x,z,s) = \bar{P}^{(3)}(0,z,s)e^{-[s+\lambda\alpha(1-C(z))]x-\int\limits_0^x \mu_3(t)dt} \tag{9.49}$$

$$\bar{V}^{(1)}(x,z,s) = \bar{V}^{(1)}(0,z,s)e^{-[s+\lambda\beta(1-C(z))]x-\int\limits_0^x \gamma_1(t)dt} \tag{9.50}$$

$$\bar{V}^{(2)}(x,z,s) = \bar{V}^{(2)}(0,z,s)e^{-[s+\lambda\beta(1-C(z))]x-\int\limits_0^x \gamma_2(t)dt} \tag{9.51}$$

where $\bar{P}^{(2)}(0,z,s)$, $\bar{P}^{(3)}(0,z,s)$, $\bar{V}^{(1)}(0,z,s)$ and $\bar{V}^{(2)}(0,z,s)$ are given by equations (9.41) to (9.44).

Again integrating equations (9.48) to (9.51) by parts with respect to x, yields

$$\bar{P}^{(2)}(z,s) = \bar{P}^{(2)}(0,z,s)\left[\frac{1-\bar{B}_2(s+\lambda\alpha(1-C(z)))}{s+\lambda\alpha(1-c(z))}\right] \tag{9.52}$$

$$\bar{P}^{(3)}(z,s) = \bar{P}^{(3)}(0,z,s)\left[\frac{1-\bar{B}_3(s+\lambda\alpha(1-C(z)))}{s+\lambda\alpha(1-C(z))}\right] \tag{9.53}$$

$$\bar{V}^{(1)}(z,s) = \bar{V}^{(1)}(0,z,s)\left[\frac{1-\bar{V}_1(s+\lambda\beta(1-C(z)))}{s+\lambda\beta(1-C(z))}\right] \tag{9.54}$$

$$\bar{V}^{(2)}(z,s) = \bar{V}^{(2)}(0,z,s)\left[\frac{1-\bar{V}_2(s+\lambda\beta(1-C(z)))}{s+\lambda\beta(1-C(z))}\right] \tag{9.55}$$

Now multiplying both sides of equations (9.48) to (9.51) by $\mu_2(x)$, $\mu_3(x)$, $\gamma_1(x)$, $\gamma_2(x)$ and integrating over x, we obtain

$$\int_0^\infty \bar{P}^{(2)}(x,z,s)\mu_2(x)dx = \bar{P}^{(2)}(0,z,s)\bar{B}_2[s+\lambda\alpha(1-C(z))] \tag{9.56}$$

$$\int_0^\infty \bar{P}^{(3)}(x,z,s)\mu_3(x)dx = \bar{P}^{(3)}(0,z,s)\bar{B}_3[s+\lambda\alpha(1-C(z))] \tag{9.57}$$

$$\int_0^\infty \bar{V}^{(1)}(x,z,s)\gamma_1(x)dx = \bar{V}^{(1)}(0,z,s)\bar{V}_1[s+\lambda\beta(1-C(z))] \tag{9.58}$$

$$\int_0^\infty \bar{V}^{(2)}(x,z,s)\gamma_2(x)dx = \bar{V}^{(2)}(0,z,s)\bar{V}_2[s+\lambda\beta(1-C(z))] \tag{9.59}$$

where

$$\bar{B}_2(s+\lambda\alpha-\lambda\alpha C(z)) = \int_0^\infty e^{-[s+\lambda\alpha(1-C(z))]x}dB_2(x)$$

$$\bar{B}_3(s+\lambda\alpha-\lambda\alpha C(z)) = \int_0^\infty e^{-[s+\lambda\alpha(1-C(z))]x}dB_3(x)$$

$$\bar{V}_1(s+\lambda\alpha(1-C(z)) = \int_0^\infty e^{-[s+\lambda\beta(1-C(z))]x} dV_1(x)$$

$$\bar{V}_2(s+\lambda\alpha(1-C(z)) = \int_0^\infty e^{-[s+\lambda\beta(1-C(z))]x} dV_2(x)$$

are the Laplace-Stieltjes transform of the second stage of service time $B_2(x)$, third stage of service time $B_3(x)$, phase one vacation time $V_1(x)$ and phase two vacation time $V_2(x)$ respectively.

Using equation (9.47), equation (9.41) reduces to

$$\bar{P}^{(2)}(0,z,s) = \bar{B}_1(a)\bar{P}^{(1)}(0,z,s) \tag{9.60}$$

Now using equations (9.56) and (9.60) in (9.42), we get

$$\bar{P}^{(3)}(0,z,s) = \bar{B}_1(a)\bar{B}_2(a)\bar{P}^{(1)}(0,z,s) \tag{9.61}$$

By using equations (9.57) and (9.61) in (9.43), we get

$$\bar{V}^{(1)}(0,z,s) = r(1-p+pz)\bar{B}_1(a)\bar{B}_2(a)\bar{B}_3(a)\bar{P}^{(1)}(0,z,s) \tag{9.62}$$

Using equations (9.58) and (9.62), we can write equation (9.44) as

$$\bar{V}^{(2)}(0,z,s) = \theta r(1-p+pz)\bar{B}_1(a)\bar{B}_2(a)\bar{B}_3(a)\bar{V}_1(b)\bar{P}^{(1)}(0,z,s) \tag{9.63}$$

Now using equations (9.57), (9.58) and (9.59), equation (9.40) becomes

$$\begin{aligned} z\bar{P}^{(1)}(0,z,s) =& 1 + [\lambda\alpha(C(z)-1)-s]\bar{Q}(s) \\ &+ (1-\theta)\bar{V}_1(b)\bar{V}^{(1)}(0,z,s) + \bar{V}_2(b)\bar{V}^{(2)}(0,z,s) \\ &+ (pz+1-p)(1-r)\bar{B}_3(a)\bar{P}^{(3)}(0,z,s) \end{aligned}$$

Using equations (9.61), (9.62) and (9.63), the above equation reduces to

$$\bar{P}^{(1)}(0,z,s) = \frac{1+[\lambda\alpha(C(z)-1)-s]\bar{Q}(s)}{Dr} \tag{9.64}$$

where

$$\begin{aligned} Dr =& z-(1-p+pz)\bar{B}_1(a)\bar{B}_2(a)\bar{B}_3(a) \\ & \times [1-r+r\bar{V}_1(b)(1-\theta+\theta\bar{V}_2(b))] \end{aligned} \tag{9.65}$$

$a = s+\lambda\alpha(1-C(z))$ and $b = s+\lambda\beta(1-C(z))$.

Substituting the value of $\bar{P}^{(1)}(0,z,s)$ from equation (9.64) into equations (9.60) to (9.63), we get

$$\bar{P}^{(2)}(0,z,s) = \bar{B}_1(a)\frac{[1+[\lambda\alpha(C(z)-1)-s]\bar{Q}(s)]}{Dr} \tag{9.66}$$

$$\bar{P}^{(3)}(0,z,s) = \bar{B}_1(a)\bar{B}_2(a)\frac{[1+[\lambda\alpha(C(z)-1)-s]\bar{Q}(s)]}{Dr} \tag{9.67}$$

$$\begin{aligned} \bar{V}^{(1)}(0,z,s) =& r(1-p+pz)\bar{B}_1(a)\bar{B}_2(a)\bar{B}_3(a) \\ & \times \frac{[1+[\lambda\alpha(C(z)-1)-s]\bar{Q}(s)]}{Dr} \end{aligned} \tag{9.68}$$

$$\begin{aligned} \bar{V}^{(2)}(0,z,s) =& \theta r(1-p+pz)\bar{B}_1(a)\bar{B}_2(a)\bar{B}_3(a)\bar{V}_1(b) \\ & \times \frac{[1+[\lambda\alpha(C(z)-1)-s]\bar{Q}(s)]}{Dr} \end{aligned} \tag{9.69}$$

Using equations (9.64), (9.66) to (9.69) in (9.46), (9.52) to (9.55), we get

$$\bar{P}^{(1)}(z,s) = \frac{[(1-s\bar{Q}(s))+\lambda\alpha(C(z)-1)\bar{Q}(s)]}{Dr}\frac{[1-\bar{B}_1(a)]}{a} \tag{9.70}$$

$$\bar{P}^{(2)}(z,s) = \frac{\bar{B}_1(a)[(1-s\bar{Q}(s))+\lambda\alpha(C(z)-1)\bar{Q}(s)]}{Dr}\frac{[1-\bar{B}_2(a)]}{a} \tag{9.71}$$

$$\begin{aligned} \bar{P}^{(3)}(z,s) =& \frac{\bar{B}_1(a)\bar{B}_2(a)}{Dr}\frac{[1-\bar{B}_3(a)]}{a} \\ & \times [(1-s\bar{Q}(s))+\lambda\alpha(C(z)-1)\bar{Q}(s)] \end{aligned} \tag{9.72}$$

$$\bar{V}^{(1)}(z,s) = \frac{r(1-p+pz)\bar{B}_1(a)\bar{B}_2(a)\bar{B}_3(a)}{Dr} \times [(1-s\bar{Q}(s)) + \lambda\alpha(C(z)-1)\bar{Q}(s)]\frac{[1-\bar{V}_1(b)]}{b} \quad (9.73)$$

$$\bar{V}^{(2)}(z,s) = \frac{\theta r(1-p+pz)\bar{B}_1(a)\bar{B}_2(a)\bar{B}_3(a)\bar{V}_1(b)}{Dr} \times [(1-s\bar{Q}(s)) + \lambda\alpha(C(z)-1)\bar{Q}(s)]\frac{[1-\bar{V}_2(b)]}{b} \quad (9.74)$$

Thus $\bar{P}^{(1)}(z,s)$, $\bar{P}^{(2)}(z,s)$, $\bar{P}^{(3)}(z,s)$, $\bar{V}^{(1)}(z,s)$ and $\bar{V}^{(2)}(z,s)$ are completely determined from equations (9.70) to (9.74) which completes the proof of the theorem.

9.6 The steady state results

In this section, we shall derive the steady state probability distribution for our queueing model. To define the steady probabilities, we suppress the argument t wherever it appears in the time-dependent analysis. This can be obtained by applying the Tauberian property,

$$\lim_{s\to 0} s\bar{f}(s) = \lim_{t\to\infty} f(t)$$

In order to determine $\bar{P}^{(1)}(z,s)$, $\bar{P}^{(2)}(z,s)$, $\bar{P}^{(3)}(z,s)$, $\bar{V}^{(1)}(z,s)$ and $\bar{V}^{(2)}(z,s)$ completely, we have yet to determine the unknown Q which appears in the numerators of the right hand sides of equations (9.70) to (9.74). For that purpose, we shall use the normalizing condition

$$P^{(1)}(1) + P^{(2)}(1) + P^{(3)}(1) + V^{(1)}(1) + V^{(2)}(1) + Q = 1$$

The steady state probabilities for an $M^{[X]}/G/1$ feedback queue with three stage heterogeneous service, server vacations with restricted admissibility are given by

$$
\begin{aligned}
P^{(1)}(1) =& \frac{\lambda\alpha E(I)E(B_1)Q}{dr}\\
P^{(2)}(1) =& \frac{\lambda\alpha E(I)E(B_2)Q}{dr}\\
P^{(3)}(1) =& \frac{\lambda\alpha E(I)E(B_3)Q}{dr}\\
V^{(1)}(1) =& \frac{\lambda\alpha r E(I)E(V_1)Q}{dr}\\
V^{(2)}(1) =& \frac{\lambda\alpha r\theta E(I)E(V_2)Q}{dr}
\end{aligned}
$$

where

$$dr = 1 - p - \lambda E(I)[\alpha(E(B_1) + E(B_2) + E(B_3)) + r\beta E(V)],$$

and $E(V) = E(V_1) + \theta E(V_2)$.

$P^{(1)}(1),\ P^{(2)}(1),\ P^{(3)}(1),\ V^{(1)}(1),\ V^{(2)}(1)$ and Q are the steady state probabilities that the server is providing first stage of service, second stage of service, third stage of service, server under phase one vacation, phase two vacation and idle respectively without regard to the number of customers in the queue.

Multiplying both sides of equations (9.70) to (9.74) by s, taking limit as $s \to 0$, applying Tauberian property and simplifying, we obtain

$$P^{(1)}(z) = \frac{\lambda\alpha(C(z)-1)[1-\bar{B}_1(f_1(z))]Q}{f_1(z)D(z)} \tag{9.75}$$

$$P^{(2)}(z) = \frac{\lambda\alpha(C(z)-1)\bar{B}_1(f_1(z))[1-\bar{B}_2(f_1(z))]Q}{f_1(z)D(z)} \tag{9.76}$$

$$P^{(3)}(z) = \frac{\lambda\alpha(C(z)-1)\bar{B}_1(f_1(z))\bar{B}_2(f_1(z))[1-\bar{B}_3(f_1(z))]Q}{f_1(z)D(z)} \tag{9.77}$$

$$V^{(1)}(z) = \frac{\lambda\alpha r(1-p+pz)(C(z)-1))\bar{B}(z)[1-\bar{V}_1(f_2(z))]Q}{f_2(z)D(z)} \tag{9.78}$$

$$V^{(2)}(z) = \frac{\lambda\alpha r\theta(1-p+pz)(C(z)-1)\bar{B}(z)\bar{V}_1(f_2(z))[1-\bar{V}_2(f_2(z))]Q}{f_2(z)D(z)} \quad (9.79)$$

where

$\bar{B}(z) = \bar{B}_1(f_1(z))\bar{B}_2(f_1(z))\bar{B}_3(f_1(z))$, $f_1(z) = \lambda\alpha(1-C(z))$,

$f_2(z) = \lambda\beta(1-C(z))$ and

$$D(z) = z - (1-p+pz)\bar{B}(z)[1-r+r\bar{V}_1(f_2(z))(1-\theta+\theta\bar{V}_2(f_2(z)))]. \quad (9.80)$$

Let $W_q(z)$ denote the probability generating function of the queue size irrespective of the state of the system. Then adding equations (9.75) to (9.79), we obtain

$$W_q(z) = P^{(1)}(z) + P^{(2)}(z) + P^{(3)}(z) + V^{(1)}(z) + V^{(2)}(z)$$

$$\begin{aligned} W_q(z) = & \frac{\lambda\alpha(C(z)-1)[1-\bar{B}_1(f_1(z))]Q}{f_1(z)D(z)} \\ & + \frac{\lambda\alpha(C(z)-1)\bar{B}_1(f_1(z))[1-\bar{B}_2(f_1(z))]Q}{f_1(z)D(z)} \\ & + \frac{\lambda\alpha(C(z)-1)\bar{B}_1(f_1(z))\bar{B}_2(f_1(z))[1-\bar{B}_3(f_1(z))]Q}{f_1(z)D(z)} \\ & + \frac{\lambda\alpha r(1-p+pz)(C(z)-1))\bar{B}(z)[1-\bar{V}_1(f_2(z))]Q}{f_2(z)D(z)} \\ & + \frac{\lambda\alpha r\theta(1-p+pz)(C(z)-1)\bar{B}(z)\bar{V}_1(f_2(z))[1-\bar{V}_2(f_2(z))]Q}{f_2(z)D(z)} \end{aligned} \quad (9.81)$$

we see that for z=1, $W_q(z)$ is indeterminate of the form 0/0. Therefore, we apply L'Hopital's rule and on simplifying, we obtain

$$W_q(1) = \frac{\alpha\lambda E(I)[E(B_1) + E(B_2) + E(B_3) + rE(V)]}{dr}Q$$

where C(1)= 1, $C'(1) = E(I)$ is mean batch size of the arriving customers, $E(B_i) = -\bar{B}_i'(0)$, $E(V_j) = -\bar{V}_j'(0)$, $i = 1, 2, 3$, $j = 1, 2$.

Therefore adding Q to above equation, equating to 1 and simplifying, we get

$$Q = 1 - \rho \tag{9.82}$$

and hence the utilization factor ρ of the system is given by

$$\rho = \frac{\alpha\lambda E(I)[E(B_1) + E(B_2) + E(B_3) + rE(V)]}{1 - p - r\lambda E(I)(\beta - \alpha)E(V)} \tag{9.83}$$

where $\rho < 1$ is the stability condition under which the steady state exists. Equation (9.82) gives the probability that the server is idle. Substituting Q from (9.82) into (9.81), we have completely and explicitly determined $W_q(z)$, the probability generating function of the queue size.

9.7 The mean queue size and the mean system size

Let L_q denote the mean number of customers in the queue under the steady state. Then

$$L_q = \frac{d}{dz}W_q(z) \ \text{ at } z = 1$$

since this formula gives 0/0 form, then we write $W_q(z)$ given in (9.81) as $W_q(z) = \dfrac{N(z)}{D(z)}$Q where

$$\begin{aligned} N(z) = & -\beta - \bar{B}(z)[-\beta + r\alpha(p(z-1)+1) \\ & \times (1 - \bar{V}_1(f_3(z))(1 - \theta + \theta\bar{V}_2(f_3(z))))] \end{aligned}$$

and $D(z)$ is given in equation (9.80).

$$\begin{aligned}
N'(z) =& -[\bar{B}_1'(f_1(z))f_1'(z)\bar{B}_2(f_1(z))\bar{B}_3(f_1(z)) \\
& + \bar{B}_1(f_1(z))\bar{B}_2'(f_1(z))f_2'(z)\bar{B}_3(f_1(z)) \\
& + \bar{B}_1(f_1(z))\bar{B}_2(f_1(z))\bar{B}_3'(f_2(z))f_1'(z)] \\
& \times [-\beta + r\alpha(p(z-1)+1)(1-\bar{V}_1(f_2(z))(1-\theta+\theta\bar{V}_2(f_2(z))))] \\
& - pr\alpha\bar{B}_1(f_1(z))\bar{B}_2(f_1(z))\bar{B}_3(f_1(z)) \\
& \times [1-\bar{V}_1(f_2(z))(1-\theta+\theta\bar{V}_2(f_2(z)))] \\
& + \alpha r(p(z-1)+1)[-\bar{V}_1'(f_2(z))f_2'(z)(1-\theta+\theta\bar{V}_2(f_2(z))) \\
& - \bar{V}_1(f_3(z))\theta\bar{V}_2'(f_2(z))f_2'(z)] \\
D'(z) =& 1 - p\bar{B}_1(f_1(z))\bar{B}_2(f_1(z))\bar{B}_3(f_1(z)) \\
& \times [1-r+r\bar{V}_1(f_2(z))(1-\theta+\theta V_2(f_2(z))] \\
& - (1-p+pz)[1-r+r\bar{V}_1(f_2(z))(1-\theta+\theta V_2(f_2(z))] \\
& \times [\bar{B}_1'(f_1(z))f_1'(z)\bar{B}_2(f_1(z))\bar{B}_3(f_1(z)) \\
& + \bar{B}_1(f_1(z))\bar{B}_2'(f_1(z))f_1'(z)\bar{B}_3(f_1(z)) \\
& + \bar{B}_1(f_1(z))\bar{B}_2(f_1(z))\bar{B}_3'(f_1(z))f_1'(z)] \\
& - (1-p+pz)\bar{B}_1(f_1(z))\bar{B}_2(f_1(z))\bar{B}_3(f_1(z)) \\
& \times [r\bar{V}_1'(f_2(z))f_2'(z)(1-\theta+\theta\bar{V}_2(f_2(z))) \\
& + r\bar{V}_1(f_2(z))\theta\bar{V}_2'(f_2(z))f_2'(z)]
\end{aligned}$$

Then, we use

$$\begin{aligned}
L_q =& \frac{d}{dz}W_q(z) \\
=& \frac{1}{\beta}\left[\frac{D'(1)N''(1)-N'(1)D''(1)}{2(D'(1))^2}\right]Q
\end{aligned} \tag{9.84}$$

where primes and double primes in (9.84) denote first and second derivative at $z = 1$ respectively. Carrying out the derivative at $z = 1$, we have

$$N'(1) = \lambda\alpha\beta E(I)[E(B_1) + E(B_2) + E(B_3) + rE(V)] \tag{9.85}$$

$$\begin{aligned} N''(1) =& \lambda^2\beta\alpha(E(I))^2[\alpha(E(B_1^2) + E(B_2^2) + E(B_3^2)) \\ &+ \beta r(E(V_1^2) + \theta E(V_2^2))] \\ &+ \lambda\alpha\beta E(I(I-1))[E(B_1) + E(B_2) + E(B_3) + rE(V)] \\ &+ 2\lambda^2\beta\alpha(E(I))^2[\alpha(E(B_1)(E(B_2) + E(B_3)) + E(B_2)E(B_3)) \\ &+ \beta r\theta E(V_1)E(V_2)] + 2\lambda^2\beta\alpha^2 r(E(I))^2 E(V) \\ &\times [E(B_1) + E(B_2) + E(B_3)] + 2\lambda r\alpha\beta pE(I)E(V) \end{aligned} \tag{9.86}$$

$$D'(1) = 1 - p - \lambda E(I)[\alpha(E(B_1) + E(B_2) + E(B_3)) + r\beta E(V)] \tag{9.87}$$

$$\begin{aligned} D''(1) =& -\lambda[2pE(I) + E(I(I-1))[\alpha(E(B_1) + E(B_2) + E(B_3)) + r\beta E(V)] \\ &- 2\lambda^2\beta\alpha r(E(I))^2 E(V)[E(B_1) + E(B_2) + E(B_3)] - \lambda^2(E(I))^2 \\ &\times [\alpha^2(E(B_1^2) + E(B_2^2) + E(B_3^2)) + \beta^2 r(E(V_1^2) + \theta E(V_2^2))] \\ &- 2\lambda^2(E(I))^2[\alpha^2(E(B_1)(E(B_2) + E(B_3)) \\ &+ E(B_2)E(B_3)) + \beta^2 r\theta E(V_1)E(V_2))] \end{aligned} \tag{9.88}$$

Then if we substitute the values $N'(1), N''(1), D'(1), D''(1)$ from equations (9.85) to (9.88) into equation (9.84), we obtain L_q in the closed form.
Further, we find the mean system size L by using Little's formula. Thus we have

$$L = L_q + \rho \tag{9.89}$$

where L_q has been found by equation (9.84) and ρ is obtained from equation (9.83).

9.8 Particular cases:

Case 1: If there is no feedback, no optional vacation and no restricted admissibility, i.e, $p = 0$, $\theta = 0$ and $\alpha = \beta = 1$.

Then our model reduces to a single server $M^{[X]}/G/1$ queue with three stage heterogeneous service and Bernoulli vacation. In this case, we find the idle probability Q, utilization factor ρ and the average queue size L_q can be simplified to the following expressions.

$$\begin{aligned}
Q =& 1 - \lambda E(I)[E(B_1) + E(B_2) + E(B_3) + rE(V_1)] \\
\rho =& \lambda E(I)[E(B_1) + E(B_2) + E(B_3) + rE(V_1)] \\
L_q =& \left[\frac{D'(1)N''(1) - N'(1)D''(1)}{2(D'(1))^2}\right] Q
\end{aligned}$$

where

$$\begin{aligned}
N'(1) =& \lambda(E(I))[E(B_1) + E(B_2) + E(B_3) + rE(V_1)] \\
N''(1) =& \lambda^2(E(I))^2[E(B_1^2) + E(B_2^2) + E(B_3^2) + rE(V_1^2)] \\
&+ \lambda E(I(I-1))[E(B_1) + E(B_2) + E(B_3) + rE(V_1)] \\
&+ 2\lambda^2(E(I))^2[E(B_1)(E(B_2) + E(B_3)) + E(B_2)E(B_3)] \\
&+ 2\lambda^2 r(E(I))^2 E(V_1)[E(B_1) + E(B_2) + E(B_3)] \\
D'(1) =& 1 - \lambda E(I)[E(B_1) + E(B_2) + E(B_3) + rE(V_1)] \\
D''(1) =& -\lambda E(I(I-1))[E(B_1) + E(B_2) + E(B_3) + rE(V_1)] \\
&- 2\lambda^2 r(E(I))^2 E(V_1)[E(B_1) + E(B_2) + E(B_3)] \\
&- \lambda^2(E(I))^2[E(B_1^2) + E(B_2^2) + E(B_3^2) + rE(V_1^2)] \\
&- 2\lambda^2(E(I))^2[E(B_1)(E(B_2) + E(B_3)) + E(B_2)E(B_3)]
\end{aligned}$$

Case 2: If there is no feedback, server has no vacation and no restricted admissibility, no second, third stages of service and $C(z) = z$ i.e, $p = 0$, r=0, $\theta = 0$, $\alpha = \beta = 1$, $E(B_2) = 0$, $E(B_3) = 0$, $E(I) = 1$ and $E(I(I-1)) = 0$.

Then our model reduces to a single server $M/G/1$ queueing system. In this case, we find the idle probability Q, utilization factor ρ and the average queue size L_q can be simplified to the following expressions.

$$
\begin{aligned}
Q =& 1 - \lambda E(B_1) \\
\rho =& \lambda E(B_1) \\
L_q =& \left[\frac{D'(1)N''(1) - N'(1)D''(1)}{2(D'(1))^2} \right] Q
\end{aligned}
$$

where

$$
\begin{aligned}
N'(1) =& \lambda E(B_1) \\
N''(1) =& \lambda^2 E(B_1^2) \\
D'(1) =& 1 - \lambda E(B_1) \\
D''(1) =& -\lambda^2 E(B_1^2)
\end{aligned}
$$

The above equations coincide with result given by Medhi (1982).

Case 3: If there is no optional vacation, no restricted admissibility and no second and third stages of service i.e, $\theta = 0$, $\alpha = \beta = 1$, $E(B_2) = 0$ and $E(B_3) = 0$.

Then our model reduces to a single server $M^{[X]}/G/1$ feedback queue with Bernoulli vacation. In this case, we find the idle probability Q, utilization factor ρ and the average queue size L_q can be simplified to the following expressions. we get

$$
\begin{aligned}
Q =& 1 - \lambda E(I)[E(B_1) + rE(V_1)] \\
\rho =& \lambda E(I)[E(B_1) + rE(V_1)]
\end{aligned}
$$

$$L_q = \left[\frac{D'(1)N''(1) - N'(1)D''(1)}{2(D'(1))^2}\right] Q$$

where

$$\begin{aligned}
N'(1) =& \lambda E(I)[E(B_1) + rE(V_1)] \\
N''(1) =& \lambda^2 (E(I))^2 [(E(B_1^2) + rE(V_1^2)] \\
& + \lambda E(I(I-1))[E(B_1) + rE(V_1)] \\
& + 2\lambda^2 r (E(I))^2 E(V_1) E(B_1) + 2\lambda r p E(I) E(V_1) \\
D'(1) =& 1 - \lambda E(I)[(E(B_1) + rE(V_1)] \\
D''(1) =& -2p\lambda E(I) + E(I(I-1))[E(B_1) + rE(V)] \\
& - 2\lambda^2 r (E(I))^2 E(V_1) E(B_1) - \lambda^2 (E(I))^2 (E(B_1^2) + r(E(V_1^2))
\end{aligned}$$

The above equations coincide with result given by Madan and Al-Rawwash (2005).

Case 4: If service and vacation times follows exponential distribution for case 1.

Then our model reduces to a single server $M^{[X]}/M/1$ queue, three stage heterogeneous service with Bernoulli vacation. Here, the exponential service rates are μ_1, μ_2, $\mu_3 > 0$ and the exponential vacation rate is $\gamma_1 > 0$, we have

$$\begin{aligned}
Q =& 1 - \rho \\
\rho =& \frac{\lambda E(I)}{\mu_1 \mu_2 \mu_3 \gamma_1} [\mu_3 \gamma_1 (\mu_2 + \mu_1) + \mu_1 \mu_2 (\gamma_1 + r\mu_3)] \\
L_q =& \left[\frac{D'(1)N''(1) - N'(1)D''(1)}{2(D'(1))^2}\right] Q
\end{aligned}$$

where

$$
\begin{aligned}
N'(1) =& \lambda E(I)[\mu_3\gamma_1(\mu_2+\mu_1)+\mu_1\mu_2(\gamma_1+r\mu_3)] \\
N''(1) =& 2\lambda^2(E(I))^2[\mu_3^2\gamma_1^2(\mu_2^2+\mu_1^2)+\mu_1^2\mu_2^2(\mu_3^2+r\gamma_1^2)] \\
&+\lambda E(I(I-1))\mu_1\mu_2\mu_3\gamma_1[\mu_3\gamma_1(\mu_2+\mu_1)+\mu_1\mu_2(\gamma_1+r\mu_3)] \\
&+2\lambda^2(E(I))^2\mu_1\mu_2\mu_3\gamma_1^2[\mu_1+\mu_2+\mu_3] \\
&+2\lambda^2 r\mu_1\mu_2\mu_3\gamma_1(E(I))^2[\mu_3(\mu_1+\mu_2)+\mu_1\mu_2] \\
D'(1) =& \mu_1\mu_2\mu_3\gamma_1-\lambda E(I)[\mu_3\gamma_1(\mu_2+\mu_1)+\mu_1\mu_2(\gamma_1+r\mu_3)] \\
D''(1) =& -\lambda E(I(I-1))\gamma_1\mu_1\mu_2\mu_3[\mu_3\gamma_1(\mu_2+\mu_1)+\mu_1\mu_2(\gamma_1+r\mu_3)] \\
&-2\lambda^2 r\gamma_1\mu_1\mu_2\mu_3(E(I))^2[\mu_2\mu_3+\mu_1\mu_3+\mu_1\mu_2] \\
&-2\lambda^2(E(I))^2[\mu_3^2\gamma_1^2(\mu_2^2+\mu_1^2)+\mu_1^2\mu_2^2(\gamma_1^2+r\mu_3^2)] \\
&-2\lambda^2(E(I))^2\gamma_1^2\mu_1\mu_2\mu_3[\mu_1+\mu_2+\mu_3]
\end{aligned}
$$

9.9 Numerical results

For the purpose of a numerical result, we use the case 4. In Table 9.1, we choose the following arbitrary values: $\mu_1 = 2, \mu_2 = 3, \mu_3 = 4$, $\gamma = 3$, $r = 0.6$, $E(I)$=0.3 and $E(I(I-1)) = 0.04$ while λ varies from 0.1 to 1.0 such that the stability condition is satisfied.

The Table 9.1 gives computed values of the idle time, the utilization factor, the mean queue size and mean system size of our queueing model.

It clearly shows that as long as increasing the arrival rate, the server's idle time decreases while the utilization factor, the mean queue size and the mean system size of our queueing model are all increases.

In Table 9.2, we choose the following values: $\mu_1 = 3$, $\mu_2 = 4$, $\mu_3 = 2$, $E(I)$=0.3, $E(I(I-1))$= 0.04, $\lambda = 3$ and $r = 0.6$ while γ varies from 1 to 10 such that the stability condition is satisfied.

Table 9.1: Computed values of various queue characteristics

λ	Q	ρ	L_q	L
0.1	0.961500	0.038500	0.003700	0.042200
0.2	0.923000	0.077000	0.009852	0.086852
0.3	0.884500	0.115500	0.018771	0.134271
0.4	0.846000	0.154000	0.031832	0.184832
0.5	0.807500	0.192500	0.046477	0.238977
0.6	0.769000	0.231000	0.066241	0.297241
0.7	0.730500	0.269500	0.090769	0.360269
0.8	0.692000	0.308000	0.120850	0.428850
0.9	0.653500	0.346500	0.157460	0.503960
1.0	0.615000	0.385000	0.201818	0.586818

Table 9.2: Computed values of various queue characteristics

γ	Q	ρ	L_q	L
1	0.050000	0.950000	3.285000	4.235000
2	0.200000	0.800000	2.550000	3.350000
3	0.250000	0.750000	1.765000	2.515000
4	0.275000	0.725000	1.493523	2.218523
5	0.290000	0.710000	1.357717	2.067717
6	0.300000	0.700000	1.276667	1.976667
7	0.307143	0.692857	1.222959	1.915816
8	0.312500	0.687500	1.184813	1.872313
9	0.316667	0.683333	1.156345	1.839678
10	0.320000	0.680000	1.134300	1.814300

The Table 9.2 gives computed values of the idle time, the utilization factor, the mean queue size and mean system size of our queueing model.

It clearly shows that as long as increasing the vacation rate, the server's idle time increases while the utilization factor, the mean queue size and the mean system size of our queueing model are all decreases.

CHAPTER TEN

$M^{[X]}/G/1$ Feedback Retrial Queue with Starting Failure and Bernoulli Vacation

CHAPTER

TEN

$M^{[X]}/G/1$ FEEDBACK RETRIAL QUEUE WITH STARTING FAILURE AND BERNOULLI VACATION

10.1 Introduction

Retrial queues have been widely used to model many problems in telephone switching systems, telecommunication networks and computers competing to gain service from a central processor unit. Most of the papers on retrial queues have considered the system without feedback. A more practical retrial queue with feedback occurs in many practical situations: for example, multiple access telecommunication systems, where messages turned out as errors are sent again, can be modeled as retrial queue with feedback. A remarkable and unavoidable phenomenon in the service facility of a queuing system is its breakdown. Kulkarni and Choi (1990) have analysed the M/G/1 retrial queue with server subjected to repairs and breakdowns. Aissani and Artalejo (1998), Artalejo (1999) and Artalejo and Gomez-Corral (2008) have considered a retrial queue in which immediately after a service completion the server searches for customer from the orbit or remains idle.

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One of the most important characteristic in the service facility of a queueing system is its starting failures. An arriving customer who finds the server idle must turn on the server. If the server is started successfully the customer gets the service immediately. Otherwise the down for the server begins and the customer must join the orbit. The server is assumed to be reliable during service. Such systems with starting failures have been studied as queueing models by Yang and Li (1994), Krishna Kumar et al. (2002b), Mokaddis et al. (2007), Ke and Chang (2009) and Sumitha and Udaya Chandrika (2012).

In this chapter, we consider $M^{[X]}/G/1$ feedback retrial queue, subject to starting failures and Bernoulli vacation. The customers arrive to the system in batches of variable size, but served one by one on a first come - first served basis. We assume that there is no waiting space and therefore if an arriving customer finds the server busy or down, the customer leaves the service area and enters a group of blocked customers called orbit in accordance with an FCFS discipline. That is, only the customer at the head of the orbit queue is allowed for access to the server where the arrival follows Poisson. As soon as the completion of service, if the customer is dissatisfied with his service, he can immediately join the retrial group as a feedback customer for receiving the same service with probability p or to leave the system forever with probability $q(= 1 - p)$. The successful commencement of service for a new customer who finds the server idle and sees no other customer in the orbit with probability δ and is α for all other new and returning customers. After the completion of each service, the server either goes for a vacation with probability β or may wait for serving the next customer with probability $1-\beta$. Repair times, service times and vacation times are assumed to be generally (arbitrary) distributed.

Here we derive time dependent probability generating functions in terms of Laplace transforms. We also derive the average orbit size, system size and average waiting time in the queue, the system. Some particular cases and numerical results are also discussed.

The rest of the chapter is organized as follows. Model description is given in section 10.2. Definitions and equations governing the system are given in section 10.3 and 10.4 respectively. The time dependent solution have been obtained in section 10.5. Corresponding steady state results have been derived explicitly in section 10.6. Average orbit size, system size and average waiting time are computed in section 10.7. Particular cases and numerical results are discussed in section 10.8 and 10.9 respectively.

10.2 Model description

We assume the following to describe the queueing model of our study.

a) Customers arrive at the system in batches of variable size in a compound Poisson process and they are provided one by one service on a first come - first served basis. Let $\lambda c_i dt$ $(i = 1, 2, \ldots)$ be the first order probability that a batch of i customers arrives at the system during a short interval of time $(t, t + dt]$, where $0 \leq c_i \leq 1$, $\sum_{i=1}^{\infty} c_i = 1$ and $\lambda > 0$ is the arrival rate of batches.

b) We assume that there is no waiting space and therefore if an arriving customer finds the server busy or down, the customer leaves the service area and enters a group of blocked customers called "orbit" in accordance with an FCFS discipline. That is, only the customer at the head of the orbit queue is allowed for access to the server.

c) As soon as the completion of service, if the customer is dissatisfied with his service, he can immediately join the retrial group as a feedback customer for receiving the same service with probability p or to leave the system forever with probability $q(= 1 - p)$. The successful commencement of service for a new customer who finds the server idle and sees no other customer in the orbit with probability δ and is α for all other new and returning customers. From this description, it is clear that at any service completion, the server becomes free and in such a case, a possible new (primary) arrival and the one (if any) at the head of the orbit, compete for service.

d) The retrial time follows a general (arbitrary) distribution with distribution function $A(s)$ and density function $a(s)$. Let $r(x)dx$ be the conditional probability density of retrial completion during the interval $(x, x + dx]$, given that the elapsed retrial time is x, so that

$$r(x) = \frac{a(x)}{1 - A(x)}$$

and therefore,

$$a(s) = r(s)e^{-\int_0^s r(x)dx}$$

e) The service follows a general (arbitrary) distribution with distribution function $B(x)$ and density function $b(x)$. Let $\mu(x)dx$ be the conditional probability density of service completion during the interval $(x, x + dx]$, given that the elapsed service time is x, so that

$$\mu(x) = \frac{b(x)}{1 - B(x)}$$

and therefore,

$$b(t) = \mu(t)e^{-\int_0^t \mu(x)dx}.$$

f) The duration of repairs follows a general (arbitrary) distribution with distribution function $F(x)$ and density function $f(x)$. Let $\eta(x)dx$ be the conditional probability density of repairs completion during the interval $(x, x + dx]$, given that the elapsed repair time is x, so that

$$\eta(x) = \frac{f(x)}{1 - F(x)}$$

and therefore,

$$f(t) = \eta(t) e^{-\int_0^t \eta(x) dx}$$

g) At the completion of each service the server may take a vacation with probability β or waits for the next customer with $1 - \beta$.

h) The server's vacation time follows a general (arbitrary) distribution with distribution function $V(t)$ and density function $v(t)$. Let $\gamma(x)dx$ be the conditional probability density of vacation completion during the interval $(x, x + dx]$, given that the elapsed vacation time is x, so that

$$\gamma(x) = \frac{v(x)}{1 - V(x)},$$

and therefore,

$$v(t) = \gamma(t) e^{-\int_0^t \gamma(x) dx}$$

h) Various stochastic processes involved in the system are assumed to be independent of each other.

10.3 Definitions

We define

$P_n(x, t)$ = Probability that at time t, the server is idle and there are n $(n > 0)$ customers in the orbit and the elapsed retrial time is x. Consequently

$P_n(t) = \int_0^\infty P_n(x,t)dx$ denotes the probability that at time t there are n customers in the orbit and the server is under idle irrespective of the value of x.

$Q_n(x,t)$ = Probability that at time t, the server is busy and there are n $(n \geq 0)$ customers in the orbit and the elapsed service time is x. Consequently $Q_n(t) = \int_0^\infty Q_n(x,t)dx$ denotes the probability that at time t there are n customers in the orbit and the server is under service irrespective of the value of x.

$R_n(x,t)$= Probability that at time t, there are n $(n > 0)$ customers in the orbit and the server is inactive due to system repair and waiting for repairs to start with elapsed repair time x. Consequently $R_n(t) = \int_0^\infty R_n(x,t)dx$ denotes the probability that at time t there are n customers in the orbit and the server is under repair irrespective of the value of x.

$V_n(x,t)$ = Probability that at time t, the server is under vacation with elapsed vacation time is x and there are n $(n \geq 0)$ customers in the orbit. Consequently $V_n(t) = \int_0^\infty V_n(x,t)dx$ denotes the probability that at time t there are n customers in the orbit and the server is under vacation irrespective of the value of x.

$P_0(t)$ = Probability that at time t, there are no customers in the orbit and the server is idle but available in the system.

10.4 Equations governing the system

The model is then, governed by the following set of differential-difference equations:

$$\frac{d}{dt}P_0(t) = -\lambda P_0(t) + (1-\beta)q\int_0^\infty Q_0(x,t)\mu(x)dx + \int_0^\infty V_0(x,t)\gamma(x)dx \quad (10.1)$$

$$\frac{\partial}{\partial x}P_n(x,t)+\frac{\partial}{\partial t}P_n(x,t)+[\lambda+r(x)]P_n(x,t)=0,\ \ n\geq 1 \tag{10.2}$$

$$\frac{\partial}{\partial x}Q_0(x,t)+\frac{\partial}{\partial t}Q_0(x,t)+[\lambda+\mu(x)]Q_0(x,t)=0 \tag{10.3}$$

$$\frac{\partial}{\partial x}Q_n(x,t)+\frac{\partial}{\partial t}Q_n(x,t)+[\lambda+\mu(x)]Q_n(x,t)=\lambda\sum_{k=1}^{n}c_kQ_{n-k}(x,t),\quad n\geq 1 \tag{10.4}$$

$$\frac{\partial}{\partial x}R_1(x,t)+\frac{\partial}{\partial t}R_1(x,t)+[\lambda+\eta(x)]R_1(x,t)=0 \tag{10.5}$$

$$\frac{\partial}{\partial x}R_n(x,t)+\frac{\partial}{\partial t}R_n(x,t)+[\lambda+\eta(x)]R_n(x,t)=\lambda\sum_{k=1}^{n}c_kR_{n-k}(x,t),\quad n\geq 2 \tag{10.6}$$

$$\frac{\partial}{\partial x}V_0(x,t)+\frac{\partial}{\partial t}V_0(x,t)+[\lambda+\gamma(x)]V_0(x,t)=0 \tag{10.7}$$

$$\frac{\partial}{\partial x}V_n(x,t)+\frac{\partial}{\partial t}V_n(x,t)+[\lambda+\gamma(x)]V_n(x,t)=\lambda\sum_{k=1}^{n}c_kV_{n-k}(x,t),\quad n\geq 1 \tag{10.8}$$

The above set of equations are to be solved subject to the following boundary conditions:

$$P_n(0,t)=(1-\beta)q\int_0^\infty Q_n(x,t)\mu(x)dx+(1-\beta)p\int_0^\infty Q_{n-1}(x,t)\mu(x)dx + \int_0^\infty R_n(x,t)\eta(x)dx+\int_0^\infty V_n(x,t)\gamma(x)dx,\ \ n\geq 1 \tag{10.9}$$

$$Q_0(0,t)=\delta\lambda c_1P_0(t)+\alpha\int_0^\infty P_1(x,t)r(x)dx \tag{10.10}$$

$$Q_n(0,t)=\alpha\lambda\int_0^\infty\sum_{k=1}^{n}c_kP_{n-k+1}(x,t)dx+\alpha\int_0^\infty P_{n+1}(x,t)r(x)dx + \delta\lambda c_{n+1}P_0(t),\ \ n\geq 1 \tag{10.11}$$

$$R_1(0,t)=\bar{\delta}\lambda P_0(t)+\bar{\alpha}\int_0^\infty P_1(x,t)r(x)dx \tag{10.12}$$

$$R_n(0,t)=\bar{\alpha}\lambda\int_0^\infty P_{n-1}(x,t)dx+\bar{\alpha}\int_0^\infty P_n(x,t)r(x)dx,\quad n\geq 2 \tag{10.13}$$

$$V_n(0,t) = \beta \int_0^\infty Q_n(x,t)\mu(x)dx, \ n \geq 0 \tag{10.14}$$

We assume that initially there are no customers in the system and the server is idle. So the initial conditions are

$$\begin{aligned} &V_n(0) = R_n(0) = 0, \ n \geq 0, \quad Q_n(0) = 0 \text{ and} \\ &P_0(0) = 1, P_n(0) = 0 \ \text{ for } \ n \geq 1. \end{aligned} \tag{10.15}$$

10.5 Generating functions of the queue length: The time-dependent solution

Now we shall find the transient solution for the above set of differential-difference equations.

Theorem: *The system of differential difference equations to describe an $M^{[X]}/G/1$ feedback retrial queue with Starting Failure and Bernoulli vacation are given by equations (10.1) to (10.14) with initial conditions (10.15) and the generating functions of transient solution are given by equation (10.62) to (10.65).*

Proof: We define the probability generating functions,

$$P(x,z,t) = \sum_{n=1}^{\infty} z^n P_n(x,t); \ P(z,t) = \sum_{n=1}^{\infty} z^n P_n(t)$$

$$Q(x,z,t) = \sum_{n=0}^{\infty} z^n Q_n(x,t); \quad Q(z,t) = \sum_{n=0}^{\infty} z^n Q_n(t)$$

$$R(x,z,t) = \sum_{n=1}^{\infty} z^n R_n(x,t); \quad R(z,t) = \sum_{n=1}^{\infty} z^n R_n(t)$$

$$V(x,z,t)=\sum_{n=0}^{\infty} z^n V_n(x,t); V(z,t)=\sum_{n=0}^{\infty} z^n V_n(t),\ C(z)=\sum_{n=1}^{\infty} c_n z^n \quad (10.16)$$

which are convergent inside the circle given by $\mid z \mid \leq 1$ and define the Laplace transform of a function $f(t)$ as

$$\bar{f}(s)=\int_0^{\infty} e^{-st} f(t)dt, \quad \Re(s)>0. \quad (10.17)$$

Taking the Laplace transform of equations (10.1) to (10.14) and using (10.15), we obtain

$$(s+\lambda)\bar{P}_0(s)=1+(1-\beta)q\int_0^{\infty} \bar{Q}_0(x,s)\mu(x)dx+\int_0^{\infty} \bar{V}_0(x,s)\gamma(x)dx \quad (10.18)$$

$$\frac{\partial}{\partial x}\bar{P}_n(x,s)+[s+\lambda+r(x)]\bar{P}_n(x,s)=0,\ n\geq 1 \quad (10.19)$$

$$\frac{\partial}{\partial x}\bar{Q}_0(x,s)+[s+\lambda+\mu(x)]\bar{Q}_0(x,s)=0 \quad (10.20)$$

$$\frac{\partial}{\partial x}\bar{Q}_n(x,s)+[s+\lambda+\mu(x)]\bar{Q}_n(x,s)=\lambda\sum_{k=1}^{n} c_k\bar{Q}_{n-k}(x,s),\ n\geq 1 \quad (10.21)$$

$$\frac{\partial}{\partial x}\bar{R}_1(x,s)+[s+\lambda+\eta(x)]\bar{R}_1(x,s)=0 \quad (10.22)$$

$$\frac{\partial}{\partial x}\bar{R}_n(x,s)+[s+\lambda+\eta(x)]\bar{R}_n(x,s)=\lambda\sum_{k=1}^{n} c_k\bar{R}_{n-k}(x,s), n\geq 2 \quad (10.23)$$

$$\frac{\partial}{\partial x}\bar{V}_0(x,s)+[s+\lambda+\gamma(x)]\bar{V}_0(x,s)=0 \quad (10.24)$$

$$\frac{\partial}{\partial x}\bar{V}_n(x,s)+[s+\lambda+\gamma(x)]\bar{V}_n(x,s)=\lambda\sum_{k=1}^{n} c_k\bar{V}_{n-k}(x,s), n\geq 0 \quad (10.25)$$

$$\bar{P}_n(0,s)=q(1-\beta)\int_0^{\infty} \bar{Q}_n(x,s)\mu(x)dx+p(1-\beta)\int_0^{\infty} \bar{Q}_{n-1}(x,s)\mu(x)dx$$
$$+\int_0^{\infty} \bar{R}_n(x,s)\eta(x)dx+\int_0^{\infty} \bar{V}_n(x,s)\gamma(x)dx,\ n\geq 0 \quad (10.26)$$

$$\bar{Q}_0(0,s) = \delta\lambda c_1 \bar{P}_0(s) + \alpha \int_0^\infty \bar{P}_1(x,s) r(x) dx \tag{10.27}$$

$$\bar{Q}_n(0,s) = \alpha\lambda \int_0^\infty \sum_{k=1}^{n} c_k \bar{P}_{n-k+1}(x,s) dx + \alpha \int_0^\infty \bar{P}_{n+1}(x,s) r(x) dx + \delta\lambda c_{n+1} \bar{P}_0(s), \quad n \geq 1 \tag{10.28}$$

$$\bar{R}_1(0,s) = \bar{\delta}\lambda \bar{P}_0(s) + \bar{\alpha} \int_0^\infty \bar{P}_1(x,s) r(x) dx, \tag{10.29}$$

$$\bar{R}_n(0,s) = \bar{\alpha}\lambda \int_0^\infty \bar{P}_{n-1}(x,s) dx + \bar{\alpha} \int_0^\infty \bar{P}_n(x,s) r(x) dx, \quad n \geq 2 \tag{10.30}$$

$$\bar{V}_n(0,s) = \beta \int_0^\infty \bar{Q}_n(x,s) \mu(x) dx, \ n \geq 0 \tag{10.31}$$

Now multiplying equations (10.19) to (10.31) by z^n and summing over n, using the generating functions defined in (10.16), we get

$$\frac{\partial}{\partial x} \bar{P}(x,z,s) + [s + \lambda + r(x)] \bar{P}(x,z,s) = 0 \tag{10.32}$$

$$\frac{\partial}{\partial x} \bar{Q}(x,z,s) + [s + \lambda - \lambda C(z) + \mu(x)] \bar{Q}(x,z,s) = 0 \tag{10.33}$$

$$\frac{\partial}{\partial x} \bar{R}(x,z,s) + [s + \lambda - \lambda C(z) + \eta(x)] \bar{R}(x,z,s) = 0 \tag{10.34}$$

$$\frac{\partial}{\partial x} \bar{V}(x,z,s) + [s + \lambda - \lambda C(z) + \gamma(x)] \bar{V}(x,z,s) = 0 \tag{10.35}$$

$$\begin{aligned} \bar{P}(0,z,s) =& (q + pz)(1 - \beta) \int_0^\infty \bar{Q}(x,z,s) \mu(x) dx \\ &+ \int_0^\infty \bar{R}(x,z,s) \eta(x) dx + \int_0^\infty \bar{V}(x,z,s) \gamma(x) dx \\ &- q(1-\beta) \int_0^\infty \bar{Q}_0(x,s) \mu(x) dx \\ &- \int_0^\infty \bar{V}_0(x,s) \gamma(x) dx \end{aligned} \tag{10.36}$$

$$\begin{aligned} z\bar{Q}(0,z,s) =& \delta\lambda C(z) \bar{P}_0(s) + \alpha \int_0^\infty \bar{P}(x,z,s) r(x) dx \\ &+ \alpha\lambda C(z) \int_0^\infty \bar{P}(x,z,s) dx \end{aligned} \tag{10.37}$$

$$\begin{aligned}\bar{R}(0,z,s) =& \lambda z\bar{\delta}\bar{P}_0(s) + \bar{\alpha}\lambda z\int_0^\infty \bar{P}(x,z,s)dx \\ &+ \bar{\alpha}\int_0^\infty \bar{P}(x,z,s)r(x)dx, \end{aligned} \tag{10.38}$$

$$\bar{V}(0,z,s) = \beta\int_0^\infty \bar{Q}(x,z,s)\mu(x)dx, \tag{10.39}$$

Using equation (10.18) in (10.36), we get

$$\begin{aligned}\bar{P}(0,z,s) = [1-(s+\lambda)\bar{P}_0(s)] + (q+pz)(1-\beta)\int_0^\infty \bar{Q}(x,z,s)\mu(x)dx \\ + \int_0^\infty \bar{R}(x,z,s)\eta(x)dx + \int_0^\infty \bar{V}(x,z,s)\gamma(x)dx \end{aligned} \tag{10.40}$$

Integrating equation (10.32) between 0 and x, we get

$$\bar{P}(x,z,s) = \bar{P}(0,z,s)e^{-[s+\lambda]x-\int\limits_0^x r(t)dt} \tag{10.41}$$

where $\bar{P}(0,z,s)$ is given by equation (10.40).

Again integrating equation (10.41) by parts with respect to x, yields

$$\bar{P}(z,s) = \bar{P}(0,z,s)\left[\frac{1-\bar{A}(s+\lambda)}{s+\lambda}\right] \tag{10.42}$$

where

$$\bar{A}(s+\lambda) = \int\limits_0^\infty e^{-[s+\lambda]x}dA(x)$$

is the Laplace-Stieltjes transform of the retrial time $A(x)$. Now multiplying both sides of equation (10.41) by $r(x)$ and integrating over x, we obtain

$$\int\limits_0^\infty \bar{P}(x,z,s)r(x)dx = \bar{P}(0,z,s)\bar{A}(s+\lambda) \tag{10.43}$$

Similarly, on integrating equations (10.33) to (10.35) from 0 to x, we get

$$\bar{Q}(x,z,s)=\bar{Q}(0,z,s)e^{-[s+\lambda-\lambda C(z)]x-\int\limits_0^x \mu(t)dt} \tag{10.44}$$

$$\bar{R}(x,z,s)=\bar{R}(0,z,s)e^{-[s+\lambda-\lambda C(z)]x-\int\limits_0^x \eta(t)dt} \tag{10.45}$$

$$\bar{V}(x,z,s)=\bar{V}(0,z,s)e^{-[s+\lambda-\lambda C(z)]x-\int\limits_0^x \gamma(t)dt} \tag{10.46}$$

where $\bar{Q}(0,z,s)$, $\bar{R}(0,z,s)$ and $\bar{V}(0,z,s)$ are given by equations (10.37) to (10.39). Again integrating equations (10.44) to (10.46) by parts with respect to x, yields

$$\bar{Q}(z,s)=\bar{Q}(0,z,s)\left[\frac{1-\bar{B}(s+\lambda-\lambda C(z))}{s+\lambda-\lambda C(z)}\right] \tag{10.47}$$

$$\bar{R}(z,s)=\bar{R}(0,z,s)\left[\frac{1-\bar{F}(s+\lambda-\lambda C(z))}{s+\lambda-\lambda C(z)}\right] \tag{10.48}$$

$$\bar{V}(z,s)=\bar{V}(0,z,s)\left[\frac{1-\bar{V}(s+\lambda-\lambda C(z))}{s+\lambda-\lambda C(z)}\right] \tag{10.49}$$

where

$$\bar{B}(s+\lambda-\lambda C(z))=\int_0^\infty e^{-[s+\lambda-\lambda C(z)]x}dB(x)$$

$$\bar{F}(s+\lambda-\lambda C(z))=\int_0^\infty e^{-[s+\lambda-\lambda C(z)]x}dF(x)$$

$$\bar{V}(s+\lambda-\lambda C(z))=\int_0^\infty e^{-[s+\lambda-\lambda C(z)]x}dV(x)$$

are the Laplace-Stieltjes transform of the service time $B(x)$, repair time $F(x)$ and vacation time $V(x)$ respectively.

Now multiplying both sides of equations (10.44) to (10.46) by $\mu(x)$, $\eta(x)$ and $\gamma(x)$ and integrating over x, we obtain

$$\int\limits_0^\infty \bar{Q}(x,z,s)\mu(x)dx=\bar{Q}(0,z,s)\bar{B}[s+\lambda-\lambda C(z)] \tag{10.50}$$

$$\int_0^\infty \bar{R}(x,z,s)\eta(x)dx = \bar{R}(0,z,s)\bar{F}[s+\lambda-\lambda C(z)] \tag{10.51}$$

$$\int_0^\infty \bar{V}(x,z,s)\gamma(x)dx = \bar{V}(0,z,s)\bar{V}[s+\lambda-\lambda C(z)] \tag{10.52}$$

Using equation (10.50) in (10.39), we can write as

$$\bar{V}(0,z,s) = \beta\bar{Q}(0,z,s)\bar{B}(s+\lambda-\lambda C(z)) \tag{10.53}$$

Using equation (10.43) in (10.37) and (10.38), we get

$$\begin{aligned} z\bar{Q}(0,z,s) =& \delta\lambda C(z)\bar{P}_0(s) + \frac{\alpha}{s+\lambda}[(s+\lambda)\bar{A}(s+\lambda) \\ &+ \lambda C(z)(1-\bar{A}(s+\lambda))]\bar{P}(0,z,s) \end{aligned} \tag{10.54}$$

$$\begin{aligned} \bar{R}(0,z,s) =& z\lambda\bar{\delta}\bar{P}_0(s) + \frac{\bar{\alpha}}{s+\lambda}[\lambda z(1-\bar{A}(s+\lambda)) \\ &+ (s+\lambda)\bar{A}(s+\lambda)]\bar{P}(0,z,s) \end{aligned} \tag{10.55}$$

Using equations (10.50) to (10.52) in (10.40), we get

$$\begin{aligned} \bar{P}(0,z,s) =& [1-(s+\lambda)\bar{P}_0(s)] + \bar{F}(a)\bar{R}(0,z,s) \\ &+ (q+pz)(1-\beta)\bar{B}(a)\bar{Q}(0,z,s) \\ &+ \bar{V}(a)\bar{V}(0,z,s) \end{aligned}$$

Using equations (10.53), (10.54), (10.55) in the above equation, we get

$$\bar{P}(0,z,s) = \frac{Nr}{Dr} \tag{10.56}$$

where

$$
\begin{aligned}
Nr =& \bar{P}_0(s)[\delta\lambda C(z)\bar{B}(a)((q+pz)(1-\beta)+\beta\bar{V}(a)) \\
& + \lambda z^2\bar{\delta}\bar{F}(a)] + z[1 - s\bar{P}_0(s)] - z\lambda\bar{P}_0(s) \qquad (10.57) \\
Dr =& z - \alpha\bar{B}(a)[(q+pz)(1-\beta)+\beta\bar{V}(a)][\bar{A}(s+\lambda) \\
& + \lambda C(z)\frac{(1-\bar{A}(s+\lambda))}{s+\lambda}] \\
& - \bar{\alpha}z\bar{F}(a)[\lambda z\frac{(1-\bar{A}(s+\lambda))}{s+\lambda} + \bar{A}(s+\lambda)] \qquad (10.58)
\end{aligned}
$$

and $a = s + \lambda - \lambda C(z)$.

Substituting the value of $\bar{P}(0,z,s)$ from equation (10.56) into equations (10.54) and (10.55), we get

$$
\bar{R}(0,z,s) = \lambda z\bar{\delta}\bar{P}_0(s) + \frac{\bar{\alpha}}{s+\lambda}[\lambda z(1-\bar{A}(s+\lambda)) + (s+\lambda)\bar{A}(s+\lambda)]\frac{Nr}{Dr} \qquad (10.59)
$$

$$
\begin{aligned}
\bar{Q}(0,z,s) =& \frac{\alpha}{s+\lambda}[(s+\lambda)\bar{A}(s+\lambda) + \lambda C(z)(1-\bar{A}(s+\lambda))]\frac{Nr}{Dr} \\
& + \frac{\delta\lambda C(z)}{z}\bar{P}_0(s) \qquad (10.60)
\end{aligned}
$$

Using equation (10.60) in (10.53), we get

$$
\begin{aligned}
\bar{V}(0,z,s) =& \frac{\alpha\beta\bar{B}(a)}{z(s+\lambda)}[(s+\lambda)\bar{A}(s+\lambda) + \lambda C(z)(1-\bar{A}(s+\lambda))]\frac{Nr}{Dr} \\
& + \beta\bar{B}(a)\frac{\delta\lambda C(z)}{z}\bar{P}_0(s) \qquad (10.61)
\end{aligned}
$$

Substituting equations (10.56), (10.59), (10.60), (10.61) in (10.42), (10.47), (10.48) and (10.49), we get

$$
\bar{P}(z,s) = \left[\frac{1-\bar{A}(s+\lambda)}{s+\lambda}\right]\frac{Nr}{Dr} \qquad (10.62)
$$

$$\bar{Q}(z,s) = \left[\frac{1-\bar{B}(a)}{a}\right][\frac{\delta\lambda C(z)}{z}\bar{P}_0(s)$$
$$+\frac{\alpha}{z(s+\lambda)}((s+\lambda)\bar{A}(s+\lambda)+\lambda C(z)(1-\bar{A}(s+\lambda)))\frac{Nr}{Dr}] \quad (10.63)$$
$$\bar{R}(z,s) = \left[\frac{1-\bar{F}(a)}{a}\right][\lambda z\bar{\delta}\bar{P}_0(s)$$
$$+\frac{\bar{\alpha}}{s+\lambda}(\lambda z(1-\bar{A}(s+\lambda))+(s+\lambda)\bar{A}(s+\lambda))\frac{Nr}{Dr}] \quad (10.64)$$
$$\bar{V}(z,s) = \left[\frac{1-\bar{V}(a)}{a}\right]\beta\bar{B}(a)[\frac{\delta\lambda C(z)}{z}\bar{P}_0(s)$$
$$+\frac{\alpha}{z(s+\lambda)}((s+\lambda)\bar{A}(s+\lambda)+\lambda C(z)(1-\bar{A}(s+\lambda)))\frac{Nr}{Dr}] \quad (10.65)$$

where Nr and Dr are given by (10.57) and (10.58). $\bar{P}(z,s)$, $\bar{Q}(z,s)$, $\bar{R}(z,s)$ and $\bar{V}(z,s)$ are completely determined from equations (10.62) to (10.65).

10.6 The steady state results

In this section, we shall derive the steady state probability distribution for our queueing model. These probabilities are obtained by suppressing the argument t wherever it appears in the time-dependent analysis. This can be obtained by applying the Tauberian property

$$\lim_{s\to 0} s\bar{f}(s) = \lim_{t\to\infty} f(t)$$

In order to determine $\bar{P}(z,s)$, $\bar{Q}(z,s)$, $\bar{R}(z,s)$ and $\bar{V}(z,s)$ completely, we have yet to determine the unknown P_0 which appears in the numerators of the right hand sides of equations (10.62) to (10.65). For that purpose, we shall use the normalizing condition

$$P(1)+Q(1)+R(1)+V(1)+P_0 = 1$$

The steady state probabilities for an $M^{[X]}/G/1$ feedback retrial queue with starting failure and Bernoulli vacation are given by

$$\begin{aligned}
P(1) =& \frac{(1-\bar{A}(\lambda)}{\lambda}\frac{D_1'}{D_2'}P_0 \\
Q(1) =& \lambda\delta E(B)P_0 + \alpha E(B)\frac{D_1'}{D_2'}P_0 \\
R(1) =& \lambda\bar{\delta}E(F)P_0 + \bar{\alpha}E(F)\frac{D_1'}{D_2'}P_0 \\
V(1) =& \lambda\delta\beta E(V)P_0 + \beta\alpha E(V)\frac{D_1'}{D_2'}P_0
\end{aligned}$$

where $P(1),\ Q(1),\ R(1),\ V(1)$ are the steady state probabilities that the server idle, server busy, server under repair and server under vacation respectively without regard to the number of customers in the orbit and P_0 is the probability that the server is idle and there are no customers in the orbit.

Multiplying both sides of equations (10.62) to (10.65) by s, taking limit as $s \to 0$, applying Tauberian property and simplifying, we obtain

$$P(z) = \left[\frac{1-\bar{A}(\lambda)}{\lambda}\right]\frac{D_1}{D_2}P_0 \tag{10.66}$$

$$\begin{aligned}
Q(z) =& [\frac{\lambda\delta C(z)}{z} + \frac{\alpha}{z}(C(z)(1-\bar{A}(\lambda)) \\
&+ \bar{A}(\lambda))\frac{D_1}{D_2}]\left[\frac{1-\bar{B}(a_1)}{a_1}\right]P_0
\end{aligned} \tag{10.67}$$

$$R(z) = [\lambda z\bar{\delta}Q + \bar{\alpha}(z(1-\bar{A}(\lambda)) + \bar{A}(\lambda))\frac{D_1}{D_2}]\left[\frac{1-\bar{F}(a_1)}{a_1}\right]P_0 \tag{10.68}$$

$$\begin{aligned}
V(z) =& [\frac{\lambda\delta C(z)}{z} + \frac{\alpha}{z}C(z)(1-\bar{A}(\lambda)) \\
&+ \bar{A}(\lambda))\frac{D_1}{D_2}]\beta\bar{B}(a_1)\left[\frac{1-\bar{V}(a_1)}{a_1}\right]P_0
\end{aligned} \tag{10.69}$$

where

$$\begin{aligned}
D_1 =& \lambda\delta C(z)\bar{B}(a_1)[(q+pz)(1-\beta) + \beta\bar{V}(a_1)] \\
&+ z^2\lambda\bar{\delta}\bar{F}(a_1) - z\lambda,
\end{aligned} \tag{10.70}$$

$$
\begin{aligned}
D_2 =& z - \alpha\bar{B}(a_1)[\bar{A}(\lambda) + C(z)(1-\bar{A}(\lambda))][(q+pz)(1-\beta) \\
&+ \beta\bar{V}(a_1)] - z\bar{\alpha}\bar{F}(a_1)[z(1-\bar{A}(\lambda)) + \bar{A}(\lambda)], \qquad (10.71)
\end{aligned}
$$

and $a_1 = \lambda - \lambda C(z)$.

Let $W_q(z)$ denote the probability generating function for the number of customers in the orbit. Then adding equations (10.66) to (10.69), we obtain

$$
W_q(z) = P(z) + Q(z) + R(z) + V(z)
$$

$$
\begin{aligned}
W_q(z) =& \left[\frac{1-\bar{A}(\lambda)}{\lambda}\right]\frac{D_1}{D_2}P_0 \\
&+[\frac{\lambda\delta C(z)}{z} + \frac{\alpha}{z}(C(z)(1-\bar{A}(\lambda)) + \bar{A}(\lambda))\frac{D_1}{D_2}]P_0\left[\frac{1-\bar{B}(a_1)}{a_1}\right] \\
&+[\lambda z\bar{\delta}P_0 + \bar{\alpha}(z(1-\bar{A}(\lambda)) + \bar{A}(\lambda))\frac{D_1}{D_2}]\left[\frac{1-\bar{F}(a_1)}{a_1}\right]P_0 \\
&+[\frac{\lambda\delta C(z)}{z} + \frac{\alpha}{z}(C(z)(1-\bar{A}(\lambda)) + \bar{A}(\lambda))\frac{D_1}{D_2}]\beta\bar{B}(a_1) \\
&\times\left[\frac{1-\bar{V}(a_1)}{a_1}\right]P_0 \qquad (10.72)
\end{aligned}
$$

we see that for z=1, $W_q(1)$ is indeterminate of the form 0/0. Therefore, we apply L'Hopital's rule and on simplifying, we get

$$
\begin{aligned}
W_q(1) =& \frac{D_1'}{\lambda D_2'}[1-\bar{A}(\lambda) + \lambda(\alpha E(B) + \bar{\alpha}E(F) + \beta\alpha E(V))]P_0 \\
&+ \lambda(\delta E(B) + \bar{\delta}E(F) + \delta\beta E(V))P_0 \qquad (10.73)
\end{aligned}
$$

where

$C(1)$= 1, $C'(1) = E(I)$ is mean batch size of the arriving customers,

$E(V) = -\bar{V}'(0)$ is mean vacation time, $E(B) = -\bar{B}'(0)$ is mean busy time and $E(F) = -\bar{F}'(0)$ is mean repair time.

Therefore adding P_0 to the above equation, equating to 1 and simplifying, we get

$$P_0 = \frac{D_2'\lambda}{D_2'\lambda + D_1'(1 - \bar{A}(\lambda) + \lambda(\alpha E(B) + \bar{\alpha}E(F) + \beta\alpha E(V))) + D_2'\lambda M} \tag{10.74}$$

where

$M = \lambda\delta E(B) + \lambda\bar{\delta}E(F) + \lambda\beta\delta E(V)$

$$\begin{aligned} D_1' =& \lambda\delta E(I)(1 + \lambda E(B)) + \delta\lambda[p(1-\beta) + \lambda\beta E(I)E(V)] \\ &+ \lambda\bar{\delta}(1 + \lambda E(F)E(I)) - \lambda\delta \end{aligned} \tag{10.75}$$

$$\begin{aligned} D_2' =& 1 - \lambda E(I)(\alpha E(B) + \bar{\alpha}E(F)) - (1 - \bar{A}(\lambda))(\alpha E(I) + \bar{\alpha}) \\ &- \alpha[p(1-\beta) + \lambda\beta E(I)E(V)] - \bar{\alpha} \end{aligned} \tag{10.76}$$

and hence the utilization factor ρ of the system is given by

$$\rho = 1 - P_0 \tag{10.77}$$

where $\rho < 1$ is the stability condition under which the steady state exists. Equation (10.74) gives the probability that the server is idle.

Substituting P_0 from (10.74) into (10.72), we have completely and explicitly determined $W_q(z)$, the probability generating function of the number of customers in the orbit.

10.7 The average orbit size and average waiting time

Let L_q denote the average number of customers in the orbit under the steady state. Then

$$L_q = \frac{d}{dz}W_q(z) \quad at\ z = 1$$

since this formula gives 0/0 form, then we write $W_q(z)$ given in (10.72) as $W_q(z) = \frac{N_1(z)}{D_1(z)}P_0 + \frac{N_2(z)}{D_2(z)}P_0$ where

$$\begin{aligned}
N_1(z) =& D_1(D_3 + \lambda\alpha D_4 D_5 + \lambda\bar{\alpha} D_6 D_7) \\
N_2(z) =& \lambda\delta C(z) D_5 + z\lambda\bar{\delta} D_6 \\
D_1(z) =& \lambda z a_1 D_2 \\
D_2(z) =& z(\lambda - \lambda C(z)) \\
D_3 =& z a_1[1 - \bar{A}(\lambda)] \\
D_4 =& \bar{A}(\lambda) + C(z)(1 - \bar{A}(\lambda)) \\
D_5 =& 1 - (1-\beta)\bar{B}(a_1) - \beta\bar{B}(a_1)\bar{V}(a_1) \\
D_6 =& z[1 - \bar{F}(a_1)] \\
D_7 =& z[1 - \bar{A}(\lambda)] + \bar{A}(\lambda) \\
D_8 =& (q + pz)(1-\beta) + \beta\bar{V}(a_1)
\end{aligned}$$

$$\begin{aligned}
L_q =& \lim_{z\to 1} \frac{d}{dz}W_q(z) \\
=& \left[\frac{D_1''(1)N_1'''(1) - N_1''(1)D_1'''(1)}{3(D_1''(1))^2}\right]Q \\
&+ \left[\frac{D_2'(1)N_2''(1) - N_2'(1)D_2''(1)}{2(D_2'(1))^2}\right]Q
\end{aligned} \tag{10.78}$$

where primes, double primes and triple primes in (10.78) denote first, second and third derivative at $z = 1$ respectively. Carrying out derivative at $z = 1$, we have

$$\begin{aligned}
N_1''(1) =& -2\lambda E(I)D_1'[1-\bar{A}(\lambda)+\lambda(\alpha(E(B)+\beta E(V))+\bar{\alpha}E(F))] \\
N_1'''(1) =& -3\lambda E(I)D_1''[1-\bar{A}(\lambda)+\lambda(\alpha(E(B)+\beta E(V))+\bar{\alpha}E(F))] \\
&+3D_1'[-\lambda(2E(I)+E(I(I-1)))(1-\bar{A}(\lambda)+\lambda E(F)\bar{\alpha}) \\
&-\lambda^2\alpha(E(B)+\beta E(V))(2(E(I)^2)(1-\bar{A}(\lambda)) \\
&+E(I(I-1)))-\lambda^2\alpha(E(I)^2)(E(B^2)+\beta E(V^2)+2\lambda E(B)E(V)\beta) \\
&-\lambda^2\bar{\alpha}E(I)(\lambda E(F^2)E(I)+2E(F)(1-\bar{A}(\lambda)))] \\
D_1''(1) =& -2\lambda^2 E(I)[\alpha-\lambda E(I)(\alpha E(B)+\bar{\alpha}E(F)) \\
&-(1-\bar{A}(\lambda))(\alpha E(I)+\bar{\alpha})-\alpha(p(1-\beta)+\lambda\beta E(V)E(I))] \\
D_1'''(1) =& -3\lambda^2 D_2'[2E(I)+E(I(I-1))]-3\lambda^2 E(I)D_2'' \\
N_2'(1) =& -\lambda^2 E(I)[\delta(E(B)+\beta E(V))+\bar{\delta}E(F)] \\
N_2''(1) =& -\lambda^3(E(I)^2)[\delta(E(B^2)+\beta E(V^2))+\bar{\delta}E(F^2)] \\
&-2\lambda^2 E(I)^2\delta[E(B)+\beta E(V)+\lambda\beta E(B)E(V)] \\
&-4\lambda^2 E(I)\bar{\delta}E(F)-\lambda^2 E(I(I-1))[\delta(E(B)+\beta E(V))+\bar{\delta}E(F)] \\
D_2'(1) =& -\lambda E(I) \\
D_2''(1) =& -\lambda[2E(I)+E(I(I-1))] \\
D_1'' =& \,\delta\lambda E(I(I-1))+2\lambda^2(E(I))^2\delta E(B) \\
&+2\delta\lambda E(I)[p(1-\beta)+\lambda\beta E(V)E(I)](1+\lambda E(B)) \\
&+\lambda^2(E(I))^2[\delta\lambda(E(B^2)+\beta E(V^2))+\lambda\bar{\delta}E(F^2)] \\
&+\lambda E(I(I-1))[\delta\lambda(E(B)+\beta E(V))+\lambda\bar{\delta}E(F)] \\
&+2\lambda\bar{\delta}+4\lambda^2\bar{\delta}E(F)E(I) \\
D_2'' =& -\lambda^2(E(I))^2[\alpha(E(B^2)+\beta E(V^2))+\bar{\alpha}E(F^2)]
\end{aligned}$$

$$
\begin{aligned}
&- \lambda E(I(I-1))[\alpha(E(B)+\beta E(V))+\bar{\alpha}E(F)] \\
&- (1-\bar{A}(\lambda))[2\lambda E(I)(\alpha E(B)E(I)+\bar{\alpha}E(F))+\alpha E(I(I-1))+2\bar{\alpha}] \\
&- 2E(I)[p(1-\beta)+\lambda\beta E(V)E(I)][\lambda\alpha E(B)+1-\bar{A}(\lambda)] \\
&- 2\lambda\bar{\alpha}E(F)E(I)
\end{aligned}
$$

where $E(B^2), E(V^2)$ and $E(F^2)$ are the second moment of service time, vacation time and the repair time respectively. $E(I(I-1))$ is the second factorial moment of the batch size of arriving customers.

Then if we substitute the values $N_1''(1), N_1'''(1), D_1''(1), D_1'''(1), N_2'(1), N_2''(1), D_2'(1), D_2''(1)$ in (10.78), we obtain L_q in the closed form.

Further, we find the mean system size L by using Little's formula. Thus we have

$$L = L_q + \rho \tag{10.79}$$

where L_q has been found by equation (10.78) and ρ is obtained from equation (10.77).

Let W_q and W denote the mean waiting time in the orbit and in the system respectively. Then by using Little's formula, we obtain

$$W_q = \frac{L_q}{\lambda}$$

$$W = \frac{L}{\lambda}$$

where L_q and L have been found in equations (10.78) and (10.79).

10.8 Particular cases

Case 1: When the server has no vacation and $C(z) = z$ i.e, β=0, $E(I)$= 1 and $E(I(I-1))$=0, then our model reduces to a single server M/G/1 retrial feedback queue with starting failure. In this case, the idle probability P_0, utilization factor ρ and the average queue size L_q can be simplified to the following expressions.

$$
\begin{aligned}
P_0 =& 1-\rho \\
\rho =& \frac{\alpha(\lambda E(B)+p)+\bar{\alpha}(1+\lambda E(F))}{\bar{A}(\lambda)} \\
L_q =& \left[\frac{D_1''(1)N_1'''(1)-N_1''(1)D_1'''(1)}{3(D_1''(1))^2}\right]P_0 \\
&+\left[\frac{D_2'(1)N_2''(1)-N_2'(1)D_2''(1)}{2(D_2'(1))^2}\right]P_0
\end{aligned}
$$

where

$$
\begin{aligned}
N_1''(1) =& -2\lambda(\lambda\delta(1+\lambda E(B))+\delta\lambda p+\lambda\bar{\delta}(1+\lambda E(F)) \\
&-\lambda\delta)[1-\bar{A}(\lambda)+\lambda\alpha E(B)+\lambda\bar{\alpha}E(F)] \\
N_1'''(1) =& -3\lambda(2\lambda^2\delta E(B)+2\delta\lambda p(1+\lambda E(B)) \\
&+\lambda^2[\delta\lambda E(B^2)+\lambda\bar{\delta}E(F^2)] \\
&+2\lambda\bar{\delta}+4\lambda^2\bar{\delta}E(F))[1-\bar{A}(\lambda)+\lambda(\alpha E(B)+\bar{\alpha}E(F))] \\
&+3(\lambda\delta(1+\lambda E(B))+\delta\lambda p+\lambda\bar{\delta}(1+\lambda E(F))-\lambda\delta) \\
&\times[-2\lambda(1-\bar{A}(\lambda)+\lambda E(F)\bar{\alpha}) \\
&-2\lambda^2\alpha E(B)(1-\bar{A}(\lambda))-\lambda^2\alpha E(B^2) \\
&-\lambda^2\bar{\alpha}(\lambda E(F^2)+2E(F)(1-\bar{A}(\lambda)))] \\
D_1''(1) =& -2\lambda^2[\alpha-\lambda(\alpha E(B)+\bar{\alpha}E(F))-(1-\bar{A}(\lambda))-\alpha p] \\
D_1'''(1) =& -6\lambda^2(1-\lambda(\alpha E(B)+\bar{\alpha}E(F))-(1-\bar{A}(\lambda))-p\alpha-\bar{\alpha}) \\
&-3\lambda^2(-\lambda^2[\alpha E(B^2)+\bar{\alpha}E(F^2)] \\
&-(1-\bar{A}(\lambda))[2\lambda(\alpha E(B)+\bar{\alpha}E(F))+2\bar{\alpha}] \\
&-2p[\lambda\alpha E(B)+1-\bar{A}(\lambda)]-2\lambda\bar{\alpha}E(F))
\end{aligned}
$$

$$N_2'(1) = -\lambda^2[\delta E(B) + \bar{\delta}E(F)]$$

$$N_2''(1) = -\lambda^3[\delta E(B^2) + \bar{\delta}E(F^2)] - 2\lambda^2\delta E(B) - 4\lambda^2\bar{\delta}E(F)$$

$$D_2'(1) = -\lambda$$

$$D_2''(1) = -2\lambda$$

The above results coincide with the results of Krishna Kumar et al. (2002b).

Case 2: When the server has no vacation, no feedback, no starting failure and $C(z) = z$ i.e, β=0, $\alpha = \delta = q = 1$, $E(I)$= 1 and $E(I(I-1))$=0, then our model reduces to a single server M/G/1 retrial queue. In this case, the idle probability P_0, utilization factor ρ and the average queue size L_q can be simplified to the following expressions.

$$\begin{aligned}
P_0 =& 1-\rho \\
\rho =& \frac{\lambda E(B)}{\bar{A}(\lambda)} \\
L_q =& \left[\frac{D_1''(1)N_1'''(1) - N_1''(1)D_1'''(1)}{3(D_1''(1))^2}\right] P_0 \\
&+ \left[\frac{D_2'(1)N_2''(1) - N_2'(1)D_2''(1)}{2(D_2'(1))^2}\right] P_0
\end{aligned}$$

where

$$\begin{aligned}
N_1''(1) =& -2\lambda^2 E(B)[1-\bar{A}(\lambda) + \lambda E(B)] \\
N_1'''(1) =& -3\lambda(2\lambda^2 E(B) + \lambda^3 E(B^2))[1-\bar{A}(\lambda) + \lambda E(B)] + 3\lambda^2 E(B)) \\
&\times [-2\lambda(1-\bar{A}(\lambda)) - 2\lambda^2 E(B)(1-\bar{A}(\lambda)) - \lambda^2 E(B^2)] \\
D_1''(1) =& -2\lambda^2[\lambda E(B) - \bar{A}(\lambda)] \\
D_1'''(1) =& -6\lambda^3(\lambda E(B) - \bar{A}(\lambda)) \\
&-3\lambda^2(-\lambda^2 E(B^2) - 2(1-\bar{A}(\lambda))\lambda E(B))
\end{aligned}$$

$$N_2'(1) = -\lambda^2 E(B)$$

$$N_2''(1) = -\lambda^3 E(B^2) - 2\lambda^2 E(B)$$

$$D_2'(1) = -\lambda$$

$$D_2''(1) = -2\lambda$$

The above results coincide with the results of Gomez-Corral (1999).

10.9 Numerical results

To numerically illustrate the results obtained in this work, we consider that the retrial times, service times and repair times are exponentially distributed with rates η, μ and r. We study different performance measures under different values of the parameters. All the values were chosen so that the stability condition is satisfied. We base our numerical example on the result found in case 2.

In Table 10.1, we choose the following values: $\mu = 4$, $\eta = 3$ and $r = 2$ while λ varies from 0.1 to 1.0 such that the stability condition is satisfied.

It clearly shows as long as increasing the arrival rate, the server's idle time decreases while the utilization factor, the mean orbit size, system size and the mean waiting time in the orbit and the system of our queueing model are all increases.

In Table 10.2, we choose the following arbitrary values: $\lambda = 0.2$, $\eta = 8$, $r = 11$ and μ varies from 1 to 10 such that the stability condition is satisfied.

It clearly shows as long as increasing the service rate, the server's idle time increases while the utilization factor, average queue size, system size and average waiting time in the orbit and the system of our queueing model are all decreases.

Table 10.1: Computed values of various queue characteristics

λ	P_0	ρ	L_q	L	W_q	W
0.1	0.973750	0.026250	0.002105	0.028355	0.021053	0.283553
0.2	0.945000	0.055000	0.008751	0.063751	0.043755	0.318755
0.3	0.913750	0.086250	0.020523	0.106773	0.068411	0.355911
0.4	0.880000	0.120000	0.038164	0.158164	0.095409	0.395409
0.5	0.843750	0.156250	0.062627	0.218877	0.125253	0.437753
0.6	0.805000	0.195000	0.095161	0.290161	0.158602	0.483602
0.7	0.763750	0.236250	0.137430	0.373680	0.196328	0.533828
0.8	0.720000	0.280000	0.191689	0.471689	0.239611	0.589611
0.9	0.673750	0.326250	0.261065	0.587315	0.290072	0.652572
1.0	0.625000	0.375000	0.350000	0.725000	0.350000	0.725000

Table 10.2: Computed values of various queue characteristics

μ	P_0	ρ	L_q	L	W_q	W
1	0.796360	0.203640	0.088290	0.291926	0.441448	1.459630
2	0.898180	0.101820	0.017430	0.119251	0.087165	0.596256
3	0.932120	0.067880	0.007360	0.075241	0.036810	0.376204
4	0.949090	0.050910	0.004150	0.055058	0.020745	0.275290
5	0.959270	0.040730	0.002720	0.043444	0.013585	0.217222
6	0.966060	0.033940	0.001950	0.035889	0.009746	0.179443
7	0.970910	0.029090	0.001490	0.030577	0.007431	0.152855
8	0.974550	0.025450	0.001180	0.026638	0.005915	0.133188
9	0.977370	0.022630	0.000970	0.023599	0.004863	0.117994
10	0.997964	0.020360	0.000820	0.021183	0.004098	0.105916

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LIST OF PUBLICATIONS

PUBLICATIONS

Papers published

1. Time dependent solution of Non-Markovian queue with two phases of service and general vacation time distribution – *Malaya Journal of Matematik*, Vol. 4, No. 1, pp. 20-29, 2013, [ISSN 2319-3786].

2. Time dependent solution of batch arrival queue with second optional service and optional re-service with Bernoulli vacation - *Mathematical Theory and Modelling*, Vol. 3, No. 1, pp. 1-8, 2013, [ISSN (Paper) 2224-5804, ISSN (Online) 2225-0522] (IC Impact Factor 5.53).

3. Transient behaviour of batch arrival feedback queue with server vacation and balking – *Proceedings of National Conference on Recent Advances in Mathematical Analysis and Applications*, Bonfring Publications, India, pp. 86-96, 2013, [ISBN 978-93-82338-69-7].

4. $M^{[X]}/G/1$ queue with two types of service subject to random breakdowns, multiple vacation and restricted admissibility – *Applied Mathematical Sciences (Ruse)*, Vol. 7, No. 53, pp. 2599 - 2611, 2013, [ISSN 1312-885X], MR 3065161.

5. Analysis of transient behaviour of batch arrival queue with two stage of service and optional re-service – *International Journal of Computational and Applied Mathematics*, Vol. 8, No. 2, pp. 161 - 171, 2013, [ISSN: 1819-4966].

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7. Two stage heterogeneous service, random breakdowns, delayed repairs and extended server vacations with Bernoulli schedule – *International Journal of Computational and Applied Mathematics*, Vol. 8, No. 3, pp. 183 - 201, 2013, [ISSN 1819-4966].

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